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Representation of the thermo-stress state of a plate based on the 3D elasticity theory

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Abstract. The general formulation of the 3D temperature static problem of the theory of elasticity is considered. A new representation of the general solution of static equations of thermoelasticity is proposed. The three-dimensional stress state of a thick or thin plate are divided into three parts: bending of the plate, symmetric compression of the plate on it's flat end surfaces and symmetric temperature problem. Third thermoelastic stress state of the plate is studied in detail. It have been reduced to a two-dimensional state after integration the stresses components along the normal to the middle surface. The displacements and stresses in the plate are expressed in terms of two two-dimensional harmonic functions and a particular solution, which is determined by a given temperature on the flat surfaces of the plate. Only the assumption has been used that the normal stresses perpendicular to the middle surface are insignificant. The introduced harmonic functions are determined from the boundary conditions on the lateral surface of the plate. The formula for the experimental determination of the sum of normal stresses has been found through the measured deflections of the end surfaces of the plate.

1 Introduction

Elastic materials are widely used in aerospace and other engineering, therefore, solving the equations of the theory of thermoelasticity and studying their stress-strain state is an important scientific problem [1–3]. Thin and thick plates, to which forces loads, given temperature fields and internal heat sources are applied, are also widely used in power engineering, technological and engineering structures [3, 4]. In [2, 5], a review of the literature on the representation of the equations of the plane theory of thermoelasticity is made using the hypotheses of plane stresses for thin plates .

The purpose of research is to use a three-dimensional thermoelastic stress state to construct a two-dimensional theory of thin and thick plates, which are under the influence of power and temperature loads, without using the hypotheses of plane stresses.

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2 Formulation of the problem and construction of basic equations

Let us consider the general formulation of a three-dimensional temperature static problem of the theory of elasticity for thin or thick plates of constant thickness h, the middle surface of which occupies a region S with a contour L and is located in the plane Oxy of the Cartesian coordinate system: $x_1 = x$, $x_2 = y$, $x_3 = z$. The lateral surface of the plate is denoted by Z. Normal loads $q_j(x, y)$ are applied to the flat end surfaces of the plate ($z = h_j$, $h_1 = h/2$, $h_2 = -h/2$) and temperatures $t_j(x, y)$, $j = \overline{1, 2}$ are set. But there are no tangential loads. We will assume that the temperature field $T(x, y, z)$ in the plate is known.

We use the symmetry of the equations of the theory of elasticity and divide the three-dimensional problem into two parts [6]. For the first task describing the symmetrical bend of the plate, and normal loads of the flat surfaces of the plate are asymmetrical:

$$\sigma_z(x, y, h_1) = g^+(x, y), \quad \sigma_z(x, y, h_2) = -g^+(x, y),$$

and for the second, which describes symmetric compression, are respectively equal

$$\sigma_z(x, y, h_1) = p^+(x, y), \quad \sigma_z(x, y, h_2) = p^+(x, y), \tag{1}$$

where $g^+ = \frac{1}{2}(q_1 - q_2)$, $p^+ = \frac{1}{2}(q_1 + q_2)$, signs "+", "–" describe the values of the corresponding functions on the upper and lower surfaces of the plate. The temperature field is known for each task. For the first problem on the lateral surface Z, the normal stresses will be asymmetric with respect to the middle surface, and for the second, they will be symmetric.

Let us divide the second problem into two problems: 2) symmetric force compression of the plate on flat end surfaces [7]; 3) a symmetric plane temperature problem, when temperatures are known $t^- = t$, $t^+ = t$, $t = \frac{1}{2}(t_1 + t_2)$. There are no forces loads.

