Methods and algorithms for determining the main quasi-homogeneous forms of polynomials and power series

Viktor Nefedov[1,*]

[1]Moscow Aviation Institute (National Research University), 4, Volokolamskoe shosse, Moscow, 125993, Russia

Abstract. Methods are proposed that allow one to determine the special forms of polynomials and power series used in solving a number of practical problems. The most important of them are the construction of necessary and sufficient conditions for an extremum for polynomials and power series, as well as checking matrices for D-stability arising in the study of ecosystem stability. This special forms (the so-called main quasi-homogeneous polynomial forms) are generalizations of the concept of a homogeneous polynomial form. They correspond to the sum of the terms of the polynomial belonging to some face of the Newton polytope of this polynomial. In some cases, the main quasi-homogeneous polynomial forms necessary for research can also be determined for power series (in particular, when constructing necessary and sufficient conditions for an extremum). In the case of polynomials, two cases are investigated separately: the selection of all the main forms of the polynomial and the selection of the main forms corresponding to the faces of the Newton polytope in its "southwestern" part (such forms also can be distinguished for an arbitrary power series), since both cases have their practical applications. Practically applicable methods are described for each of these cases. Several methods are considered sequentially (starting with a simple enumeration and ending with a method with a significant reduction in the number of options in the enumeration). The last (most economical) method is described as a practically realizable algorithm. A practically realizable rather economical algorithm for solving an auxiliary problem is described—finding the set of corner points of the Newton polytope.

1 Introduction

Let us briefly recall the main stages of the study of the polynomial $p(x)$, where $x=(x_1,\dots,x_n)$, to an extremum, for definiteness—to a minimum. Let $x^{(0)}$—some stationary point for which $p'(x^{(0)}) = 0_{(n)} \in \mathbb{R}^n$. The next step is to check the sufficient minimum condition. For this, the quadratic form $\varphi(h) = \langle p''(x^{(0)})h, h\rangle$, where $h \in \mathbb{R}^n$, is investigated. Then, if this form is positive definite, i.e. $\varphi(h) > 0$ for all $h \neq 0_{(n)}$ (which in fact means that $p(x)$ is strongly convex in a neighborhood of $x^{(0)}$) then $x^{(0)}$ is a local minimum point of the polynomial $p(x)$. If $\exists h^{(0)} \in \mathbb{R}^n : \varphi(h^{(0)}) < 0$, then it is not a local minimum point. And if $\varphi(h) \geq 0$ for all $h \in \mathbb{R}^n$, and at the same time $\exists h^{(0)} \in \mathbb{R}^n$: $h^{(0)} \neq 0_{(n)}$, $\varphi(h^{(0)}) = 0$, then it says: "more subtle research methods are required", which are usually not given. Thus, even in the case of a weak convexity in the neighborhood of $x^{(0)}$, the proposed schemes "do not work". Mean-

---

[*] e-mail: nefedovvn54@yandex.ru

while, cases with $\varphi(h) \equiv 0$, when $p''(x^{(0)})$ is a zero matrix are possible and for these cases sufficient conditions for a minimum are also described [1, 2]. In the case $\varphi(h) \equiv 0$, we can only assert that there is no quadratic form, but other forms can exist, i.e. a generalization of the notion of a homogeneous form of a polynomial is required. By such a generalization we mean the sum of the terms of the polynomial corresponding to some face of the Newton polytope [3] of the polynomial $p(x)$. It should be noted that Newton's polytopes are a means for studying a wide class of problems (see, for example, [3–6]). In particular, Newton's theory of polyhedra connects the geometry of polyhedra with algebraic geometry [6]. Let us introduce some notation. For a vector $k \in \mathbb{Z}^n_{\geq}$, where $\mathbb{Z}_{\geq} = \mathbb{N} \cup \{0\}$, a polynomial $p(x)$, and a set $N \subseteq \mathbb{Z}^n_{\geq}$ we denote: $x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$; $coef(p,k) = a$—coefficient at the term ax^k in $p(x)$; $N_p = \{k \in \mathbb{Z}^n_{\geq} \mid coef(p,k) \neq 0\}$—is the carrier of the polynomial $p(x)$; CoN_p—is the convex hull of the set N_p (Newton's polytope of the polynomial $p(x)$); $p_N(x) = \sum_{k\in N} coef(p,k)x^k$. We will call a polynomial $\varphi(x)$ an A-quasi-homogeneous (polynomial) form, where $A \in \mathbb{N}^n$, if $\varphi \not\equiv 0$, $\exists B \in \mathbb{Z}_{\geq}$: $\forall k \in N_\varphi$ $\langle A, k\rangle = B$; we will call $\varphi(x)$ a quasi-homogeneous form, if $\exists A \in \mathbb{N}^n$, such that $\varphi(x)$ is an A-quasi-homogeneous form. Polynomial $p(x)$ will be called: nonnegative, if $\forall x \in \mathbb{R}^n$ $p(x) \geq 0$; nondegenerate in the weak sense, if $x_{i_1} \neq 0, \ldots, x_{i_r} \neq 0 \Rightarrow p(x) \neq 0$, where $\{x_{i_1}, \ldots, x_{i_r}\}$—is the set of variables from which $p(x)$ essentially depends. We will write for the polynomials $q(x), p(x)$: $q \preccurlyeq p$, if (a) $N_q \subseteq N_p$; (b) $\forall k \in N_q$ $coef(q,k) = coef(p,k)$. For the polynomials $p(x), \varphi(x)$ and the vector $A \in \mathbb{N}^n$ we say that $\varphi(x)$ is the main A-quasi-homogeneous form of $p(x)$ if (a) $\varphi \preccurlyeq p$; (b) $N_\varphi = \{k \in N_p \mid \langle A, k\rangle = B_A\}$, $B_A = \min\{\langle A, k\rangle \mid k \in N_p\}$. We say that a polynomial $\varphi(x)$ is a main quasi-homogeneous form of $p(x)$ if $\exists A \in \mathbb{N}^n$ such that $\varphi(x)$ is a main A-quasi-homogeneous form of $p(x)$.

