Spectral representation of stochastic integration operators

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Abstract. The spectral representation for stochastic integration operators with respect to the Wiener process is proposed in the form of a composition of spectral characteristics used in the spectral form of mathematical description for control systems. This spectral representation can be defined relative to the various orthonormal bases. For given deterministic square-integrable kernels, the spectral characteristic of a stochastic integration operator is determined as an infinite random matrix. The main applications of such a representation suppose solving linear stochastic differential equations and modeling multiple or iterated Stratonovich stochastic integrals. Specific formulas are provided that allow to represent the spectral characteristic for the stochastic integration operator, the kernel of which is the Heaviside function, relative to Walsh functions and trigonometric functions.

1 Introduction

The article considers a stochastic integration operator with respect to the Wiener process [1], it is associated with a spectral characteristic. For the chosen orthonormal basis of the square-integrable function space, it is an infinite random matrix depended on this orthonormal basis. It is proposed to represent the spectral characteristic of the stochastic integration operator by the spectral characteristic of the integration operator and the spectral characteristic of the multiplier, which are used in the spectral form of mathematical description in the control theory [2].

The main idea of the spectral form of mathematical description is to represent functions and random processes in the form of ordered sets of their expansion coefficients for the chosen orthonormal basis, i.e., in the form of spectral characteristics [2]. The relations for calculating such spectral characteristics and transforming them are implemented as packages of elementary and specialized algorithms and programs of the spectral method [2–5]. For the spectral representation of stochastic integration operators, it is proposed to use orthonormal bases that are usually applied in the spectral form of mathematical description: Legendre polynomials, Walsh and Haar functions, trigonometric functions, etc.

The main application of the spectral representation of stochastic integration operators is the modeling of solutions to linear stochastic differential equations and systems of such equations [6]. In particular, it can find the application for the representation and modeling of multiply or iterated Stratonovich stochastic integrals [7, 8]. It can also be used for modeling solutions to nonlinear stochastic differential equations and systems of such equations using high-order numerical methods based on the Taylor–Stratonovich expansion [8–10].

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2 Definitions

Let be $(\Omega, \mathcal{S}, \mathrm{P})$ a complete probability space. The random process is a function $f(t,\omega)\colon [0,T]\times\Omega\to\mathrm{R}$ that is $\mathcal{S}$-measurable for each $t\in[0,T]$, and we denote it by $f(t)$ for brevity. We restrict ourselves to random processes that satisfy the condition

$$\mathrm{E}\int_0^T f^2(t)\,dt < \infty, \tag{1}$$

where E is the mathematical expectation. Realizations of the random process $f(t)$ are elements of the space $L_2([0,T])$.

We call the spectral characteristic of the random process $f(t)$ defined relative to the orthonormal basis $\{q_i(t)\}_{i=0}^{\infty}$ of the space $L_2([0,T])$ the ordered set of expansion coefficients

$$F_i = \int_0^T q_i(t)\,f(t)\,dt, \quad i=0,1,2,\ldots, \tag{2}$$

and this spectral characteristic is represented as the infinite random column matrix F. In fact, the column matrix F defines the random process $f(t)$:

$$f(t) = \sum_{i=0}^{\infty} F_i q_i(t), \quad t\in[0,T], \tag{3}$$

or more strictly: the column matrix F defines the equivalence class that consists of random processes $f(t)$, for which

$$\lim_{n\to\infty}\mathrm{E}\int_0^T \left(f(t) - \sum_{i=0}^{n} F_i q_i(t)\right)^2 dt = 0.$$

For the orthonormal basis $\{q_i(t)\}_{i=0}^{\infty}$, we require the additional condition: the quadratic integrability of products $q_l(t)\,q_j(t)$, $l,j=0,1,2,\ldots$

Let be $\mathcal{A}$ the linear operator (Hilbert–Schmidt operator):

$$\mathcal{A}f(t) = \int_0^T k(t,\tau)\,f(\tau)\,d\tau, \quad f(\cdot)\in L_2([0,T]), \tag{4}$$

where $k(t,\tau)$ is the kernel satisfying the condition

$$\int_0^T\int_0^T k^2(t,\tau)\,d\tau\,dt < \infty.$$

Its spectral characteristic defined relative to the orthonormal basis $\{q_i(t)\}_{i=0}^{\infty}$ is the infinite matrix A with entries [2, 6]

$$A_{ij} = \int_0^T q_i(t)\,\mathcal{A}q_j(t)\,dt = \int_0^T\int_0^T q_i(t)\,k(t,\tau)\,q_j(\tau)\,d\tau\,dt, \quad i,j=0,1,2,\ldots$$