Let us consider in detail the third problem. It is described by normal stresses symmetric with respect to the middle surface, and in conditions (1) must be set $p^+ = 0$. On the lateral surface of the plate Z, we have the following three-dimensional boundary conditions:

$$[\sigma_y \sin^2\alpha + \sigma_x \cos^2\alpha + \tau_{xy} \sin 2\alpha]\,|_Z = \tau_{1n}|_Z, \quad \left.\left[\frac{\sin 2\alpha}{2}(\sigma_y - \sigma_x) + \tau_{xy}\cos 2\alpha\right]\right|_Z = \tau_{2n}|_Z,$$

$$[\tau_{xz}\cos\alpha + \tau_{yz}\sin\alpha]\,|_Z = \tau_{3n}|_Z, \tag{2}$$

where $\tau_{jn}(x, y, -z) = \tau_{jn}(x, y, z)$, $j = \overline{1, 2}$, $\tau_{3n}(x, y, -z) = -\tau_{3n}(x, y, z)$ are normal and tangential loads, α is the angle between the normal to the contour L and the axis Ox. From conditions (1), (2) it follows that normal stresses will be symmetrical: $\sigma_j(x, y, -z) = \sigma_j(x, y, z)$, $j = \overline{1, 3}$.

The general representation of thermoelastic stresses is expressed for the three considered problems in terms of deformations [1, 3]

$$\sigma_k = 2G\left[\varepsilon_k + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T(x, y, z)\right], \quad \tau_{kj} = G\gamma_{kj}, \quad k \neq j, \tag{3}$$

where $e = \frac{1-2\nu}{E}\Theta + 3\alpha T$ is volume deformation, $\Theta = \sigma_1 + \sigma_2 + \sigma_3$.

We substitute relation (3) into the equilibrium equations and write down the equations of stationary thermoelasticity in displacements [1, 3]

$$(1-2\nu)\nabla^2 u_k + \frac{\partial e}{\partial x_k} = 2(1+\nu)\alpha\frac{\partial T}{\partial x_k}, \quad k=\overline{1,3}, \tag{4}$$

$$\nabla^2 T(x,y,z) = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{5}$$

where ν is the Poisson's ratio, α is the coefficient of thermal expansion.

We use the general solution of the Lamé equations [8], add to it the well-known thermoelastic potential [1, 3] and write down a new representation of the general solution of the thermoelasticity equations (4), which describes the thermoelastic stress state of the plate

$$u_x = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \quad u_y = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}, \quad u_z = \frac{\partial P}{\partial z} - 4(1-\nu)\Phi, \tag{6}$$

where $P = z(\Phi + \beta\Omega) + \Psi$; Φ, Ψ, Q are three-dimensional harmonic displacement functions; $\Omega = \int_0^z T\,dz$; $\beta = \frac{1+\nu}{2(1-\nu)}\alpha$; Ω, T are known functions that satisfy equation (5); $T(x,y,-z) = T(x,y,z)$. The biharmonic function P, Q satisfie the equations

$$\Delta P + \frac{\partial^2}{\partial z^2}P = 2\frac{\partial}{\partial z}(\Phi + \beta\Omega), \tag{7}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is two-dimensional Laplace operator.

We use the displacement representation (6) and find the deformations. Using formula (3), we determine the general expression for the normal stresses

$$\sigma_j = 2G\left[\frac{\partial^2 P}{\partial x_j^2} - (-1)^j\frac{\partial^2 Q}{\partial x_1\,\partial x_2} - 2\nu\frac{\partial\Phi}{\partial x_3} - 2\beta T\right], \quad j=\overline{1,2},$$

$$\sigma_3 = 2G\left[\frac{\partial^2 P}{\partial x_3^2} - 2(2-\nu)\frac{\partial\Phi}{\partial x_3} - 2\beta T\right], \tag{8}$$

and shear stresses

$$\tau_{12} = G\left[2\frac{\partial^2 P}{\partial x_1\,\partial x_2} + \frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\right],$$

$$\tau_{j3} = G\left[\frac{\partial}{\partial x_j}\left[2\frac{\partial P}{\partial x_3} - 4(1-\nu)\Phi\right] - (-1)^j\frac{\partial^2 Q}{\partial x_{3-j}\,\partial x_3}\right], \quad j=\overline{1,2} \tag{9}$$

where $G = E/2(1+\nu)$, E are shear and Young's moduli. Let us express the volume deformation e and the sum of normal stresses Θ

$$e = -2(1-2\nu)\frac{\partial}{\partial z}\Phi + 2\beta T, \quad \Theta = -2E\left(\frac{\partial}{\partial z}\Phi + \frac{\alpha T}{1-\nu}\right). \tag{10}$$