The use of quasi-homogeneous forms gives the following necessary and sufficient conditions for an extremum.

Statement 1 [1]**.** Let $p(x)$ be a polynomial, $p(0_{(n)}) = 0$, $p'(0_{(n)}) = 0_{(n)}$, and all main quasi-homogeneous forms of $p(x)$ are non-negative and non-degenerate in the weak sense. Then $0_{(n)}$ is a point of a local minimum.

Statement 2 [1]**.** Let $0_{(n)}$ be a point of local minimum of the polynomial $p(x)$, $p(0_{(n)}) = 0$, $p'(0_{(n)}) = 0_{(n)}$. Then all main quasi-homogeneous forms of $p(x)$ are non-negative.

Example 1. (1) Let $p_1(x_1, x_2) = x_1^2x_2^6 - 1.98x_1^4x_2^5 + x_1^6x_2^4 + x_2^{10} - 10x_1x_2^9 - 20x_1^8x_2^4$. Then the main quasi-homogeneous forms of this polynomial are: (a) $\varphi(x_1, x_2) = x_1^2x_2^6 - 1.98x_1^4x_2^5 + {} + x_1^6x_2^4 = 0.99x_1^2x_2^4(x_2 - x_1^2)^2 + 0.01x_1^2x_2^4(x_2^2 + x_1^4)$—non-negative and non-degenerate in the weak sense (here $A = \langle 1, 2\rangle$; if $x_1 \neq 0$, $x_2 \neq 0$, then $\varphi(x_1, x_2) > 0$); (b) $x_2^{10}, x_2^{10} + x_1^2x_2^6, x_1^2x_2^6$, $x_1^6x_2^4$ are non-negative and non-degenerate in the weak sense. Then, by virtue of statement 1, $0_{(2)}$ is the point of the local minimum of the polynomial $p_1(x_1, x_2)$. (2) Let $p_2(x_1, x_2) = {} = x_1^2x_2^6 - 2.01x_1^4x_2^5 + x_1^6x_2^4 + x_2^{10} - 10x_1x_2^9 - 20x_1^8x_2^4$. Then the main quasi-homogeneous form $\eta(x_1, x_2) = x_1^2x_2^6 - 2.01x_1^4x_2^5 + x_1^6x_2^4$ of the polynomial $p_2(x_1, x_2)$ is not non-negative ($\eta(1, 1) = -0.01$), and, by virtue of statement 2, $0_{(2)}$ is not a local minimum point.

Remark 1. The definition of the main quasi-homogeneous forms of the polynomial was given in relation to the problem of studying stationary points for local minimality. In this regard, a vector $A \in \mathbb{N}^n$ was used, and with this choice of A, the quasi-homogeneous forms corresponded to the faces of the Newton polyhedron in its "southwestern" part. But when solving more general problems (for example, when studying a polynomial for definiteness in a positive orthant [7, 8]), quasi-homogeneous forms corresponding to all faces of the Newton polytope may be required, and in this case we will use $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$ instead of $A \in \mathbb{N}^n$ (see [9]).

Remark 2. In the case $A \in \mathbb{N}^n$, we can pose the problem of finding the set of main quasi-homogeneous forms not only for the polynomial, but also for the power series, which is the result of the expansion of the analytic function in powers of the variables in the vicinity of the stationary point, since in this case their number will be finite and it will be required a finite number of "first" terms from this expansion. Moreover, statements 1, 2 remain valid for the power series $p(x)$ (see [2]).

2 Methods for finding the main quasi-homogeneous forms of a polynomial

The simplest is

Method 1. Let us first consider the case when it is required to select all main A-quasi-homogeneous forms of the polynomial $p(x) \not\equiv 0$ for $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$ (see remark 1). We iterate over all non-empty subsets $N \subseteq N_p$ (i.e. $N \in 2^{N_p} \backslash \{\emptyset\}$). Then, to check whether $p_N(x)$ is the main quasi-homogeneous form of the polynomial $p(x)$, it is enough to verify whether there exists an integer solution to the system of equalities and inequalities (with respect to the unknown vector $e \in \mathbb{Z}^n$):

$$\langle e, k\rangle = \langle e, k^{(0)}\rangle, \quad k \in N \backslash \{k^{(0)}\}; \quad \langle e, k\rangle \geq \langle e, k^{(0)}\rangle + 1, \quad k \in N_p \backslash N, \tag{1}$$

where here and below, $k^{(0)}$ is any fixed element from N (if $|N| = 1$, then there are no equalities in (1)). Then, if there is an integer solution $e = A$ of system (1), then $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$ (if $A = 0_{(n)}$, then inequalities in (1) do not hold), and $p_N(x)$ is the main A-quasi-homogeneous form of the polynomial $p(x)$; otherwise, $p_N(x)$ is not the main quasi-homogeneous form of $p(x)$. This follows directly from the definition of the main quasi-homogeneous form.