Next, we define the random linear operator $\mathcal{A}_W$:

$$\mathcal{A}_W f(t) = \int_0^T k(t,\tau)\,f(\tau)\,dw(\tau), \quad f(\cdot)\in L_2([0,T]), \tag{5}$$

where $w(t)$ is a standard Wiener random process defined on the probability space $(\Omega,\mathcal{S},\mathrm{P})$, and the infinite random matrix $A^{\mathcal{V}}$ with entries

$$A_{ij}^{\mathcal{V}} = \int_0^T q_i(t)\,\mathcal{A}_W q_j(t)\,dt = \int_0^T\int_0^T q_i(t)\,k(t,\tau)\,q_j(\tau)\,dw(\tau)\,dt, \quad i,j=0,1,2,\ldots, \tag{6}$$

is called the spectral characteristic of the random linear operator $\mathcal{A}_W$.

3 Spectral Representation of Stochastic Integration Operators

Let us introduce an additional notation. Let V be the infinite three-dimensional matrix with entries

$$V_{ijk} = \int_0^T q_i(t)\,q_j(t)\,q_k(t)\,dt, \quad i,j,k = 0,1,2,\ldots$$

In the spectral form of mathematical description, this matrix is called the spectral characteristic of the multiplier [2, 6].

Theorem 1. Let $A^{\mathcal{V}}$ be the spectral characteristic of the random linear operator $\mathcal{A}_W$ with the kernel $k(t,\tau)$, A be the spectral characteristic of the Hilbert–Schmidt operator with the same kernel, V be the spectral characteristic of the multiplier, $\mathcal{V}$ be an infinite random column matrix, whose elements are independent random variables with the standard normal distribution. Then spectral characteristics $A^{\mathcal{V}}$ and A satisfy the relation

$$A^{\mathcal{V}} = A(V\mathcal{V}),$$

or in the expanded form

$$A^{\mathcal{V}}_{ij} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} A_{il}V_{ljk}\mathcal{V}_k.$$

Proof. Let us write the formula (6) for entries of the spectral characteristic $A^{\mathcal{V}}$ of the random linear operator $\mathcal{A}_W$, representing the kernel $k(t,\tau)$ as a series relative to the orthonormal basis $\{q_i(t)\}_{i=0}^{\infty}$:

$$k(t,\tau) = \sum_{k,l=0}^{\infty} A_{kl}\,q_k(t)\,q_l(\tau), \quad t,\tau \in [0,T],$$

i.e.,

$$\begin{aligned}
A^{\mathcal{V}}_{ij} &= \int_0^T q_i(t) \int_0^T \sum_{k,l=0}^{\infty} A_{kl}\,q_k(t)\,q_l(\tau)\,q_j(\tau)\,dw(\tau)\,dt = \\
&= \lim_{n,m\to\infty} \sum_{k=0}^{n}\sum_{l=0}^{m} A_{kl} \int_0^T q_i(t)\,q_k(t)\,dt \int_0^T q_l(\tau)\,q_j(\tau)\,dw(\tau) = \\
&= \lim_{n,m\to\infty} \sum_{k=0}^{n}\sum_{l=0}^{m} A_{kl}\delta_{ik} \int_0^T q_l(\tau)\,q_j(\tau)\,dw(\tau) = \lim_{m\to\infty}\sum_{l=0}^{m} A_{il}\int_0^T q_l(\tau)\,q_j(\tau)\,dw(\tau), \\
&\qquad i,j = 0,1,2,\ldots,
\end{aligned}$$

where δ_{ik} is the Kronecker delta.

Further, we rewrite the product $q_l(t)\,q_j(t)$ as a series relative to the orthonormal basis $\{q_i(t)\}_{i=0}^{\infty}$:

$$q_l(t)\,q_j(t) = \sum_{k=0}^{\infty} V_{ljk}\,q_k(t),$$

then

$$\begin{aligned}
A^{\mathcal{V}}_{ij} &= \lim_{m\to\infty}\sum_{l=0}^{m} A_{il}\int_0^T \sum_{k=0}^{\infty} V_{ljk}q_k(\tau)\,dw(\tau) = \lim_{n,m\to\infty}\sum_{l=0}^{m} A_{il}\sum_{k=0}^{n} V_{ljk}\int_0^T q_k(\tau)\,dw(\tau) = \\
&= \lim_{n,m\to\infty}\sum_{l=0}^{m} A_{il}\sum_{k=0}^{n} V_{ljk}\mathcal{V}_k = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} A_{il}V_{ljk}\mathcal{V}_k, \quad i,j = 0,1,2,\ldots,
\end{aligned}$$

where $\mathcal{V}_k$ are independent random variables with the standard normal distribution. ◂

Note that the definition of the random linear operator $\mathcal{A}_W$ can be extended to the space of random processes $f(t)$ satisfying condition (1). However, it should be highlighted that the stochastic integral in this case is understood in the Stratonovich sense [8–10]. Then the following theorem can be formulated.

