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An analytical solution of boundary problems of shell deformation mechanics

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Abstract. This paper proposes an algorithm for the analytical (with a controllable error) solution of boundary-value problems of shell deformation mechanics. Decomposition of partial differential equations is per-formed. The obtained ordinary differential equations in matrix form are solved in the form of converging matrix series in the corresponding coordinates. Atthe point of intersection of the coordinates of ordinary differential equations, the algebraic equations of their connection are satisfied. The error control of the solution of the boundary value problem is carried out by comparing the results when the grid of the de-composition of equations is refined. The study solved the problems of deformation a hinged cylindrical shell and a shallow shell fixed on the edges loaded with external pressure.

1 Introduction

The development of new constructions for transport infrastructure for safe and regular delivery of goods to Mars is one of the tasks of the development of cosmonautics for the period after 2030 [1]. The development of novel promising calculation methods for finding the optimal required characteristics of various structures is an integral part of the development of new technology [2]. As shown in [3], the use of analytical algorithms for solving boundary problems in the mechanics of deformation of thin shells can significantly increase the accuracy of calculations, especially in stress concentrators. The essence of the method is that linear differential equations with partial derivatives written in canonical form, that is, for quantities characterizing the state of the section of the shell, are taken as a mathematical model of the mechanics of shell deformation. Generally, the methods for modeling shell structures in rocket and space, aviation technology are based on FEM [4, 5]. Contrary to them, the proposed method makes it possible to simulate the stress concentration in thin-walled structures with the required accuracy. Partial differential equations [6, 7] are reduced to ordinary differential equations [8, 9], which can be solved analytically [10–12], via decomposition.

2 Materials and Methods

According to the method proposed, the system of p equations of p desired functions $z_1 = z_1(\alpha,\beta), \dots, z_m = z_m(\alpha,\beta)$:

$$L_i(z_1, \dots, z_m) = f_i(\alpha,\beta), \quad i = (1 \dots m). \tag{1}$$

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With boundary conditions:

$$l_j(z_1,\ldots,z_m) = \varphi_j(\alpha,\beta), \quad j = (1\ldots r) \tag{2}$$

by representing the operator in the form of h addends $L_i = \sum_{k=1}^{h} L_{ik}$, $i = (1\ldots p)$ and representing right-hand sides of the equations was a sum of unknown functions $f_i(\alpha,\beta) = \sum_{k=1}^{h} f_i^{(k)}(\alpha,\beta)$ can be reduced to a system of simpler equations and coupling equations:

$$L_{ik}(z_1^{(k)},\ldots,z_m^{(k)}) = f_i^{(k)}(\alpha,\beta). \tag{3}$$

With boundary conditions:

$$l_j(z_1^{(k)},\ldots,z_m^{(k)}) = \varphi_j(\alpha,\beta). \tag{4}$$

And coupling equations:

$$z_1^{(1)}(\alpha,\beta) = z_i^{(k)}(\alpha,\beta). \tag{5}$$

Via decomposition, the system of partial differential equations is reduced to system of equations [13] written in the following form:

$$\begin{gathered} F\left(\alpha,\beta,z_i^{(k)},\frac{\partial z_i^{(k)}}{\partial\alpha},\ \frac{\partial^2 z_i^{(k)}}{\partial\alpha^2},\ \ldots,\ \frac{\partial^n z_i^{(k)}}{\partial\alpha^n},f_i^{(k)}\right) = 0, \\ F\left(\alpha,\beta,z_i^{(k)},\frac{\partial z_i^{(k)}}{\partial\beta},\frac{\partial^2 z_i^{(k)}}{\partial\beta^2},\ldots,\frac{\partial^n z_i^{(k)}}{\partial\beta^m},f_i^{(k)}\right) = 0. \end{gathered} \tag{6}$$