To solve the problem of finding an integer solution to system (1), one can consider a problem $\max\{|e_1|, \ldots, |e_n|\} \to \min$ with constraints (1) or an equivalent LPP (add a new variable u)

$$u \to \min; \quad e_1 \leq u, \ -e_1 \leq u, \ \ldots, \ e_n \leq u, \ -e_n \leq u \quad (\Leftrightarrow \max\{|e_1|, \ldots, |e_n|\} \leq u), \tag{2}$$

under additional constraints (1). To solve the LPP (1), (2), it is better to use the simplex method, since any solution obtained by this method will be an extreme point of a convex polyhedral set given by a system of equalities and inequalities with integer coefficients, and, therefore, the components of this solution are rational numbers. Note further that if e is a solution to system (1), then $\forall \lambda \geq 1$ λe is also a solution to this system. But then, taking any rational solution e of system (1), and multiplying it by the least common multiple of the denominators of the nonzero components of this solution, we obtain an integer solution to system (1).

As already noted above, for solving some problems one is interested in main A-quasi-homogeneous forms not for all $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$, but only for $A \in \mathbb{N}^n$. In this case, instead of (1), one should consider the system (1),

$$e_i \geq 1, \quad i = 1, 2, \ldots, n, \tag{3}$$

to find some solution of which, instead of LPP (1), (2), one can consider the LPP of the form (1),

$$e_1 \to \min; \quad e_i \geq 1, \quad i = 1, 2, \ldots, n. \tag{4}$$

Remark 3. The simplest examples show that the same quasi-homogeneous form can be the main A-quasi-homogeneous for different $A \in \mathbb{Z}^n \backslash 0_{(n)}$, including for $A \in \mathbb{N}^n$ and for $A \in \mathbb{Z}^n \backslash \mathbb{N}^n$. For instance, consider a polynomial $p(x_1, x_2) = x_1^2 + x_2^2 + x_1^2 x_2^2$, for which x_2^2 is the main A-quasi-homogeneous form for $A = \langle 2, 1\rangle$, and also for $A = \langle 1, -1\rangle$ and for $A = \langle 1, 0\rangle$. Thus, for this example, solving LPP (1), (2), we may obtain $A = \langle 1, -1\rangle$ and the question of the existence of $A \in \mathbb{N}^n$ remains open. When using LPP (1), (4), this situation does not arise.

Acting in this way, we can define all main A-quasi-homogeneous polynomial forms of the polynomial $p(x)$ (either for $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$, or, in accordance with remark 1, only for $A \in \mathbb{N}^n$). However, in the case when the quantity $|N_p|$ is large enough, the number of elements in 2^{N_p} is extremely large, which makes it impossible to apply this approach. In this regard, the question arises of reducing the number of subsets that should be checked by solving the problem of finding at least one integer solution in LPP (1), (2) (or, respectively, in (1), (4)). This will require some definitions and notation. We will use the following binary relations for $a, b \in \mathbb{R}^n$: $a \leq b \Leftrightarrow a_i \leq b_i$, $i = 1, 2, \ldots, n$; $a < b \Leftrightarrow a_i < b_i$, $i = 1, 2, \ldots, n$. Let Y be a finite set from $\mathbb{R}^n$. We denote: $P(Y) = \{y \in Y \mid \forall y' \in Y \; [y' \leq y \Rightarrow y' = y]\}$—the set of Pareto-optimal points from Y; $S(Y) = \{y \in Y \mid \forall y' \in Y \; [y' < y \Rightarrow y' = y]\}$—the set of Slater-optimal points from Y. We also introduce generalizations of these sets. Let $z = (z_1, \ldots, z_n) \in \{0, 1\}^n$. Let us denote for $a, b \in \mathbb{R}^n$: $a \leq_z b \Leftrightarrow (-1)^{z_i} a_i \leq (-1)^{z_i} b_i$, $i = 1, 2, \ldots, n$; $a <_z b \Leftrightarrow (-1)^{z_i} a_i < (-1)^{z_i} b_i$, $i = 1, 2, \ldots, n$. We further denote $P_z(Y) = \{y \in Y \mid \forall y' \in Y \; [y' \leq_z y \Rightarrow y' = y]\}$, $S_z(Y) =$ $= \{y \in Y \mid \forall y' \in Y \; [y' <_z y \Rightarrow y' = y]\}$, $P^*(Y) = \bigcup_{z \in \{0,1\}^n} P_z Y)$, $S^*(Y) = \bigcup_{z \in \{0,1\}^n} S_z(Y)$. In addition, for $A \in \mathbb{R}^n$ denote $z^A = (z_1^A, \ldots, z_n^A) \in \{0, 1\}^n$, where $z_i^A = \begin{cases} 1, & if \quad A_i < 0, \\ 0, & if \quad A_i \geq 0. \end{cases}$ Obviously, $\forall a, b, A \in \mathbb{R}^n$: $a \leq_{z^A} b \Rightarrow \langle A, a \rangle \leq \langle A, b \rangle$, and if $A_i \neq 0, i = 1, 2, \ldots, n$, then $a \leq_{z^A} b$, $a \neq b \Rightarrow \langle A, a \rangle < \langle A, b \rangle$.

We present the following almost obvious

Statement 3. Let Y be a finite set from $\mathbb{R}^n, A \in \mathbb{R}^n$, $Y^* = Arg\min\{\langle A, y \rangle \mid y \in Y\}$. Then

(a) if $A_i > 0$, $i = 1, 2, \ldots, n$, then $Y^* \subseteq P(Y)$;

(b) if $A_i \neq 0$, $i = 1, 2, \ldots, n$, then $Y^* \subseteq P_{z^A}(Y)$;

(c) if $A_i \geq 0$, $i = 1, 2, \ldots, n$, $A \neq 0_{(n)}$, then $Y^* \subseteq S(Y)$;

(d) if $A \neq 0_{(n)}$, then $Y^* \subseteq S_{z^A}(Y)$.

