Stability of the Stationary Flow of an Incompressible Liquid

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Abstract. The work presents four methods for assessing the stability of stationary vortex structures in multiple continuous regions. First one is based on linearized Navier - Stokes equations using modal analysis. The second method considers non-linear equations and is focused on the stability of the stationary flow at the inner point of the region. The third method is focused on the qualitative analysis of the dependence of an input power, performance and dissipation function. The fourth method is based on the analysis of the potential energy of the flowing liquid. Each of the mentioned methods will be described by a mathematical model, from which it will be possible to assess the influence of boundary conditions and the shape of the area on the mentioned stability.

1 Introduction

An important property of liquids is fluidity, which allows the liquid to form diverse vortex structures[1, 2]. The goal of current research is to study the conditions on which basis it is possible to control the formation of vortex structures. There are only two options. The first, used for many years, is based on inserting various types of obstacles to the flowing liquid and thus influencing its flow. The second, more sophisticated method uses the injection of liquids into the area on the principle of active flow control [3, 4, 5, 6, 7]. The effort is influenced by non-stationary vortex structures and the provision of flow or, conversely, to induce non-stationary vortex structures to fulfil a certain function in the sense of finding new technologies. Both methods are thus related to the analysis of stationary flow stability. Stability conditions can be investigated by various methods:

1. On the analysis of eigenvalues of a linearized problem, including the influence of boundary conditions
2. Based on the wavevector principle, by describing the flow at a selected point in space
3. By analyzing the performance of the flowing liquid
4. By analyzing the potential energy of the flowing liquid

We will perform the mentioned analyzes in a multiply continuous region with a volume $V$ bounded by the surfaces $\Gamma, S, \Theta, \Gamma$ is the shell on which apply the boundary conditions $\mathbf{v} = 0$; $S$ is the union of surfaces through which liquid flows into the area. The boundary condition

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\( \mathbf{v} = \mathbf{v}(\mathbf{x}) \) applies on this surface. \( \Theta \) is the union of the surfaces through which the liquid flows out of the region. On this surface, the condition \( \sigma_{ij} n_j = 0 \) applies. Where \( \sigma_{ij} = -\hat{p} \delta_{ij} + (\eta + \eta_T) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \) and \( \mathbf{n} \) is the vector of the external normal to the surface, \( \hat{p} = p + \frac{2}{3} k \), \( k \) is the kinetic energy of turbulence, \( \eta \) molecular dynamic viscosity, \( \eta_T \) turbulent viscosity.

2 The eigen value analysis

We consider the centred Navier-Stokes equations of the form:

\[
\rho \frac{\partial v_i}{\partial t} + \rho \frac{\partial}{\partial x_j} (v_i v_j) - \frac{\partial \sigma_{ij}}{\partial x_j} = \rho g_i, \quad (1)
\]

and a continuity equation in the form:

\[
\frac{\partial v_i}{\partial x_i} = 0. \quad (2)
\]

Assume that there is a stationary flow where:

\[
v_{0i} = v_{0i}(x_j); \hat{p}_0 = \hat{p}_0(x_j). \quad (3)
\]

Let the perturbation belong to these variables:

\[
c_i = c_i(x_j, t); h = h(x_j, t). \quad (4)
\]

The resulting velocity and pressure will be composed of the following parts:

\[
v_i = v_{0i}(x_j) + c_i(x_j, t); \hat{p} = \hat{p}_0(x_j) + h(x_j, t) \quad (5)
\]

While it holds:

\[
|v_{0i}| \gg |c_i| \text{ and } \hat{p}_0 \gg h \quad (6)
\]

Neglecting non-linear terms

\[
\frac{\partial}{\partial x_j} (c_i c_j) \to 0 \quad (7)
\]

in the equation (1) after substituting (5), linearized equations can be written neglecting the right-hand side in the form:

\[
\rho \frac{\partial c_i}{\partial t} + \rho \frac{\partial}{\partial x_j} (v_{0i} c_j + c_i v_{0j}) - \frac{\partial S_{ij}}{\partial x_j} = 0, \quad (8)
\]

\[
S_{ij} = -h \delta_{ij} + (\eta + \eta_{or}) \left( \frac{\partial c_i}{\partial x_j} + \frac{\partial c_j}{\partial x_i} \right), \quad (9)
\]

\[
\frac{\partial c_i}{\partial x_i} = 0. \quad (10)
\]