According to the problem statement and conditions (1), there will be zero stresses on the plate surfaces. Let us take into account the symmetry conditions, relations (8), (10) and the fact that the normal stresses $\sigma_3 \ll \sigma_1$, $\sigma_3 \ll \sigma_2$, after its integration, we obtain

$$\frac{\partial P^+}{\partial z} = 2(2-\nu)\Phi^+ + 2\beta\Omega^+, \tag{11}$$

where $\Omega^+ = \int_0^{h_1} T\,dz$.

We use representations (9) and write down the conditions for the absence of tangential loads on the plate surfaces

$$\frac{\partial}{\partial x_j}\left[\frac{\partial P^+}{\partial x_3} - 2(1-\nu)\Phi^+\right] - \frac{(-1)^j}{2}\frac{\partial^2 Q^+}{\partial x_{3-j}\,\partial x_3} = 0, \quad j = \overline{1,2}. \tag{12}$$

Let us take into account relation (11) and simplify equations (12)

$$4\frac{\partial}{\partial x_j}(\Phi^+ + \beta\Omega^+) = (-1)^j\frac{\partial^2 Q^+}{\partial x_{3-j}\,\partial x_3}, \quad j = \overline{1,2}. \tag{13}$$

From equations (13), after mathematical transformation, we obtain that the values of the functions on the surface of the plate will be harmonious functions:

$$\Delta(\Phi^+ + \beta\Omega^+) = 0, \quad \Delta\frac{\partial Q^+}{\partial z} = 0. \tag{14}$$

After integration of equation (7) along the normal to the middle surface, taking into account relations (11), (14), we write down the key equations of the theory of plates on the basic functions $\tilde{P}$, $\tilde{Q}$, Φ^+

$$\Delta\tilde{P} = -4(1-\nu)\Phi^+, \quad \Delta\tilde{Q} = -2\frac{\partial}{\partial z}Q^+, \tag{15}$$

where $\tilde{P} = \int_{-h_1}^{h_1} P\,dz$, $\tilde{Q} = \int_{-h_1}^{h_1} Q\,dz$. Equations (13)–(15) and the known function Ω^+ are sufficient to represent these functions.

We had used equations (6)–(10) and reduced the three-dimensional problem for a symmetrically loaded thick plate to a two-dimensional problem, without had used the hypotheses of the plane stress state. We used only the statement physically substantiated for the plate that the normal stresses perpendicular to the middle surface are insignificant in comparison with the longitudinal stresses.

We use relations (8), (9), (14), (15) and construct a two-dimensional mathematical theory of a symmetrically loaded heat-sensitive thick plate. To do this, we substitute three-dimensional stresses (8), (9) into the well-known expressions of normal and tangential efforts [7] and after integrating them over the variable we obtain:

$$T_1 = 2G\left[\frac{\partial^2\tilde{P}}{\partial x^2} + \frac{\partial^2\tilde{Q}}{\partial x\,\partial y} - 4\nu\Phi^+ - 4\beta\Omega^+\right], \quad T_2 = 2G\left[\frac{\partial^2\tilde{P}}{\partial y^2} - \frac{\partial^2\tilde{Q}}{\partial x\,\partial y} - 4\nu\Phi^+ - 4\beta\Omega^+\right],$$

$$S_{12} = S_{21} = 2G\left[\frac{\partial^2\tilde{P}}{\partial x\,\partial y} + \frac{1}{2}\left(\frac{\partial^2\tilde{Q}}{\partial y^2} - \frac{\partial^2\tilde{Q}}{\partial x^2}\right)\right]. \tag{16}$$

Relationships (15) (16) coincide with the key equations of the plane stress state found in [7], when $\Omega^+ = 0$.