A consequence of statement 3 is

Statement 4. Let some polynomial $p(x) \not\equiv 0$. Then

(a) $\forall A \in \mathbb{N}^n$ for the main A-quasi-homogeneous form $\varphi_A(x)$ of this polynomial, $N_{\varphi_A} \subseteq P(N_p)$ holds;

(b) $\forall A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$ for the main A-quasi-homogeneous form $\varphi_A(x)$ of this polynomial, $N_{\varphi_A} \subseteq S_{z^A}(N_p) \subseteq S^*(N_p)$ holds;

(c) $\forall A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$, if $A_i \neq 0$, $i = 1, 2, \ldots, n$, then for the main A-quasi-homogeneous form $\varphi_A(x)$ of this polynomial, $N_{\varphi_A} \subseteq P_{z^A}(N_p) \subseteq P^*(N_p)$ holds.

Statement 5. Let Y be a finite set from $\mathbb{R}^n$, $A \in \mathbb{R}^n$. Then

(a) $\forall y \in Y$, $\forall z \in \{0, 1\}^n$ $\exists y^* \in P_z(Y) \subseteq S_z(Y)$: $y^* \leq_z y$;

(b) $\forall y \in Y$ $\exists y^* \in P_{z^A}(Y) \subseteq S_{z^A}(Y)$: $y^* \leq_{z^A} y$, $\langle A, y^* \rangle \leq \langle A, y \rangle$.

Using statements 4, 5, we obtain the possibility of reducing the number of options $N \in 2^{N_p} \backslash \{\emptyset\}$ for enumerating all main quasi-homogeneous forms of the polynomial $p(x)$. In the case of finding all main A-quasi-homogeneous forms for $A \in \mathbb{N}^n$, it is sufficient to consider LPP (1), (4) for $N \in 2^{N_p} \backslash \{\emptyset\}$, and in (1) we can replace N_p by $P(N_p)$ (by virtue of statement 5(b)). Accordingly, in the case of finding all main A-quasi-homogeneous forms for $A \in \mathbb{Z}^n \backslash 0_{(n)}$, it is sufficient to consider LPP (1), (2) for $N \in 2^{S^*(N_p)} \backslash \{\emptyset\}$, and in (1) we can replace N_p with $S^*(N_p)$ (by virtue of statement 5(b)). This alone can result in a significant reduction in the total amount of computation.

To further reduce the number of calls to the solution of LPP (1), (2) (or (1), (4)) one more concept is needed. Let Y be a convex set from $\mathbb{R}^n$. A point $y \in Y$ is called a corner point of Y if condition $y = \alpha y^{(1)} + (1 - \alpha) y^{(2)}$, where $y^{(1)}, y^{(2)} \in Y$, $\alpha \in (0, 1)$, implies $y^{(1)} = y^{(2)} = y$. For a finite set $Y \subset \mathbb{R}^n$ we denote: $\Psi(Y)$—the set of corner points of the CoY (CoY is the convex hull of the set Y), $\Omega(Y) = \Psi(Y) \cap P(Y)$.

Statement 6. Let Y be a finite set from $\mathbb{R}^n$. Then $\Psi(Y) \subseteq P^*(Y)$. Statement 6 is a consequence of the fact that for any corner point $v \in \Psi(Y)$ there is a vector $A \in \mathbb{R}^n$ such that $A_i \neq 0$, $i = 1, 2, \ldots, n$, and $\forall y \in \Psi(Y)\backslash\{v\}$ $\langle A, y\rangle < \langle A, v\rangle$ (hence $v \in Arg\min\{\langle A, y\rangle \mid y \in Y\} \subseteq$ $\subseteq P_{z^A}(Y) \subseteq P^*(Y)$; see statement 3(b)).

Method 2. Let us first consider the case when it is required to select all main A-quasi-homogeneous forms of the polynomial $p(x) \not\equiv 0$ for $A \in \mathbb{N}^n$. Let us show that in this case it will be sufficient to consider LPP (1), (4) for all nonempty subsets N of the set $\Omega(N_p) = \Psi(N_p) \cap P(N_p)$, that is, for $N \in 2^{\Omega(N_p)}\backslash\{\emptyset\}$. Moreover, the set N_p in (1) can be replaced by $\Omega(N_p)$. Then in the case when for LPP $e_1 \to \min$ under constraints

$$\langle e, k\rangle = \langle e, k^{(0)}\rangle,\quad k \in N\backslash\{k^{(0)}\};\quad \langle e, k\rangle \geq \langle e, k^{(0)}\rangle + 1,\quad k \in \Omega(N_p)\backslash N;\quad e_i \geq 1,\quad i = 1, 2, \ldots, n, \tag{5}$$

where $N \in 2^{\Omega(N_p)}\backslash\{\emptyset\}$, there is an integer solution $e = A$, for the set

$$\tilde{N} = \{k \in P(N_p) \mid \langle A, k\rangle = \langle A, k^{(0)}\rangle\} \tag{6}$$

the following holds: $p_{\tilde{N}}(x)$ is the main A-quasi-homogeneous form of the polynomial $p(x)$. Indeed, suppose it is not. Then $\exists \tilde{k} \in \Omega(N_p)\backslash N$: $\langle A, \tilde{k}\rangle < \langle A, k^{(0)}\rangle$ (here we used the fact that the minimum value of a linear function on a bounded convex polyhedral set is attained at the corner points of this set (in this case, due to $A \in \mathbb{N}^n$ belonging to $P(N_p)$), and this contradicts the fact that $e = A$ is a solution to system (5).