Boundary conditions:

\[
\Gamma: c_i = 0; \ S: c_i = 0; \ \Theta: S_{ij} n_i = 0 \quad (11)
\]

We move on to the problem of eigenvalues if we put:

\[
c_i = \varphi_i(x_j) e^{st}; h = \psi e^{st}. \quad (12)
\]
\[
s = \alpha + i\omega; \alpha, \omega \in \mathbb{R}
\]

After substituting into (8), (10) we obtain the system of equations for the determination of \(\varphi_i(x_j), \psi(x_j)\) with the homogenous boundary conditions:

\[
s \rho \varphi_i + \rho \frac{\partial}{\partial x_j}(v_{0i}\varphi_j + \varphi_i v_{0j}) - \frac{\partial \Phi_{ij}}{\partial x_j} = 0,
\]

\[
\frac{\partial \varphi_i}{\partial x_j} = 0; \quad \Phi_{ij} = -\psi \delta_{ij} + 2(\eta + \eta_0)\varphi_{ij}; \quad \varphi_{ij} = \frac{1}{2} \left( \frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right)
\]

Boundary conditions:

\[
\Gamma: \varphi_i = 0; \quad S: \varphi_i = 0; \quad \Theta: \Phi_{ij}n_j = 0
\]

When we provide scalar multiplication of equation (14) by the complex conjugate function \(\varphi_i^*\) and than we use the Gauss ostrogradsky theorem – for example:

\[
\int_V \frac{\partial}{\partial x_j}(v_{0i}\varphi_j + \varphi_i v_{0j})\varphi_i^* dV
\]

\[
= \int_{\Theta} (v_{0i}\varphi_j + \varphi_i v_{0j})\varphi_i^* n_j d\Theta - \int_V (v_{0i}\varphi_j + \varphi_i v_{0j}) \frac{\partial \varphi_i^*}{\partial x_j} dV
\]

we get for the eigen value the equation:

\[
s = \frac{1}{\int_V \varphi_i \varphi_i^* dV} \left[ -\int_{\Theta} (v_{0i}\varphi_j + \varphi_i v_{0j}) \varphi_i^* n_j d\Theta \right. \]

\[
+ \int_V (v_{0i}\varphi_j + \varphi_i v_{0j}) \varphi_i^* dV - 2 \int_V (\eta + \eta_T)\varphi_{ij}\varphi_{ij}^* dV \right]
\]

When \(\alpha < 0\), the system is asymptotically stable, when \(\alpha = 0\) it is in the stability range, when \(\alpha > 0\) non-stationary fluid movement occurs. Its amplitude will be limited by the effect of non-linear terms.

It is clear from the expression (14) that the last term, representing the effect of the dissipation function, has a significantly positive effect on stability. From the expression (15), we can conclude that the system is asymptotically stable when \(v_{0i} = 0\). Thus, the stationary velocity field \(v_{0i}(x_j)\) has a fundamental influence on the formation of unstable vortex structures. Its effect inside the domain \(V\) and on the boundaries can be controlled by new fixed elements inserted into the domain \(V\). This measure changes the velocity field \(v_{0i}(x_j)\) and thus also the eigenshapes of the oscillation \(\varphi_i(x_j)\). The same effect can also be achieved by controlled injection of a jet of liquid into the area \(V\). This again allows us to achieve a change in \(v_{0i}(x_j)\), and thus also a change in the proper forms of the oscillation and the values of \(\alpha\) and \(\omega\).
Assessing stability based on the analysis of eigenvalues and waveforms is quite complicated. A better qualitative idea of stability is provided by the methodology based on the assessment of stability at a selected point in the area using its wave vector. This methodology is based on non-linear equations, assuming the decomposition of the velocity and pressure fields according to (5). In this case, the nonlinear Navier-Stokes equations will have the form:

\[ \frac{\rho \partial c_i}{\partial t} + \rho \frac{\partial v_{0i}}{\partial x_j} c_j + \rho \frac{\partial c_i}{\partial x_j} v_{0j} - \frac{\partial S_{ij}}{\partial x_j} + \rho \frac{\partial c_i}{\partial x_j} c_j = 0. \]  \hfill (16)

