On the question of the no symmetry of the stress tensor for open systems

Evelina Prozorova[1]

[1]Mathematics and Mechanics Faculty, St. Petersburg State University, 28, Universitetsky pr., Stary Peterhof, 198504, Russia

Abstract. The paper discusses the influence of stress no symmetry in continuum mechanics and in stochastic processes. The no symmetry of the stress tensor arises in connection with the influence of the moment and leads to the ambiguity of the solutions of the equations of classical mechanics and the transformation of two-dimensional structures into three-dimensional in the absence of compensating forces. The model for a continuous medium was constructed earlier from phenomenological considerations and from kinetic theory. The article proposes a modification of the classical Boltzmann equation with the inclusion of the diffusion process and the moment in addition to the convective operator. The importance of these terms in generalized kinetic equations and in stochastic processes (Brownian motion and Landau damping) is assumed. In the last equations the probabilistic terms are replaced by deterministic ones.

1 Introduction

It is known that a significant difference between theory and experiment occurs in those cases when the gradients of physical quantities (velocity, temperature, density, charge, etc.) are significant. The discrepancy between experiment and theory appears in problems of free oscillations, during wave propagation and forced high-frequency (ultrasonic) vibrations. Usually, kinetic and gas-dynamic mathematical modeling is carried out within the framework of a continuous medium model with the choice of a type as vector field description, i.e. mathematical points are considered and conservation laws are written out. An abstract point does not allow one to explore the structure, since there is no structure at the point. Extended information can be obtained on the basis of the generalized kinetic equation [1], the study of which is performed by the Fourier method. The equation is written out for a physically defined point. The analysis of the equation shows that this consideration does not take into account the effect of the moment in non-equilibrium processes and does not take into account the effect of flows through the boundary of the elementary volume, since the description of the process through the distribution function does not allow to take them into account through the boundary [2]. With high-frequency vibrations and sufficiently short wavelengths, the influence of the microstructure of the material inevitably affects.

The most common model that takes into account the microstructure of the material is the Cosserat moment model or similar models [3]. The proposed theories can appear only in the presence of a finite linear structural length parameter. In this case, the proper moment of the environment is actually taken into account. However, the law of conservation of moment is often not considered in this case. To construct conservation laws when using the Ostrogradsky–Gauss theorem and replacing the integral over the surface with an integral over the volume, we neglect the integral term outside, i.e. we neglect the circulation along the sides of the elementary volume (in the two-dimensional case, this is clearly seen) [4].

Circulation means the presence of rotation, which in turn means the presence of a moment of force (angular momentum). As a result, we have a symmetric stress tensor, a symmetric velocity tensor, etc. It does not follow from the Boltzmann equation that the pressure in the Euler and Navier–Stokes equations is equal to one third of the sum of the pressures on the corresponding coordinate axes. Using Pascal's law for equilibrium, the pressure is chosen equal to one third of the pressure on the coordinate pads. However, the theory remains the same when determining different pressures at each of the sites, i.e. p_x, p_y, p_z.

The use of one pressure is possible under equilibrium conditions (Pascal's law), but for nonequilibrium conditions the fact is not obvious. Attention is drawn to this in the textbook of L.G. Loityansky [5]. The role of the moment is clearly traced in meteorology through the formation of vortex formations of various types. In the theory of elasticity, when considering the relationship between the components of the strain tensor and the stress tensor, is used the experimental fact of the change in the components p_x, p_y, p_z (stresses) normal to the elementary area (stresses) that are proportional to the sum of the stresses of other components, and they all differ. It should be noted that in the nonequilibrium case there will always be a moment around the axis outside the plane. If it is not balanced, the problem will become three-dimensional.