With partial derivatives with respect to only one of the variables in the following form:

$$F\left(\alpha,\beta,z_i^{(k)},\frac{\partial z_i^{(k)}}{\partial\alpha},\frac{\partial^2 z_i^{(k)}}{\partial\alpha^2},\frac{\partial^2 z_i^{(k)}}{\partial\alpha\,\partial\beta},\ldots,\frac{\partial^r z_i^{(k)}}{\partial\alpha^r},\frac{\partial^s z_i^{(k)}}{\partial\alpha^t\,\partial\beta^{s-t}},f_i^{(k)}\right) = 0 \tag{7}$$

and

$$F\left(\alpha,\beta,z_i^{(k)},\frac{\partial z_i^{(k)}}{\partial\beta},\frac{\partial^2 z_i^{(k)}}{\partial\beta^2},\frac{\partial^2 z_i^{(k)}}{\partial\alpha\,\partial\beta},\ldots,\frac{\partial^u z_i^{(k)}}{\partial\beta^u},\frac{\partial^v z_i^{(k)}}{\partial\alpha^w\,\partial\beta^{v-w}},f_i^{(k)}\right) = 0. \tag{8}$$

In these equations, there coupling equations, partial derivatives with respect to a single variable and mixed derivatives.

Mixed derivatives of equations (8) and (9) are considered as partial derivatives with respect to a single variable:

$$\frac{\partial^s z_i^{(k)}}{\partial\alpha^t\,\partial\beta^{s-t}} = \frac{\partial^t\left(\dfrac{\partial^{s-t} z_i^{(k)}}{\partial\beta^{s-t}}\right)}{\partial\alpha^t} \quad \text{or} \quad \frac{\partial^v z_i^{(k)}}{\partial\alpha^w\,\partial\beta^{v-w}} = \frac{\partial^{v-w}\left(\dfrac{\partial^w z_i^{(k)}}{\partial\alpha^w}\right)}{\partial\beta^{v-w}}. \tag{9}$$

Thus, these equations are reduced to the form in (2), (3). Now, equations (2) and (4) can be written in the matrix form:

$$z'_\alpha(\alpha,\beta) = A_\alpha(\alpha,\beta)\,z_\alpha(\alpha,\beta) + f_\alpha(\alpha,\beta). \tag{10}$$

Similarly, equations (2) and (3) are written:

$$z'_\beta(\alpha,\beta) = A_\beta(\alpha,\beta)\,z_\beta(\alpha,\beta) + f_\beta(\alpha,\beta). \tag{11}$$

Evidently, the solution of equations (4) will be as accurate as $\frac{\partial^{s-t} z_i^{(k)}}{\partial \beta^{s-t}}$, while solution of equations (5) will be as accurate as $\frac{\partial^{w} z_i^{(k)}}{\partial \alpha^{w}}$. If differentiated equations (1) are to be used as the coupling equations, then these solutions can be used to satisfy coupling equations. Lesser-order derivatives of functions and the functions themselves do not have to be determined from (4) and (5). They are calculated from (2) and (3).

Such an approach to mixed derivatives implies the need to have many additional boundary functions for the function $\frac{\partial^{s-t} z_i^{(k)}}{\partial \beta^{s-t}}$, $\frac{\partial^{w} z_i^{(k)}}{\partial \alpha^{w}}$ and their derivatives for equations (4) and (5), accordingly. If boundary conditions cannot be expressed naturally, one should satisfy coupling equation in boundary points or in some additional points.

Analytical solutions formulas [14, 15] for ordinary differential equations can also be used for equations similar to (7), (8). As $z(\alpha,\beta)$ is a function of two variables, columns of parameters on the interval boundaries should be interpreted as columns of arbitrary function in formulas [11]. By fixing the value of one of the variables in (7), (8), that is, assuming that $z(\alpha,\beta_1)$, $z(\alpha,\beta_2)\ldots$ and $z(\alpha_1,\beta)$, $z(\alpha_2,\beta)\ldots$, one can solve the differential equations along the corresponging coordinate line for the function of one variable by using [16]. So, one can construct a mesh of solution lines by intermittently fixing one coordinate, then the other (and considering them as parameters of the corresponding equations).