Let us assume the solution of the Navier-Stokes equations (16) and the continuity equation (10) in the region of chosen point, in the form:

\[ c_i = A_i e^{st + iK_kx_k}; h = P e^{st + iK_kx_k}; \]

\[ s = \alpha + i\omega; \alpha, \omega \in \mathbb{R}; A_i, P \in \mathbb{C}, \]

where \( K = K_k(K_1, K_2, K_3) \) is wave vector. After the substitution in (16) and (10) we obtain:

\[ \rho s A_i + i\rho K_j v_{0j} A_i + \rho \frac{\partial v_{0i}}{\partial x_j} A_i + \eta K_j K_i A_i + i\rho A_i A_j K_j e^{st + iK_kx_k} = 0, \]

\[ A_i K_i = 0. \]

If we consider the continuity equation (19), which informs that the velocity vector is perpendicular to the eigenvector, the nonlinear term in equation (18) will be equal to zero. By successively multiplying equation (18) by the elements \( A_i, K_i \), we obtain the relation for the real and imaginary part of the eigenvalue \( s = \alpha + i\omega \). If we decompose the tensor \( \frac{\partial v_{0i}}{\partial x_j} \) into a symmetric and an antisymmetric part

\[ \frac{\partial v_{0i}}{\partial x_j} = V_{ij} + \Omega_{ij}, \]

it holds:

\[ \alpha = -\frac{1}{2} V_{ij} A_i^* A_j + A_i A_j^* - \frac{\eta}{\rho} K \cdot K, \]

\[ \omega = -K \cdot v_0 + \frac{i}{2} \Omega_{ij} A_i^* A_j + A_i A_j^* \]

\[ = -K \cdot v_0 + i\Omega \cdot \frac{A \times A^*}{A \cdot A^*}, \]

where \( \Omega \) is the stationary angular velocity vector. It is clear from expression (21) that the symmetric part of the tensor \( \frac{\partial v_{0i}}{\partial x_j} \) fundamentally affects the stability of vortex structures. The effect of kinematic viscosity always has a stabilizing influence, but only depending on the magnitude of the eigenvector \( k \) for which it applies:

\[ k = \frac{2\pi}{\Lambda} = \sqrt{K \cdot K}. \]

From the above, it can be concluded that the eigenshapes of the oscillation with small wavelengths \( \Lambda \) will be more damped. Therefore, low-wavelength eigenshapes of the oscillation will be susceptible to the emergence of instability. The amplitude of the eigenvalue of the pressure function will be given by the relation:

\[ p = i\rho (V_{ij} + \Omega_{ij}) \frac{K_i A_j}{K K^*}. \]  \hfill (24)
A simple example will serve to explain the influence of individual members [9]. Let's assume that in a given area, for example, in the shape of a diffuser or a confuser, the $v_{01}$ component predominates. For $\alpha$ and $\omega$ in the given case, the relations apply:

$$\alpha = -\frac{\partial v_{01}}{\partial x_1} - \frac{\eta}{\rho} \frac{4\pi^2}{\Lambda^2}, \quad \omega = 2\pi \frac{v_{01}}{\Lambda}$$

(25)

As a result of the condition (25) are upcoming the following qualitative relations:

Convergent section (confuser - stability)

$$\frac{\partial v_{01}}{\partial x_1} > 0,$$

(26)

Divergent section (diffuser)

$$\frac{\partial v_{01}}{\partial x_1} < 0,$$

(27)