The existing theory is connected with the fact that when deriving conservation laws in the theory of elasticity, the contribution of the distributed moment to the conditions of equilibrium of forces is excluded. As a result, the law of balance of forces and moments of forces are considered separately. The inaccuracy in determining the velocities in the stress tensor does not greatly affect the results at low velocities. The peculiarity of classical equations is the absence of "nesting" of simpler equations into more complex ones. In stochastic processes of open systems, the movement of fast molecules is accompanied by a change in the position of the center of inertia, which is accompanied by the appearance of a moment. For the processes under study, a binary interaction is selected that depends only on the distance between particles. The appearance of a moment leads to a change in the direction of velocities and the formation of local structures. It can be assumed that the moment will make a significant contribution to the equation of state for liquids and gases at high pressures (virial equation), since there are fast and slow molecules even for equilibrium distribution functions. The effect of the velocity distribution is most clearly manifested for locally equilibrium conditions in the Langevin and Fokker–Planck equations and in the Landau damping in plasma, in the self-organization of nanostructures. In the general case, during the dynamic formation of the structure, the position of the center of inertia changes, which entails a change in the angular momentum. In fact, when considering the collision integral in the Boltzmann equation, the distribution function changes not only in space, but also in velocities. These changes are taken into account in the listed equations.

A modification of the theory is proposed that eliminates the noticed inaccuracies. In this work, a mathematical analysis of the equations of continuum mechanics, including the system of equations of the theory of elasticity with the original asymmetric stress tensor, is carried out. It turned out that in the plane case for four unknowns in the classical formulation we have three equations: two equations from the stress equilibrium condition and one equation - the moment equilibrium condition.

Unknown voltages: σ_{xx}, σ_{yy}, τ_{xy}, τ_{yx}. Thus, we need to close the problem with an additional condition. In the classical version, this condition is the symmetry condition of the stress tensor. However, you can think of other closure options. The absence of symmetry, as noted, is also indicated by the use of the Ostrogradsky–Gauss theorem for a medium with rotation, although a stationary medium is usually considered. It is possible to obtain a solution to a problem with an no symmetric tensor using an iterative procedure. To do this, you must first solve the problem with a symmetric tensor, leave one solution that was obtained, then use the second equation and solve it as a differential one. This will be one of the possible solutions. In three-dimensional problems, three equations of stress equilibrium and three equations for moments are solved. The algorithm for solving a three-dimensional problem is the same as for a two-dimensional one. Sometimes, when solving problems of the theory of elasticity, the first invariants are used, but this can be done if the tensor is symmetric. When considering vortex motion, the tensor is not symmetric (Love's method) [5].

The equilibrium condition is met if there are no internal and external forces, however, any surface forces can be converted by changing variables into internal forces and deformations will occur. It can be shown that the stress tensor will be symmetric only under equilibrium conditions, but it is impossible to transfer the results to nonequilibrium processes. In equilibrium in continuum mechanics, equilibrium means the uniformity of the distribution of all macroparameters. However, any additional external influence leads to the emergence of certain disturbances and creates a distributed moment, which creates additional force.

The processing of numerical calculations in early works was carried out without averaging the values along the sides of the elementary volume. In the modern theory of elasticity, problems are solved by the finite element method, after which the results are averaged. The finite volume method is often used to numerically solve the Euler and Navier–Stokes equations. However, its formulation is based on differential equations with a symmetric stress tensor. In the mechanics of liquid and gas, it is not customary to average the calculation results over the calculated cells. Consequently, since the mesh has finite dimensions, in old works, the asymmetry of the tensor is partially taken into account in numerical calculations. With a large number of steps, averaging gives significant errors. The main argument in favor of choosing a symmetric tensor is the use of the force equilibrium equation without the influence of the moment. However, the gradient of the moment is power. Consequently, the requirement of simultaneous fulfillment of the laws of the balance of forces and moments of forces changes the equations. In modern experiments with rods, measurements are taken at the center of the rod and do not provide information about the action of the moment, since they are actually one-dimensional measurements. For a model with a symmetric tensor, it is necessary to fulfill the law of conservation of the moment (equality of the mixed derivatives of the projections of the corresponding moments) or the conjugation condition. However, this is rarely done.