Let us show that the main$A$-quasi-homogeneous forms of the polynomial $p(x)$ for all $A \in \mathbb{N}^n$ will be distinguished in this way. We need

Statement 7. Let $N \subset \mathbb{R}^n$ be a finite set,$A \in \mathbb{R}^n$, $N_1 = Arg\min\{\langle A, x\rangle \mid x \in N\}$. Further let $k \in N_1$. Then $\min\{\langle A, x\rangle \mid x \in N\} = \min\{\langle A, x\rangle \mid x \in \Psi(N)\}$ and $\exists k^{(1)}, \ldots, k^{(r)} \in N_1 \cap \Psi(N)$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$:

$$\sum_{i=1}^{r} \alpha_i k^{(i)} = k,\quad \sum_{i=1}^{r} \alpha_i = 1,\quad \alpha_i > 0,\quad i = 1, 2, \ldots, r. \tag{7}$$

Evidence. It is known (see, for example, [10, 11]) that there are $k^{(1)}, \ldots, k^{(r)} \in \Psi(N)$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, for which (7) is true. Note that due to (7) we have: $B = \langle A, k\rangle = \sum_{i=1}^{r} \alpha_i \langle A, k^{(i)}\rangle$, whence, by virtue of $\langle A, k^{(i)}\rangle \geq B$, $\alpha_i > 0$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, we obtain $\langle A, k^{(i)}\rangle = B$, $i = 1, 2, \ldots, r$. Thus, $k^{(1)}, \ldots, k^{(r)} \in N_1 \cap \Psi(N)$.

Corollary 1. Let the conditions of statement 7 be satisfied. Then $Arg\min\{\langle A, y\rangle \mid y \in \Psi(N)\} =$ $= \Psi(N) \cap N_1$.

Remark 4. Let the conditions of statement 7 be satisfied and $A_i \neq 0$, $i = 1, 2, \ldots, n$. Then due to Statement 3 the condition $\exists k^{(1)}, \ldots, k^{(r)} \in N_1 \cap \Psi(N)$ can be replaced by $\exists k^{(1)}, \ldots, k^{(r)} \in$ $\in N_1 \cap P_{z^A}(N) \cap \Psi(N)$, and in the case $A_i > 0$, $i = 1, 2, \ldots, n$,—by $\exists k^{(1)}, \ldots, k^{(r)} \in N_1 \cap \Omega(N)$.

Let for some $\breve{N} \subseteq P(N_p)$, $\breve{A} \in \mathbb{N}^n$ (where $\breve{N} \neq \emptyset$) $p_{\breve{N}}(x)$ is the main $\breve{A}$-quasi-homogeneous form of the polynomial $p(x)$, that is, $\breve{N} = Arg\min\{\langle \breve{A}, y\rangle \mid y \in N_p\}$. Let us show that it will be isolated by the method 2. We denote $N = \breve{N} \cap \Omega(N_p)$. Obviously (see statement 7 and remark 4) that $N \neq \emptyset$. Let $k^{(0)} \in N$. Then system (5) has an integer solution (in particular, the $e = \breve{A}$). Let us show that for any solution $e = A$ of system (5) (not even necessarily integer) for the set $\tilde{N}$ satisfying (6), $\tilde{N} = \breve{N}$ holds (and thus $p_{\breve{N}}(x)$ is distinguished by the method under consideration). (a) Let us show that $k \in \breve{N} \Rightarrow k \in \tilde{N}$. Since $\breve{N} = Arg\min\{\langle \breve{A}, k\rangle \mid k \in N_p\}$, by virtue of statement 7 and remark 4 $\exists k^{(1)}, \ldots, k^{(r)} \in \Omega(N_p) \cap \breve{N} = N$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, which holds (7). Moreover, according to the choice of A, we have: $k^{(i)} \in N \Rightarrow \langle A, k^{(i)}\rangle = \langle A, k^{(0)}\rangle$, $i = 1, 2, \ldots, r$, whence, by virtue of (7), we obtain $\langle A, k\rangle = \langle A, k^{(0)}\rangle$, i.e. $k \in \tilde{N}$. (b) Now let us show that $k \in \tilde{N} \Rightarrow k \in \breve{N}$. Note that from (5) it follows: $\langle A, k^{(0)}\rangle = \min\{\langle A, k\rangle \mid k \in \Omega(N_p)\} =$ $= \min\{\langle A, k\rangle \mid k \in N_p\}$ (since the minimum of the linear function is attained on $\Psi(N_p)$ and by virtue of $A_i = e_i \geq 1$, $i = 1, 2, \ldots, n$, simultaneously on $P(N_p)$), $N = Arg\min\{\langle A, k\rangle \mid k \in \Omega(N_p)\} = = \Omega(N_p) \cap Arg\min\{\langle A, k\rangle \mid k \in N_p\}$, $Arg\min\{\langle A, k\rangle \mid k \in N_p\} \subseteq P(N_p)$ and therefore $\tilde{N} = = Arg\min\{\langle A, k\rangle \mid k \in N_p\}$, $\Omega(N_p) \cap \tilde{N} = N$, whence, by virtue of statement 7 and remark 4, $\exists k^{(1)}, \ldots, k^{(r)} \in \Omega(N_p) \cap \tilde{N} = N$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, which holds (7). Moreover, $\langle \breve{A}, k\rangle = \sum_{i=1}^{r} \alpha_i <$ $< \breve{A}, k^{(i)}\rangle = \langle \breve{A}, k^{(1)}\rangle$, due to $k^{(1)}, \ldots, k^{(r)} \in N = \breve{N} \cap \Omega(N_p) \subseteq \breve{N} = Arg\min\{\langle \breve{A}, y\rangle \mid y \in N_p\}$, that is, $k \in \breve{N}$.