Note that if we substitute relations (16) into the equilibrium equation of the plate in the efforts [9], then we obtain the equation in partial derivatives of the fourth order

$$\Delta\Delta\tilde{P} = 4\beta(1-\nu)\,\Delta\Omega^+. \tag{17}$$

Equation (17) also follows from the obtained relations (14), (15).

3 Representation of thermoelastic stresses in terms of harmonic functions

Let us take into account the first harmonic equation of relations (14) and determine the representation of the function Φ^+

$$\Phi^+ = -h\frac{\partial^2\varphi}{\partial y^2} - \beta\Omega^+, \tag{18}$$

where φ is unknown harmonic function. We use expression (18), relation (13) and obtain the following simple dependence:

$$\frac{\partial Q^+}{\partial z} = 4h\frac{\partial^2\varphi}{\partial x\,\partial y}. \tag{19}$$

Let us take into account representations (18), (19) and write down the general solution of equations (15)

$$\tilde{P} = 2(1-\nu)h\left[y\frac{\partial\varphi}{\partial y} + \beta\omega_1\right] + hg_1(x,y), \quad \tilde{Q} = -4yh\frac{\partial\varphi}{\partial x} + hg_2(x,y), \tag{20}$$

where ω_1 is a particular solution of the equation

$$h\,\Delta\omega_1 = 2\Omega^+, \tag{21}$$

g_j, $j = \overline{1,2}$ are harmonic functions that can be represented

$$g_1 = (1+\nu)h\left[\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\right], \quad g_2 = (1+\nu)h\left[\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}\right], \tag{22}$$

ϕ, ψ are harmonic functions.

We substitute functions (20), (22) into relations (16) and see that the function ϕ is not included in the representation of efforts (15), so that it can be ignored.

Thus, we can write functions (20) in such a simple form:

$$\tilde{P} = 2(1-\nu)h\left[y\frac{\partial\varphi}{\partial y} + \beta\omega_1\right] - (1+\nu)h\frac{\partial\psi}{\partial x},$$
$$\tilde{Q} = -4hy\frac{\partial\varphi}{\partial x} + (1+\nu)h\frac{\partial\psi}{\partial y}, \quad \Phi^+ = -h\frac{\partial^2\varphi}{\partial y^2} - \beta\frac{h}{2}\Delta\omega_1, \tag{23}$$

where functions ω_1, Ω^+ describe the effect of temperature on the stress state of the plate, and harmonic functions φ, ψ correspond to the plane stress state.

Let us express the efforts in terms of functions that are determined by the known temperature field. Let us substitute expressions (23) into relations (16) and obtain simple formulae

$$T_1 = -Eh\alpha\frac{\partial^2\omega_1}{\partial y^2}, \quad T_2 = -Eh\alpha\frac{\partial^2\omega_1}{\partial x^2}, \quad S_{12} = S_{21} = Eh\alpha\frac{\partial^2\omega_1}{\partial x\,\partial y}. \tag{24}$$

Let's introduce the biharmonic function $U = 2E\left(y\frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x} - \frac{\alpha}{2}\omega_1\right)$. From relations (16), (23), (24) we obtain a general representation of thermoelastic stresses, which coincides in form with the known expressions for the stresses of the plane problem of the elasticity theory [7]

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x\,\partial y}. \tag{25}$$

We have written the sum of normal stresses (25) and have got

$$\sigma_x + \sigma_y = 4\mathrm{E}\frac{\partial^2\varphi}{\partial y^2} - \frac{2\alpha E}{h}\Omega^+. \tag{26}$$