It can be assumed that the moment will make a significant contribution to the equation of state for liquids and gases at high pressures, since there are fast and slow molecules even for equilibrium distribution functions. The movement of fast molecules is accompanied by a change in the position of the center of inertia, which is accompanied by the appearance of a moment. The appearance of a moment leads to a change in the direction of velocities and the formation of local structures. This is shown in [6–8]. It seems unnecessary to separate the divergent and rotational components for speed. At low speeds and the previous level of computer technology, this had to be done, especially since the speeds for with and without vortex in no compressible fluid coincide. The physical sense is also well tracked. Currently, there is no need for such a division. The main goal of the work is to show the role of moment in all processes of collective interaction of particles in open systems, while in classical mechanics closed systems are considered and the role of moment is often not taken into account. In the proposed version of the theory, the reversibility of the equations remains the same, but the role of boundary conditions increases. They also receive little attention in modern theory. For example, for the Liouville equation.

2 Examples of the influence of the moment in the theory of elasticity

The problems are taken from the book [9]. Consider the problem when a concentrated force is given on a half-plane. Let us find the stress distribution with and without taking into account the influence of the moment (figure 1(a)). Consider two equilibrium equations forces and the equation of equilibrium of moments

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0, \quad \frac{\partial \sigma_z}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0,$$
$$x\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) - z\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z}\right) + \sigma_{zx} - \sigma_{xz} = 0.$$

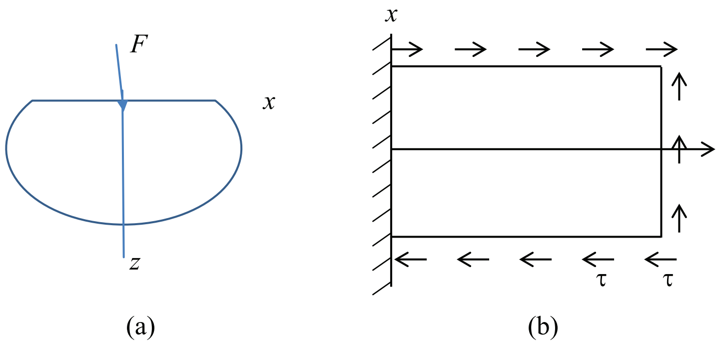

Figure 1. Formulation of the problem one (a), and the problem two (b)

Suggested that $\sigma_{zx} = -A\frac{x}{x^2+z^2}$, option for case $\sigma_{xz} = \sigma_{zx}$ and

$$\sigma_z = -A\frac{z^3}{(x^2+z^2)^2}$$

was considered in [4]. First, we solve the problem when the symmetry condition of the stress tensor is satisfied $\sigma_{xz} = \sigma_{zx}$

$$\frac{\partial \sigma_{zx}}{\partial z} = -A\frac{2xz}{(x^2+z^2)^2},$$
$$\sigma_x = -2Az\int_0^x \frac{x\,dx}{(x^2+z^2)^2} - Az\int_0^y \frac{dy}{(y+z^2)^2} = Az\frac{1}{(z^2+y)} + f(z),$$
$$\sigma_x = Az\frac{1}{(z^2+x^2)} + f(z),$$
$$\frac{\partial \sigma_{xz}}{\partial x} = -A\left(\frac{1}{x^2+z^2} - \frac{2x^2}{(x^2+z^2)^2}\right) = -A\left(-\frac{z^2-x^2}{(x^2+z^2)^2}\right),$$
$$\sigma_z = -A\int_0^z \left(\frac{1}{x^2+z^2} - \frac{2x^2}{(x^2+z^2)^2}\right)dz =$$
$$= -A\frac{1}{x}\operatorname{arctg} z - A\left(\frac{z}{x(1+z^2)} + \frac{1}{x}\operatorname{arctg} z\right) = -A\frac{z}{x(1+z^2)} + f(x).$$

Let σ_x and σ_{zx} remain the same. Then from the equation for the moments it is possible to determine the difference a new value σ_{xz}

$$\frac{\partial \sigma_{xz}}{\partial x} - \frac{\sigma_{xz}}{x} + \frac{\partial \sigma_{zz}}{\partial z} = 0,$$
$$x\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}\right) + \sigma_{zx} - \sigma_{xz} = 0.$$

Then a new value σ_z ($\sigma_z = \sigma_{zz}$) determined. An example is given in order to demonstrate the many possible solutions in the two-dimensional case. Consequently, the solution of the two-dimensional problem under the condition of symmetry of the stress tensor is only one of the options for solving the problem. Algorithm for finding stresses with an no symmetric tensor easily programmed using previously obtained results. Consider the combined action of shear stresses τ (figure 1(b)).