Let us now consider the case when it is required to select all main A-quasi-homogeneous forms for $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$. Let us show that in this case it will be sufficient to consider system LPP (1), (2) for all nonempty subsets N of the set $\Psi(N_p)$, that is, for $N \in 2^{\Psi(N_p)} \backslash \{\emptyset\}$. Moreover, the set N_p in (1) can be replaced by $\Psi(N_p)$. Then in the case when for $N \in 2^{\Psi(N_p)} \backslash \{\emptyset\}$ for the system considered in this way

$$\langle e, k\rangle = \langle e, k^{(0)}\rangle, \quad k \in N \backslash \{k^{(0)}\}; \quad \langle e, k\rangle \geq \langle e, k^{(0)}\rangle + 1, \quad k \in \Psi(N_p) \backslash N, \tag{8}$$

there is an integer solution $e = A$, which is found by a solution of the LPP (2), (8) for the set

$$\tilde{N} = \{k \in S^*(N_p) \mid \langle A, k\rangle = \langle A, k^{(0)}\rangle\} \tag{9}$$

the following holds: $p_{\tilde{N}}(x)$ is the main A-quasi-homogeneous form of the polynomial $p(x)$. Indeed, suppose it is not. Then either $\exists \tilde{k} \in \Psi(N_p)$: $\langle A, \tilde{k}\rangle < \langle A, k^{(0)}\rangle$ (here we used the fact that the minimum value of a linear function on a bounded convex polyhedral set is attained at the corner points of this set), which contradicts the fact that $e = A$ is a solution to system (8), or $\langle A, k^{(0)}\rangle = \min\{\langle A, k\rangle \mid k \in N_p\}$. In the latter case, by virtue of statement 4(b), $Arg\min\{\langle A, k\rangle \mid k \in N_p\} \subseteq S^*(N_p)$, whence it follows that $Arg\min\{\langle A, k\rangle \mid k \in N_p\} = \tilde{N}$, and $p_{\tilde{N}}(x)$ is the main A-quasi-homogeneous form of the polynomial $p(x)$.

Let us show that the main A-quasi-homogeneous forms of the polynomial $p(x)$ for all $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$ will be distinguished in this way. Let for $\breve{N} \subseteq P(N_p)$ (where $\breve{N} \neq \emptyset$) $p_{\breve{N}}(x)$ is the main $\breve{A}$-quasi-homogeneous form of the polynomial $p(x)$ for some $\breve{A} \in \mathbb{Z}^n \backslash \{0_{(n)}\}$, i.e. $Arg\min\{\langle \breve{A}, k\rangle \mid k \in N_p\} = \breve{N}$. Let us show that it will be distinguished by the described method. We denote $N = \breve{N} \cap \Psi(N_p)$. Obviously, $N \neq \emptyset$ (by virtue of statement 7, $\breve{N} \neq \emptyset \Rightarrow N \neq \emptyset$), that is, $N \in 2^{\Psi(N_p)} \backslash \{\emptyset\}$. Let $k^{(0)} \in N$. Then, using the fact that $\Psi(N_p) \backslash N = \Psi(N_p) \backslash \breve{N} \subseteq N_p \backslash \breve{N}$, we obtain that system (8) has an integer solution (in particular, the vector $\breve{A}$). Let us show that for any solution $e = A$ of system (8) (even not necessarily integer) for the set $\tilde{N}$ satisfying (9), $\tilde{N} = \breve{N}$ holds (and thus $p_{\breve{N}}(x)$ is distinguished by the method under consideration). (a) Let us show that $k \in \breve{N} \Rightarrow k \in \tilde{N}$. Since $\breve{N} = Arg\min\{\langle \breve{A}, k\rangle \mid k \in N_p\}$, by virtue of statement 7, $\exists k^{(1)}, \ldots, k^{(r)} \in \Psi(N_p) \cap \breve{N} = N$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, which holds (7). Moreover, according to the choice of A, we have: $\langle A, k^{(i)}\rangle = \langle A, k^{(0)}\rangle$, $i = 1, 2, \ldots, r$, whence, by virtue of (7), we obtain $\langle A, k\rangle = \langle A, k^{(0)}\rangle$, i.e. $k \in \tilde{N}$ (by statement 4(b), $k \in \breve{N} \Rightarrow k \in S^*(N_p)$). (b) Let us show that $k \in \tilde{N} \Rightarrow k \in \breve{N}$. Note that from (8) it follows: $\langle A, k^{(0)}\rangle = \min\{\langle A, k\rangle \mid k \in \Psi(N_p)\} = = \min\{\langle A, k\rangle \mid k \in N_p\}$. Using (8) and corollary 1, we find that $N = Arg\min\{\langle A, k\rangle \mid k \in \Psi(N_p)\} = = \Psi(N_p) \cap Arg\min\{\langle A, k\rangle \mid k \in N_p\}$, and hence, by virtue of $k^{(0)} \in N$, as well as of statement 4(b), $\tilde{N} = Arg\min\{\langle A, k\rangle \mid k \in N_p\}$, $\Psi(N_p) \cap \tilde{N} = N$. Since $k \in \tilde{N}$, by virtue of statement 7 $\exists k^{(1)}, \ldots, k^{(r)} \in \Psi(N_p) \cap \tilde{N} = N$, $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, which holds (7). Moreover, $\langle \breve{A}, k\rangle = \sum_{i=1}^{r} \alpha_i \langle \breve{A}, k^{(i)}\rangle = \langle \breve{A}, k^{(1)}\rangle$, due to $k^{(1)}, \ldots, k^{(r)} \in N = \breve{N} \cap \Psi(N_p) \subseteq \breve{N}$, that is, $k \in \breve{N}$.