Theorem 2. Let X and F be spectral characteristics of random processes $x(t)$ and $f(t)$, respectively, satisfied the relation $x(t) = \mathcal{A}_W f(t)$ be the random linear operator with the spectral characteristic $A^{\mathcal{V}}$. Then spectral characteristics X, F and $A^{\mathcal{V}}$ satisfy the relation

$$X = A^{\mathcal{V}} F,$$

or in the expanded form

$$X_i = \sum_{j=0}^{\infty} A_{ij}^{\mathcal{V}} F_j.$$

Proof. Let us write the formula (2) for entries of the spectral characteristic X:

$$X_i = \int_0^T q_i(t)\, x(t)\, dt = \int_0^T q_i(t)\, \mathcal{A}_W f(t)\, dt, \quad i = 0, 1, 2, \ldots,$$

and represent the random process $f(t)$ in the form (3), and also use the definition (6) for entries of the spectral characteristic $A^{\mathcal{V}}$:

$$X_i = \int_0^T q_i(t)\, \mathcal{A}_W \sum_{j=0}^{\infty} F_j q_j(t)\, dt = \lim_{n\to\infty} \sum_{j=0}^{n} F_j \int_0^T q_i(t)\, \mathcal{A}_W q_j(t)\, dt =$$
$$= \lim_{n\to\infty} \sum_{j=0}^{n} A_{ij}^{\mathcal{V}} F_j = \sum_{j=0}^{\infty} A_{ij}^{\mathcal{V}} F_j, \quad i = 0, 1, 2, \ldots \quad \blacktriangleleft$$

Consider the particular case $k(t, \tau) = 1(t - \tau)$, where $1(t - \tau)$ is the Heaviside function. Here, $\mathcal{A}_W$ is the stochastic integration operator, the spectral characteristic of which we denote by $P^{-1,\mathcal{V}}$. Therefore, if

$$x(t) = \int_0^t f(\tau)\, dw(\tau), \quad t \in [0, T],$$

where the stochastic integral is understood in the Stratonovich sense, and X and F are spectral characteristics of random processes $x(t)$ and $f(t)$, respectively, then

$$X = P^{-1,\mathcal{V}} F = P^{-1}(V\mathcal{V})F,$$

where P^{-1} is the spectral characteristic of the integration operator [2, 6], i.e., the spectral characteristic of the linear operator (4) for $k(t, \tau) = 1(t - \tau)$.

The main application of the above theorems is the represention of solutions to linear stochastic differential equations and systems of such equations [6], including the representation of multiply or iterated Stratonovich stochastic integrals [7, 8]. The advantage of this representation is based on the fact that the calculation for entries of the spectral characteristic P^{-1}, as well as entries of the spectral characteristic V relative to the various orthonormal bases, such as Legendre polynomials, Walsh and Haar functions, trigonometric functions, is often used to solve analysis and synthesis problems for different control systems by the spectral form of mathematical description [2, 11], i.e., algorithms for the calculation are known.

For example, for the Walsh functions

$$q(i, t) = \frac{1}{\sqrt{T}} \begin{cases} 1 & \text{for} \quad i = 0, \\ \prod_{\{k\colon \gamma_k = 1\}} r(k, t) & \text{for} \quad i = 1, 2, \ldots, \end{cases}$$

where $\gamma_k \in \{0, 1\}$ are coefficients in the binary representation of the number i, and $r(k, t)$ are the Rademacher functions defined by $r(k, t) = 2^{\lfloor 2^k \pi t/T \rfloor}$ ($\lfloor . \rfloor$ is the floor function, $k = 1, 2, \ldots$), we have

$$P^{-1}_{00} = \frac{T}{2}, \quad P^{-1}_{k,2^m+k} = -P^{-1}_{2^m+k,k} = \frac{T}{2^{m+2}},$$
$$P^{-1}_{ml} = 0 \quad \text{otherwise}, \quad m, l = 0, 1, 2, \ldots, \quad k = 0, 1, \ldots, 2^m - 1;$$
$$V_{ijk} = \frac{1}{\sqrt{T}} \begin{cases} 1 & \text{for} \quad i \oplus j \oplus k = 0, \\ 0 & \text{for} \quad i \oplus j \oplus k \neq 0, \end{cases} \quad i, j, k = 0, 1, 2, \ldots,$$

where $\oplus$ means the bitwise addition of numbers modulo 2 (XOR bitwise operation). Moreover, it is not difficult to obtain an explicit representation for the product $E^{\mathcal{V}} = V\mathcal{V}$ taking into account the group property of the Walsh functions [12]:

$$E^{\mathcal{V}}_{ij} = \frac{1}{\sqrt{T}} \mathcal{V}_{i\oplus j}, \quad i, j = 0, 1, 2, \ldots,$$

i.e., $E^{\mathcal{V}}$ is the Walsh–Toeplitz infinite matrix [13].