(instability) in the case, that it holds :

$$-\frac{\partial v_{01}}{\partial x_1} - \frac{\eta}{\rho} \frac{4\pi^2}{\Lambda^2} > 0,$$

(28)

3 Energetic methods

Each of the terms of the Navier-Stokes equation represents a force acting on a unit volume of liquid. Multiplying the equation by the position vector gives energy per unit volume and multiplying it by the velocity of the liquid gives power. On the basis of potential energy and power, it is also possible to make considerations about the stability of vortex structures. Considering flow solution software, they can be used very effectively to actively control unsteady flow. The methods are evident from the following mathematical models [11, 12, 13].

If we multiply the Navier-Stokes equations for an incompressible fluid by the position vector $x$, we get:

$$\rho \frac{\partial v_i}{\partial t} x_i + \rho \left( \frac{\partial v_i}{\partial x_j} v_j \right) x_i + \frac{\partial p}{\partial x_i} x_i - \frac{\partial \Pi_{ij}}{\partial x_j} x_i = \rho g_i x_i.$$  

(29)

Considering the equation of continuity (2), the individual terms of equation (29) can be modified [8, 10] as follows:

$$\frac{\partial v_i}{\partial t} x_i = \frac{1}{2} div (\mathbf{v} x_j x_j) = \frac{1}{2} div (\mathbf{v} |\mathbf{x}|^2),$$

(30)

$$\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t},$$

(31)

$$\frac{\partial v_i}{\partial x_j} v_j x_i = div [(\mathbf{v} \cdot \mathbf{x}) \mathbf{v}] - v^2,$$

(32)

$$\frac{\partial p}{\partial x_i} x_i = gradp \cdot \mathbf{x} = div(px) - 3p,$$

(33)

$$g_i x_i = \frac{1}{4} div (\mathbf{g} \cdot \mathbf{x}) \cdot \mathbf{x}.$$  

(34)
Based on the above relations, equations (29) can be rewritten in the form:

\[ \text{div} \left[ \frac{\rho}{2} \frac{\partial \mathbf{v}}{\partial t} |\mathbf{x}|^2 + \rho (\mathbf{v} \cdot \mathbf{x}) \mathbf{v} + p \mathbf{x} - \sigma - \frac{\rho}{2} (\mathbf{g} \cdot \mathbf{x}) \mathbf{x} \right] = \rho v^2 + 3p, \]  

\[ (35) \]

\[ \sigma_i = \Pi_{ij}x_j, \]  

\[ (36) \]

\[ \mathbf{\sigma} = [2 \text{grad}(\mathbf{v} \cdot \mathbf{x}) - 2 \mathbf{v} + \text{rot} \mathbf{v} \times \mathbf{x}] \eta. \]  

\[ (37) \]

Since the right-hand side of the equation is a positive number, the energy shown in square brackets will flow out of the field \( V \). Going to the limit when calculating the divergence of the function \( \mathbf{a} \) from the relation:

\[ \text{div} (\mathbf{a}) = \left( \lim_{\Delta V \to 0} \int_{S \cup \Theta \cup \Gamma} \mathbf{a} \cdot \mathbf{n} d\beta \right) / \Delta V; \beta = S \cup \Theta \cup \Gamma. \]  

\[ (38) \]

It can be determined that the divergence of the expression in square brackets (35), in the case that the domain \( V \) is formed by a spherical region with the radius of the sphere \( r = R \). In this case, the following applies:

\[ \left( \lim_{\Delta t \to 0} \int_{\beta} \left( \frac{\rho}{2} \frac{\partial \mathbf{v}}{\partial t} R^2 + \rho v_r^2 R + pR - 2\eta \frac{\partial v_r}{\partial r} R \right) d\beta \right) / \Delta V = \rho v^2 + 3p, \]  

\[ (39) \]