Method 3. Let us first consider the case when it is required to select all main A-quasi-homogeneous forms for $A \in \mathbb{Z}^n \backslash \{0_{(n)}\}$.

Statement 8. Let $N_p \neq \emptyset$ (that is, $p(x) \not\equiv 0$) hold for some polynomial $p(x)$. Let $k^{(1)}, \ldots, k^{(r)} \in \Psi(N_p)$, $k^{(0)} = \dfrac{1}{r}(k^{(1)} + \ldots + k^{(r)}) \in riCoN_p$, $r \in \mathbb{N}$. Then any set of points from N_p, including the points $k^{(1)}, \ldots, k^{(r)}$, cannot enter any main quasi-homogeneous polynomial form of $p(x)$ other than the polynomial $p(x)$ itself.

Evidence. Suppose that such a form has been found, i.e. $\exists A \in \mathbb{Z}^n\backslash 0_{(n)}$, $B \in \mathbb{Z}$: $\langle A, k^{(1)}\rangle = B, \dots, \langle A, k^{(r)}\rangle = B$, $\forall k \in N_p$ $\langle A, k\rangle \geq B$, and $\langle A, k^*\rangle > B$ holds for some $k^* \in N_p$. We can assume that $k^* \in \Psi(N_p)$ (since the linear function reaches its extreme values at the corner points of the bounded polytope). Then $\langle A, k^{(0)}\rangle = B$. On the other hand, since $k^{(0)} \in riCoN_p$, then [12] $\exists \alpha_k > 0$, $k \in \Psi(N_p)$: $\sum_{k\in\Psi(N_p)} \alpha_k = 1$, $k^{(0)} = \sum_{k\in\Psi(N_p)} \alpha_k k$, and by virtue of $\alpha_{k^*} > 0$ we obtain $\langle A, k^{(0)}\rangle > B$, that is, came to a contradiction.

Remark 5. When $A \in \mathbb{N}^n$ statement 6 can be strengthened. Let $\bar{k} \in \mathbb{Z}^n_{\geq}$, where $\bar{k}_i = \max\{k_i \mid k \in N_p\}$, $i = 1, 2, \dots, n$. Then the condition $k^{(0)} \in riCoN_p$ can be replaced by a weaker one: $k^{(0)} \in riCo\{N_p \cup \{\bar{k}\}\}$. The proof is similar, taking into account the fact that $\bar{k} \in \Psi(N_p \cup \{\bar{k}\})$.

Statement 8 points to the following possible organization of the process (algorithm) for the selection of all main quasi-homogeneous polynomial forms of the polynomial $p(x)$ different from this polynomial (for example, in the case of $intCoN_p \neq \emptyset$ any main quasi-homogeneous polynomial form of the polynomial $p(x)$ is different from it).

Algorithm 1 (finding all main quasi-homogeneous polynomial form of the polynomial $p(x)$)

Step 1. Select $\Psi(N_p), S^*(N_p)$. Let's number the points from $\Psi(N_p)$, i.e. let $\Psi(N_p) = \{k^{(1)}, \dots, k^{(m)}\}$.

Step 2. Further work of the algorithm is divided into m stages. At the 1st stage, we list all main quasi-homogeneous polynomial forms with the occurrence of $k^{(1)}$ in them. At the 2nd stage—with the occurrence of $k^{(2)}$, but without $k^{(1)}$. At the 3rd stage—with the entry of $k^{(3)}$, but already without $k^{(1)}$, $k^{(2)}$, etc.

Step 3. The described multistage process is conveniently represented as a set of growing trees (forest). Each point $k^{(i)} \in \Psi(N_p)$ is the root of the tree along which all main quasi-homogeneous polynomial forms will be located with a survey entry of the point $k^{(i)}$ in them, but in the absence of points $k^{(1)}, \dots, k^{(i-1)}$. As a result, sets of pairwise disjoint sets of main quasi-homogeneous polynomial forms will be obtained.

Let us describe a tree with the root $k^{(1)}$ (the rest are described similarly). We will consider $k^{(1)}$ as the top of the 0th level of this tree. Level 1 vertices adjacent to $k^{(1)}$ are all $k \in \Psi(N_p)\backslash\{k^{(1)}\}$ for which $\frac{1}{2}(k + k^{(1)}) \notin riCoN_p$ (in this case, by virtue of statement 8, all obviously inappropriate cases are cut off), which we arrange in ascending order of numbers. Each vertex $\bar{k}$ of the 1st level is adjacent to the vertices of the 2nd level ("sons" of the vertex $\bar{k}$). Moreover, any vertex $\tilde{k}$ that is a "son" of a vertex $\bar{k}$ is taken from the set of level 1 vertices lying to the right of $\bar{k}$ (to avoid repetitions) and such that $\frac{1}{3}(k^{(1)} + \bar{k} + \tilde{k}) \notin riCoN_p$ (in this case, by virtue of statement 8, all obviously inappropriate cases are cut off), which we again arrange in the order of increasing numbers. We continue this process as long as possible. During this process, for each current vertex $\check{k}$ of the tree under construction, select the set of vertices included in the chain connecting $\check{k}$ with $k^{(1)}$ (it is the only one in the tree) and check for the set of vertices $N \subseteq \Psi(N_p)$, included into this chain, the existence of an integer solution in LPP (2), (8). In the case of the existence of a solution, we single out the next main quasi-homogeneous polynomial form $\tilde{N}$ by formula (9), where $k^{(0)}$ is any representative from N.