We determine the harmonic functions φ, ψ from the boundary conditions (2) specified on the lateral surface of the plate, which, after integrating them, will be reduced to the conditions on the contour L of the region S

$$[\sigma_y \sin^2\alpha + \sigma_x \cos^2\alpha + \tau_{xy}\sin 2\alpha]\,|_L = \sigma_n,$$
$$\left[\frac{\sin 2\alpha}{2}(\sigma_y - \sigma_x) + \tau_{xy}\cos 2\alpha\right]\Bigg|_L = \tau_n, \tag{27}$$

where $\sigma_n = \dfrac{1}{h}\displaystyle\int_{-h_1}^{h_1} \sigma_n(x,y,z)\,dz\Big|_L$, $\tau_n = \dfrac{1}{h}\displaystyle\int_{-h_1}^{h_1} \tau_n(x,y,z)\,dz\Big|_L$, α is the angle between the contour normal L and the axis Ox. The third condition (2) is identically satisfied.

We will assume that the partial solution of the equation (21) is known function ω_1. The obtained expressions for stresses (25) and boundary conditions (27) make it possible to solve various boundary value problems for thermoelastic plates. Methods for satisfying boundary conditions (27) were developed in [10–12].

We find the displacements u_x, u_y in the plate after averaging formulas (6)

$$u_x = \frac{1}{h}\left[\frac{\partial\tilde{P}}{\partial x} + \frac{\partial\tilde{Q}}{\partial y}\right] = 2(1+\nu)\left\{\frac{\partial^2\psi}{\partial y^2} - y\frac{\partial^2\varphi}{\partial x\,\partial y}\right\} - 4\frac{\partial\varphi}{\partial x} + 2(1-\nu)\beta\frac{\partial\omega_1}{\partial x},$$
$$u_y = \frac{1}{h}\left[\frac{\partial\tilde{P}}{\partial y} - \frac{\partial\tilde{Q}}{\partial x}\right] = 2(1-\nu)\frac{\partial\varphi}{\partial y} - 2(1+\nu)\left[y\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\psi}{\partial x\,\partial y}\right] + 2(1-\nu)\beta\frac{\partial\omega_1}{\partial y}.$$

From the last formula (6) we will find a deflection of the upper flat surface of the plate

$$u_z^+ = -2h\nu\frac{\partial^2\varphi}{\partial y^2} + (1+\nu)\alpha\Omega^+. \tag{28}$$

We use formula (26) and write the deflection (28) in another form

$$u_z^+ = -\frac{h\nu}{2\mathrm{E}}(\sigma_x + \sigma_y) + (1+2\nu)\alpha\Omega^+. \tag{29}$$

If we do not take into account the temperature, then formula (29) will have a simple form

$$u_z^+ = -\frac{h\nu}{2E}(\sigma_x + \sigma_y). \tag{30}$$

Formulas (29), (30) can be used to experimental determination of the sum of normal stresses in the plate. They allow experimentally determine the stress concentration on the free contour ($\sigma_n = 0$) of the hole in the plate. We take into account that the equality is performed on the load-free contour of the hole

$$\sigma_\tau = \sigma_x + \sigma_y, \tag{31}$$

where σ_τ is the unknown stress along the contour of the hole, directed perpendicular to the normal to the contour. The stresses is determined based on equations (29), (31)

$$\sigma_\tau = -\frac{2E}{h\nu}u_z^+ + \frac{2(1+2\nu)}{h\nu}E\alpha\Omega^+. \tag{32}$$

We experimentally have measured the deflection u_z^+ and have defined the temperature $\Omega^+ = \displaystyle\int_0^{h_1} T\,dz$. By formula (32) we find the concentration of stresses σ_τ along the free contour of the hole.

4 Conclusions

On the basis of the three-dimensional theory of elasticity, a two-dimensional theory of thermoelastic, symmetrically loaded thick plates is constructed without the use of hypotheses about flat stresses distributions. The found representations of stresses and displacements are exactly equal to the corresponding averaged values of the three-dimensional theory of thermoelasticity. The obtained formulas imply the representation of the stresses of the plane problem of the theory of elasticity. A linear relationship between the normal deflection of the flat surface of the plate and by the value of the sum of normal stresses has been established. This formula is used to experimentally determine the concentration of stresses along the free contour of the hole in the plate.

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