When we choose, as an example, the shape of sphere as the control volume \( \Delta V \), it will hold:

\[ \beta = 4\pi R^2; \Delta V = \frac{4}{3} \pi R^3, \]  

\[ (40) \]

and than the value of divergence (39) can be calculated as:

\[ 3 \left( \rho v_r^2 + p - 2\eta \frac{\partial v_r}{\partial r} \right) = \rho v^2 + 3p. \]  

\[ (41) \]

From there we obtain the relation between the resulting velocity at any point of the domain \( V \) and its radial component defined in the spherical coordinate system. Applies to:

\[ v = \sqrt{3} \sqrt{1 - \frac{2\eta}{\rho v_r^2} \frac{\partial v_r}{\partial r} v_r}. \]  

\[ (42) \]

Expression (35) is very important in determining the stability of vortex structures. If the flow is to be stationary, without time-dependent vortex structures, the following must apply:

\[ \text{div} \left[ \rho (\mathbf{v} \cdot \mathbf{x}) \mathbf{v} + p \mathbf{x} - \mathbf{\sigma} - \frac{\rho}{2} (\mathbf{g} \cdot \mathbf{x}) \mathbf{x} \right] = \rho v^2 + 3p, \]  

\[ (43) \]

The difference between the left and right sides of equation (44) can be easily determined by using, for example, the Ansys Fluent software, and thus determine the places of the largest deviations, which will be the source of the instability of vortex structures. By integrating through field \( V \) we obtain:

\[ \int_s [\rho (\mathbf{v} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{n}) + p \mathbf{x} \cdot \mathbf{n} - \mathbf{\sigma} \cdot \mathbf{n}]dS + \int_\Gamma [p \mathbf{x} \cdot \mathbf{n} - \mathbf{\sigma} \cdot \mathbf{n}]d\Gamma + \int_\Theta \rho (\mathbf{v} \cdot \mathbf{x})(\mathbf{v} \cdot \mathbf{n})d\theta = \int_V (\rho v^2 + 3p)dV + \rho \mathbf{g} \cdot \mathbf{z} \mathbf{\tau} V, \]  

\[ (44) \]
where \( z_T \) is the coordinate of the center of gravity of the field \( V \). In further considerations, we neglect the influence of gravity. If the system is to be stable, the left side of expression (45) must be positive. Equation (45) is particularly significant in that it states the influence of the same conditions on the stability of vortex structures. It is well seen on the example of the area of the diffuser shape, which can be described in a spherical coordinate system. See the following Fig. 1.

\[
-\int_S \sigma \cdot \mathbf{n} dS - \int_\Gamma \sigma \cdot \mathbf{n} d\Gamma + \rho R_2 \int_\Theta v_r^2 d\Theta = \int_V (\rho v^2 + 3p) dV + R_1 \int_S (\rho v_r^2 + p) dS.
\]

(45)

If we apply equation (45) to area \( V \) according to Fig. 1, it holds:

\[
-\int_S \sigma \cdot \mathbf{n} dS - \int_\Gamma \sigma \cdot \mathbf{n} d\Gamma + \rho R_2 \int_\Theta v_r^2 d\Theta = \int_V (\rho v^2 + 3p) dV + R_1 \int_S (\rho v_r^2 + p) dS.
\]

It is clear from the expression that at high velocities at the entrance to the domain \( V \) on the surface \( V \), the right-hand side of equation (46) will take on high values that cannot be compensated for by integration over \( \Theta \). Stability in this case can be ensured by viscous forces especially on the surface \( \Gamma \) and the effects of turbulent viscosity on the surface \( S \). If their effect is not sufficient, unstable vortex structures will arise in the domain \( V \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Area \( V \) in the diffuser shape.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Boundary conditions and dimensions of diffuser}
\end{figure}
From equation (45), it can also be concluded that its structures are actively controlled, for example by injecting a stream of liquid into the field \( V \) by creating new input areas \( S \). However, their effect will be reflected again adversely on the right-hand side of equation (45). The injection will make sense only if it affects the velocity field inside the region \( V \). Namely, the velocity components \( v_x \) and \( v_\phi \) by reducing them, if they will significantly influence the formation of vortex structures. If the \( v_x, v_\phi \) components are small, the injection of the liquid stream will have an adverse effect on the stability.