Another example is the use of trigonometric functions

$$q(i, t) = \frac{1}{\sqrt{T}} \begin{cases} 1 & \text{for} \quad i = 0, \\ \sqrt{2} \cos \dfrac{i\pi t}{T} & \text{for} \quad i = 2, 4, 6, \ldots, \\ \sqrt{2} \sin \dfrac{(i+1)\pi t}{T} & \text{for} \quad i = 1, 3, 5, \ldots, \end{cases}$$

as an orthonormal basis (Fourier basis):

$$P^{-1}_{00} = \frac{T}{2}, \quad P^{-1}_{0m} = -P^{-1}_{m0} = \frac{\sqrt{2}T}{(m+1)\pi}, \quad P^{-1}_{l-1,l} = -P^{-1}_{l,l-1} = \frac{T}{l\pi},$$
$$k = 1, 2, 3, \ldots, \quad m = 2k - 1, \quad l = 2k;$$

$$V_{ijk} = V_{jik} = \frac{1}{\sqrt{T}} \begin{cases} \delta_{jk} & \text{for } i = 0, \\ \dfrac{\sqrt{1 + \delta_{ij}}\,\delta_{j-i,k} + \delta_{i+j,k}}{\sqrt{2}} & \text{for } i = 2n, \\ \dfrac{\delta_{i+j,k} - \delta_{j-i-2,k}}{\sqrt{2}} & \text{for } i = 2n-1 \text{ and } j = 2l, \\ \dfrac{\sqrt{1+\delta_{ij}}\,\delta_{j-i,k} - \delta_{i+j+2,k}}{\sqrt{2}} & \text{for } i = 2n-1 \text{ and } j = 2l-1, \end{cases} \quad \begin{array}{l} i, k = 0, 1, 2, \ldots, \\ j = i, i+1, i+2, \ldots, \\ n, l = 1, 2, 3, \ldots, \end{array}$$

and for the product $E^{\mathcal{V}} = V\mathcal{V}$ we can write the following expression if $i \leq j$:

$$E^{\mathcal{V}}_{ij} = E^{\mathcal{V}}_{ji} = \frac{1}{\sqrt{T}} \begin{cases} \mathcal{V}_j & \text{for } i = 0, \\ \dfrac{\sqrt{1 + \delta_{ij}}\,\mathcal{V}_{j-i} + \mathcal{V}_{i+j}}{\sqrt{2}} & \text{for } i = 2n, \\ \dfrac{\mathcal{V}_{i+j} - \mathcal{V}_{j-i-2}}{\sqrt{2}} & \text{for } i = 2n-1 \text{ and } j = 2l, \\ \dfrac{\sqrt{1+\delta_{ij}}\,\mathcal{V}_{j-i} - \mathcal{V}_{i+j+2}}{\sqrt{2}} & \text{for } i = 2n-1 \text{ and } j = 2l-1, \end{cases} \quad \begin{array}{l} i = 0, 1, 2, \ldots, \\ j = i, i+1, i+2, \ldots, \\ n, l = 1, 2, 3, \ldots \end{array}$$

Above formulas are sufficient to represent multiply or iterated Stratonovich stochastic integrals of arbitrary multiplicity with unit weight functions (for integrals of multiplicity 2, relations to calculate entries of the spectral characteristic P^{-1} are sufficient). For other weight functions, the spectral characteristic V is additionally required.

4 Conclusion

The article gives the spectral representation for stochastic integration operators with respect to the Wiener process in the form of a composition of spectral characteristics used in the spectral form of mathematical description for control systems. Results of the article can be applied for the exact or approximate representation of multiply or iterated Stratonovich stochastic integrals and for approximate modeling in the implementation of methods for the numerical solution to stochastic differential equations with high orders of strong convergence [7, 8, 14], where the transition from infinite matrices to finite ones using "truncation" [2, 6] is necessary. Taking into account how the Stratonovich stochastic integral is related to the Itô stochastic integral [8–10], as well as in the general case with the θ-integral [10], $0 \leq \theta \leq 1$, we can obtain the corresponding spectral representations. Obtained results can be generalized to represent the stochastic integration operator with fractional order by defining the kernel $k(t, \tau)$, for example, taking into account the results from [15]. Obtained results can also be applied for modeling optimal control systems with random disturbances [16–18].

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