The solution of e equations with a symmetric tensor is the function

$$\tau_{xz} = \tau_{zx} = \varphi(x), \quad \sigma_z = \varphi' z + R(x).$$

From the equation for the moments and iterating, we obtain

$$\tau_{xz} = \varphi(x) + f(x).$$

The functions $\varphi(x)$, $f(x)$ are determined from the boundary conditions.

Algorithm for finding stresses with an asymmetric tensor easily programmed using previously obtained results. Earlier, the no symmetry of the stress tensor was proved in [8], proceeding from the generalization of the Ostrogradsky–Gauss theorem for continuum mechanics. The proposed study complements the previously obtained results. Thus, the symmetry condition for the stress tensor is one of the possible conditions for the closure of the system of equations of the theory of elasticity. Consequently, for structural particles, in addition to the system of internal moments, there are moments associated with the action of stress gradients, and they need be taken into account.

3 The influence of the angular momentum in stochastic processes

Consider the classical problems of nonequilibrium statistical mechanics [10–14]. One of the problems, the problem of Brownian motion, is associated with the motion of a heavy colloidal particle placed in a liquid consisting of light particles. The solution to the problem is associated with the introduction of a probabilistic process. Let's try to replace it with a deterministic one. The process of motion of a Brownian particle is referred to as stochastic. Let us first discuss the changes in the Boltzmann equation taking into account the influence of the moment. For the generalized kinetic equation in the Landau representation, it is also required to introduce new terms.

The classical kinetic theory (modified and classical) also does not follow Pascal's law. Consider the classical Boltzmann equation (the law of momentum does not hold)

$$f(t + dt, r + \xi_j\, dt, \xi_j + F_j\, dt)\, dr\, d\xi_j = f(r_i, \xi_j, t)\, dr\, d\xi_j + \left(\frac{\partial f}{\partial t}\right)_{coll} dt,$$
$$u(t, x) = \frac{1}{n}\int \xi f(t, x, \xi)\, d\xi, \quad P_{ij} = m\int c_j c_i f(t, x, \xi)\, d\xi,$$
$$q_i = m\int c^2 c_i f(t, x, \xi)\, d\xi, \quad c = \xi - u.$$

f—is the distribution function, r—is the radius vector; x—point coordinate; ξ—is the velocity of a point, is the molecular weight, and, according to the definition of the distribution function f_N, the probability of finding the system at points (x, ξ) in the intervals $dx_i\, d\xi_i$ is

$$f_N(t, x_1, x_2, \ldots, x_N, \xi_1, \xi_2, \ldots, \xi_N)\, dx_1 \ldots dx_N\, d\xi_1 \ldots d\xi_N.$$

Taking into account rotation and diffusion, the equation has the form

$$f\left(t + dt, r + \xi_i\, dt + r \times \omega, \xi_i + F_i\, dt + \frac{\partial M}{\partial r} dt\right) dr\, d\xi_i + G_2(t + dt, r + \xi_i\, dt + r \times \omega\, dt),$$

$$\xi_i + F_i\, dt + \frac{\partial M}{\partial r} dt = f(t, r, \xi_i)\, dr\, d\xi_i + G_1(t, r, \xi_i) + \left(\frac{\partial f}{\partial t}\right)_{coll} dt.$$

M is the moment associated with the collective action of all particles on each other as a result of the displacement of the center of inertia, which is the result of the movement of particles with different speeds. G_1 and G_2—flows through the boundaries of the considered elementary volume. These values $G_i = m\xi_i \frac{\partial f}{\partial r}$. They are responsible for the additional terms obtained by S.V. Vallander from phenomenological considerations [15]. When calculating macroparameters through the distribution function and projecting values on the coordinate axis, the symmetry of some quantities not may be violated. This can happen when calculating the pressure and the pressure tensor:

$$P_{ij} = m \int c_i c_j f(t, x, \xi)\, d\xi.$$

The symmetry of the stress tensor is postulated on the basis of this form. In aeromechanics, the projections of the calculated values used, and not the indices of the velocities included in the formula. Therefore, there is no way to speak unambiguously about symmetry.