In the intersection points, coupling equations should be applied. By solving the coupling equations for unknown functions $f(\alpha,\beta)$, we get the expressions for their averaged values in the following form:

$$C_{\alpha_i} z_\alpha(\alpha,\beta_i) + C_{\beta_j} z_\beta(\alpha_j,\beta) + C_{ij} q, \tag{12}$$

where C_{α_i}, C_{β_j}, C_{ij}—coefficient matrices, q—external load column. By plugging them in formulas [16] instead of f_i, we can write a system of algebraic equations. Thus, the boundary problem is finally solved by solving the obtained system of algebraic equations.

3 Results

A series of analysis are carried out to test the algorithm [16]. The circumferential and meridian bending moments are calculated in a cylindrical panel (figure 1) along two lines (red in the figure 1). The panel is simple supported.

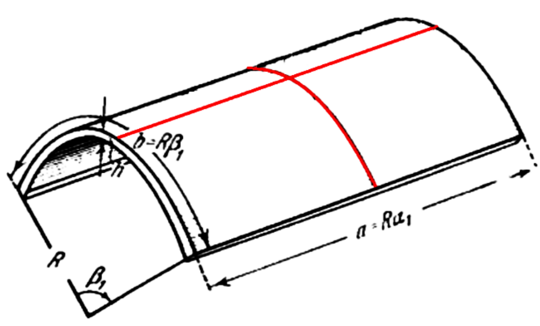
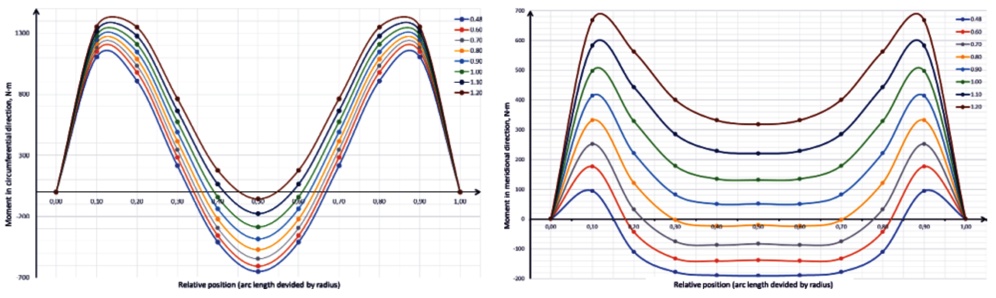


Figure 1. Cylindrical plate

The dimensions of the panel are: in the direction α is 1.5 m, the length of the arc in the direction β is 1.5 m. The radius are varied from 0.48 m (half-cylinder) to 50 m (almost flat plate). The bending moments M_1—meridional and M_2—circumferential are calculated using double rows according to the eq. (13)

$$
\begin{aligned}
M_{1k} &= \frac{16\mu}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^3(m^2+\nu\lambda^2 n^2)\sin\left(\frac{m\pi\alpha}{\alpha_1}\right)\sin\left(\frac{m\pi\beta}{\beta_1}\right)}{n(m^2+\lambda^2 n^2)^2[(m^2+\lambda^2 n^2)^4+\mu m^4]},\\
M_{2k} &= \frac{16\mu}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^3(\lambda^2 n^2+\nu m^2)\sin\left(\frac{m\pi\alpha}{\alpha_1}\right)\sin\left(\frac{m\pi\beta}{\beta_1}\right)}{n(m^2+\lambda^2 n^2)^2[(m^2+\lambda^2 n^2)^4+\mu m^4]}.
\end{aligned}
\tag{13}
$$