A diffuser flowing through an incompressible liquid was chosen as a demonstration example. The geometry of the area is clear from Fig. 2, from which the boundary conditions are visible. The initial condition was a stationary velocity and pressure field. The task was solved as a planar one, with the STT k-\( \omega \) turbulence model, at a time step of \( 5 \times 10^{-4} \) sec. Images on Fig. 3 show both the velocity and pressure fields in time, with a marked point on the details of the time dependence of the absolute value of the velocity and pressure value. Fig. 4 shows plotted contours of pressure and absolute at the point marked in Fig. 3.

![Figure 3. Pressure and velocity magnitude distribution with corresponding time dependencies](image1)

![Figure 4. Pressure and velocity magnitude distribution in the chosen point in the time dependence (marked in Fig. 3 as a place of plotted contour)](image2)

From the representation of the time dependence (Fig.3), it is clear that even with stationary boundary conditions, the response to the initial conditions in the diffuser is non-stationary, which corresponds to the stated theory.
Another option for assessing the stability of vortex structures is performance analysis. This is obtained by multiplying equation (1) by the velocity $v_i$. We thus obtain the performances of the individual members, related to the unit of volume. After editing, we get:

$$\frac{1}{2} \frac{\partial}{\partial t} (v^2) + div \left( Yv - \frac{1}{\rho} S \right) + \frac{1}{\rho} \sigma = div [(g \cdot x)v].$$  \hspace{1cm} (46)

$$Y = \rho + \frac{1}{2} v^2; \sigma = \Pi_{ij} v_{ij}; S = S(S_i); S_i = \Pi_{ij} v_j,$$  \hspace{1cm} (47)

where $Y$ is the specific energy, $\sigma$ is the density of entropy production.

From equation (49) the stability conditions can again be easily determined. If the flow is to be stationary, i.e. $\frac{\partial c_i}{\partial t} = 0$, the condition must be fulfilled

$$div \left( Yv - \frac{1}{\rho} S \right) + \frac{1}{\rho} \sigma = 0,$$  \hspace{1cm} (49)

while neglecting gravity. This condition can be determined by computational simulation. The value of the left side indicates the degree of instability at a given point $V$. The influence of the boundary conditions can be obtained by integrating (50) over the domain $V$. Using the Gauss-Ostrogradsky theorem, we obtain:

$$\frac{1}{\rho} \int_V \sigma dV = \int_S \left( \frac{1}{\rho} S - Yv \right) n dS + \int_\Theta \left( \frac{1}{\rho} S - Yv \right) n d\Theta$$  \hspace{1cm} (50)

The left side of the equation represents the effect of the dissipation function, which compensates for the adverse effect of the power supplied by surface and convective forces on the area $S$. With an excess of power and a low value of the stationary dissipation function, an unsteady flow characterized by non-stationary vortex structures and non-stationary pressure changes will arise. It again follows from expression (51) that by controlling the injection into the region $V$, only the angular velocities of the liquid in the places of the maximum absolute values of the left side of equation (50) can be eliminated.

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4 Conclusion

Four methods of qualitative and quantitative stability analysis of stationary vortex structures were presented. In the first, it is described by an analytical relation (15), from which the influence of boundary conditions and the shape of the region on the stability of vortex structures can be assessed. The viscosity of the liquid has a significantly positive effect.

The second method analyses the emergence of non-stationary elements at a selected point of the system. Depending on the wave vector and the stationary speed, the stability conditions and frequency of non-stationary oscillations are defined, see expressions (21) and (22).

The third method is based on the definition of potential energy. Its advantage is the possibility to analyse the emergence of non-stationary vortex structures both at a selected point and in the entire area, which enables the expression (43).
The fourth method is based on performance analysis and allows analysis both at a given point of the area using divergence, and in the entire area using the application of the Gauss-Ostrogradsky theorem. Expression (49) shows a significant influence of the density of the dissipation function $\sigma$.

References