The theory of Brownian motion is one of the main branches of the statistical theory of open systems. In the theory of Brownian motion elementary objects are small particles, while in kinetic theory, the main objects are molecules. Both models are macromodels, but the level of description of the structure of the environment is different. The kinetic equation corresponds to a more detailed description. We believe that the environment is in equilibrium. We will consider two approaches to solve problems: the equation for a single particle and for an ensemble of particles (the Fokker–Planck equation) To take into account the atomic structure of a liquid, Langevin introduced an additional force into the equations of motion

$$F_L = mv(t), \quad F = -m\gamma v, \quad \gamma = m\eta, \quad \eta = \rho v.$$

In the classical case, equilibrium is possible between Brownian particles and the medium; the particles can be distributed evenly. However, such an assumption can be considered unlikely due to the distribution of particles over velocities and the formation of new moments for individual particles due to the motion of the center of inertia. The fact is that in this case the action of the moment creates a force that distributes the particles not only in terms of velocities, but also in coordinates. The proposed modified Fokker–Planck equation has the form:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial r} - \frac{1}{m}\frac{\partial U}{\partial r}\frac{\partial f}{\partial v} + \frac{1}{m}\frac{\partial M}{\partial r}\frac{\partial f}{\partial v} = D\frac{\partial^2 y}{\partial v^2} + \frac{\partial}{\partial v}(\gamma v f).$$

Thus, in the kinetic theory for a gas, for the Landau damping and for the motion of Brownian particles, the nonuniform distribution of particles in velocities and coordinates is supported by the angular momentum and creates fluctuations in physical quantities that must be taken into account. The action of the moment is local in space and time. Therefore, we can say that the process will still remain "Markov" as in the probabilistic approach.

Let us turn to Landau damping in plasma. The peculiarity of plasma is not the binary interaction of two particles, but the collective one. In calculations, ions can be considered as heavy particles and, consequently considered as Brownian particles.

Therefore, let's move on to the simplest problem. Let us consider oscillations in a plasma without collisions, that is, let us proceed to the study of waves propagating in a plasma, the frequency of which is high in comparison with the frequency of pair collisions of electrons and ions. In this case, there are several options to consider. Landau collisional damping for large Knudsen numbers; for small Knudsen numbers in unbounded plasma; for small Knudsen numbers in a confined plasma. They differ from each other. When studying oscillations, we will consider small deviations from equilibrium Consider an unbounded plasma for small Knudsen numbers $L \ll \lambda$. Diffusion works here. Let us consider the dispersion and damping of longitudinal oscillations of an electron plasma under the influence of the thermal motion of plasma particles [1].

Let us investigate a variant of a limited plasma, a free-molecular flow with a region of wavelengths (values of wave numbers) for which the contribution corresponding to Landau damping is the main one.

$$l \to L^0 \quad (l \gg L), \qquad r_D \ll \gamma \ll \sqrt{r_D L} \quad (r_D \ll L \ll l).$$

We must use the Vlasov kinetic equations with a self-consistent field. Since we are interested in high-frequency oscillations, for which $\omega\tau \gg 1$, where τ is the average time between pair collisions of particles, we can ignore the integrals of particle collisions in the kinetic equations. Longitudinal oscillations of an electron plasma in the classical case are described by the following two equations (collisionless case, Vlasov equation)

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial r} + e\delta E\frac{\partial f_0}{\partial p} = 0,$$

$$div\,\delta E = 4\pi \int dp\,\delta f,$$

$$\varepsilon_l(\omega, k)\,e \int dc\,\delta f(p, k, \omega) = i \int dr e^{-ikr} e \int dp \frac{\delta f(p, r, t_0)}{\omega - kv}.$$