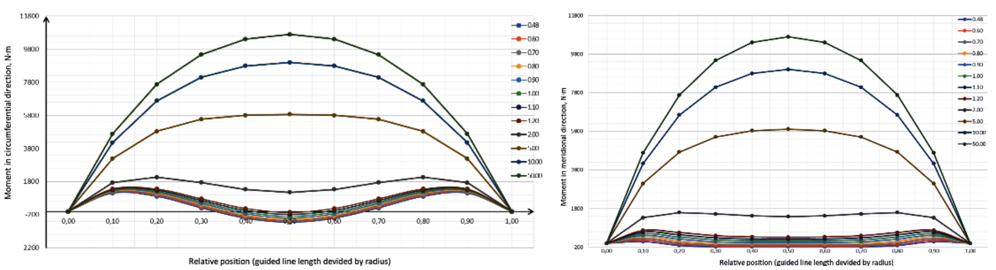
The diagrams below show the moment diagrams: in figures 2(a) and (b) in the range of radius from 0.48 m to 1.20 m, in figures 3(a) and (b) up to 50 m.



(a) (b)

Figure 2. Result for cylindrical simply supported panel: (a) moment in circumferential direction; (b) moment in meridional direction

As shown for some values of the radius, for example 1.0 m, moments become negative, this means that values gave different sign at center and near boundary condition zone. As the radius increases, the panel straightens into a square plate and the moment diagrams become the same in the center part.



(a) (b)

Figure 3. Result for cylindrical simply supported panel: (a) moment in circumferential direction; (b) moment in meridional direction

For a cylindrical panel, the decomposition solution was compared to the solution in double trigonometric series. Table 1 compares the results of calculating deflection and bending moments.

Table 1. Comparison of deflections and bending moments of a cylindrical shell

Column Header Goes Here	Solution in double trigonometric series	Decomposition solution	Difference, %
Deflection, m	$14.23 \cdot 10^{-5}$	$14.19 \cdot 10^{-5}$	2.3
Meridional bending moment, N · m	131.0	130.5	3.5
Circumferential bending moment, N · m	−285.9	−284.8	3.8

Table 2. Results of mesh density influence evaluation

Iteration number	1	2	3	4	5	6	7	8	9	10
Mesh density, meridians × × parallels	5 × 5	7 × 7	9 × 9	11 × 11	13 × 13	17 × 17	19 × 19	21 × 21	23 × 23	25 × 25
Deflection, mm	3.16409	3.16700	3.17151	3.17349	3.17433	3.1747	3.1749	3.17496	3.1749	3.1750
Relative difference, %	0.0919	0.1424	0.0621	0.0267	0.0119	0.0054	0.0025	0.0010	0.0003	0.0000
Bending moment, N	27.315	27.174	27.205	27.238	27.260	27.274	27.284	27.291	27.296	27.300
Relative difference, %	−0.5187	0.1140	0.1196	0.0808	0.0534	0.0364	0.0257	0.0187	0.0140	0.0108

Also, we solved a boundary problem of deformation of a shallow shell fixed at the edges and loaded with external pressure. Displacements w and bending moments M of a double-curvature shell are calculated in its pole. On the shell, a mesh of meridians and parallels was imposed to separate the sides of the shell into intervals. The calculation results are shown in table 2.

Aside from the absolute values of deflections and moments, to evaluate convergence and the solution errors, we also calculated the difference of the calculated value at two subsequent iterations divided by the value obtained at current iteration. In table 2, this quantity is referred to as the relative difference. The proposed approach to solving the problems of shell mechanics and its program implementation enjoys fast convergence of the solution. The solution error can be reduced by increasing the density of the decomposition mesh.

4 Conclusion

The computational algorithm of the proposed analytical method is extremely simple: an analytical solution of ordinary differential equations is determined by formulas in the form of convergent matrix series [7] on intervals with a controlled error, a system of algebraic equations is formed and solved. The problem of calculation stability is overcome by dividing the main (boundary) interval into sub-intervals. A special feature of this algorithm is that no constant integration of differential equations is required.

The efficiency of parametric investigations with variable calculation is ensured by repeated solution of the system of algebraic equation with another load and/or boundary conditions without having to solve the system of differential equations again.

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