Proposed option

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial r} + e\delta E\frac{\partial f_0}{\partial p} + \frac{\partial \delta M}{\partial r}\frac{\partial f_0}{\partial p} + \frac{\partial}{\partial r} D \frac{\partial \delta f}{\partial r} = 0,$$

$$div\,\delta E = 4\pi \int dp\,\delta f,$$

$$\varepsilon_l(\omega, k)e \int dc\delta f(p, k, \omega) = i \int dr e^{-ikr} e \int dp \frac{\delta f(p, r, t_0)}{\omega - kv} + i \int dr e^{-ikr} \int dp \frac{\delta M(p, r, t)}{\omega - kv}\frac{\partial f_0}{\partial p}.$$

Qualitatively, we can say that for this case, diffusion plays a small role and, since part of the energy is converted into rotational motions (the action of the moment), the reversible operator will act as a dissipative one. Note that at the initial moment, the distributed moment of force also exists and concentrates a certain amount of energy. For monochromatic waves of large amplitude, the action (principles) can lead to the formation of a vertical velocity component, forming complex plane flows. Despite the collisionless nature of the movement binary collisions exist [16]. They create additional dissipation. It should be noted that the generalized equation for a unified description of kinetic and gas-dynamic processes is suitable for "weak" interactions. As before, the contribution of the moments in the motion of molecules is not taken into account. Most likely, the difference between the most probable and average values is due precisely to the lack of taking into account the rotational movements for which the moment is responsible.

4 Conclusion

Open systems are among the most real systems encountered in practice. Building models of such systems is an important task. The paper analyzes the effect of stress no symmetry in models of continuum mechanics and in stochastic processes of open systems. The no symmetry of the stress tensor associated with the work of the moment leads to ambiguity in the solutions of the equations of classical mechanics and the transformation of two-dimensional structures into three-dimensional in the absence of compensating forces.

The substantiation of the no symmetry of the strain tensor on the basis of continuum mechanics and kinetic theory is considered. A modification of the classical Boltzmann equation with the inclusion of a diffusion process and a moment is proposed. The importance of these terms in generalized kinetic equations and in stochastic processes (Brownian motion and Landau damping) is assumed. In the last equations, the probabilistic terms are replaced by deterministic ones.

References

[1] Y.L. Klimontovich, *Statistical theory of open systems*, vol. 1, 2 (Lenand, Moscow, 2019)
[2] E.V. Prozorova, JP Journal of Heat and Mass Transfer, **17**, 1 (2019)
[3] E. Cosserat, F. Cossert, *Theories des corps deformables* (Hermann, Paris, 1909)
[4] E. Prozorova, *Ostrogradsky–Gauss Theorem for Problems of Gas and Fluid*, J. Phys.: Conf. Ser., **1334**, 012009 (2019)
[5] L.G. Loytsyansky, *Mechanics of liquid and gas* (Nauka, Moscow, 1970)
[6] E. Prozorova, WSEAS Transaction on Applied and Theoretical Mechanics, **15** (2020)
[7] E. Prozorova, WSEAS Transaction on Fluid Mechanics, **15** (2020)
[8] E. Prozorova, WSEAS transactions on Applied and Theoretical Mechanics, **16** (2021)
[9] E.A. Bulanov, *Moment stresses in the mechanics of solid, bulk and liquid bodies* (University book, Moscow, 2012)
[10] R. Balescu, *Equilibrium and nonequilibrium statistic mechanics*, (A Wiley-Intersciences Publication John Willey and Sons, New-Yourk–London, 1975)
[11] P. Resibois, M. De Lener, *Classical kinetic theory of fluids* (John Wiley and Sons, New-York–London, 1977)
[12] N.G. Van Kampen, *Stochastic processes in physics and chemistry* (North-Holland, Amsterdam, 1991)
[13] *Physics of Simple Liquids*, ed. by H.N.V. Temperley, I.S. Rowlinson, G.S. Rushbrooke (North Holland Publishing Company, Amsterdam, 1968)
[14] A.A. Vlasov, *Nonlocal statistical mechanics* (Nauka, Moscow, 1978)
[15] S.V. Vallander, M.P. Elovskih, Reports of the ANAS, **79**, 37 (1951)
[16] F. Goodman, G. Vachmann, *Dynamics of gas scattering by a surface* (Mir, Moscow, 1980)