Let us now consider the case when it is required to select all main A-quasi-homogeneous forms for $A \in \mathbb{N}^n$. In this case, we modify algorithm 1. At step 1, we will now use $\Omega(N_p)\backslash riCo\{N_p \cup \{\bar{k}\}\}$, where $\bar{k} \in \mathbb{Z}^n_{\geq}$, $\bar{k}_i = \max\{k_i \mid k \in N_p\}$, $i = 1, 2, \dots, n$, instead of $\Psi(N_p)$. Accordingly, at step 3, instead of LPP (2), (8), we use LPP $e_1 \to \min$ under constraints (5), and find $\tilde{N}$ by formula (6) (instead of (9)). In addition, we replace everywhere in the algorithm $riCoN_p$ on the $riCo\{N_p \cup \{\bar{k}\}\}$.

3 Method for finding the set of corner points $\Psi(Y)$

Consider the following problem. Let $Y \subset \mathbb{R}^n$ be a finite set. It is required to select $\Psi(Y)$—the set of corner points of the convex polytope CoY. We will use

Statement 9. Let $Y \subset \mathbb{R}^n$ be a finite set. For a point $v \in Y$ to have $v \in \Psi(Y)$ it is necessary and sufficient that

$$\exists e \in \mathbb{R}^n: \quad e \neq 0_{(n)}, \quad \forall y \in Y \backslash \{v\} \langle y - v, e \rangle < 0. \tag{10}$$

Corollary 2. Using the fact that a linear function attains its extremal values on a convex polytope at the corner points, in condition (10) we can replace Y with any finite set V such that $\Psi(Y) \subseteq V \subseteq Y$, and since $\Psi(Y) \subseteq P^*(Y)$, then in (10) one can, for example, change Y to $P^*(Y)$.

Corollary 3. Using the fact that if the inequalities in (10) hold for some $e \in \mathbb{R}^n$, these inequalities will also hold for λe, where $\lambda > 0$, condition (10) can be reformulated in terms of LPP. Suppose we are in the conditions of statement 9 and a finite set V is such that $\Psi(Y) \subseteq V \subseteq Y$ (for example, $V = P^*(Y)$). Then, for the point $v \in V$ to hold $v \in \Psi(Y)$, it is necessary and sufficient that for the LPP (11): $t(v) < 0$, where

$$u \to \min(= t(v)); \quad \langle y - v, e \rangle \leq u, \quad y \in V \backslash \{v\}, \quad -1 \leq e_i \leq 1, \quad i = 1, 2, \ldots, n. \tag{11}$$

The following simple technique allows significantly reduce the number of calls to the solution of the LPP (11) to find $\Psi(Y)$. Let $Y \subset \mathbb{R}^n$ be a finite set, $g_i(x)$, $i = 1, 2, \ldots, k$, be real functions defined on Y. Introduce the sets $Y_0 = Y$, $Y_i = \{x \in Y_{i-1} \mid g_i(x) = \min\{g_i(x') \mid x' \in Y_{i-1}\}\}$, $i = 1, 2, \ldots, k$. Let us denote $\mathrm{lexArg}[Y, g_1, \ldots, g_k] = Y_k$.

Statement 10. Let $Y \subseteq \mathbb{R}^n$ be a finite set, $A \in \mathbb{R}^n$, $Y_A = \mathrm{lexArg}[Y, \langle A, x \rangle, x_1, \ldots, x_n]$. Then Y_A consists of a single point $y^A \in \Psi(Y)$.

Remark 5. Statement 10 remains valid in the case $Y_A = \mathrm{lexArg}[Y, \langle A, x \rangle, (-1)^{i_1} x_1, \ldots$ $\ldots, (-1)^{i_n} x_n]$, where $i_1, \ldots, i_n \in \{0, 1\}$.

Consider the point $y^\Sigma = (1 \backslash |Y|) \sum_{y \in Y} y \in \mathrm{ri}CoY$ (instead of Y we can use $P^*(Y)$ or any set V, satisfying condition $\Psi(Y) \subseteq V \subseteq Y$).

Algorithm 2 (finding the set $\Psi(Y)$)

Step 1. Put $U = P^*(Y)$, $W = \emptyset$, $V = U \cup W$. At each stage of the algorithm we have

$$W \subseteq \Psi(Y) \subseteq V = U \cup W \subseteq P^*(Y). \tag{12}$$

Step 2. Consider any point $v \in U$. Let $A = v - y^\Sigma$ (obviously, $A \neq 0_{(n)}$). Define $\{y^A\} =$ $= Y_A = \mathrm{lexArg}[Y, \langle A, x \rangle, x_1, \ldots, x_n]$ (see statement 10) and set $W := W \cup \{y^A\}$. If $y^A = v$, then assign $U := U \backslash \{v\}$ and go to step 3. If $y^A \neq v$, then solve the LPP (11) and assign $U := U \backslash \{v\}$. If $t(v) < 0$, then $v \in \Psi(Y)$ (see corollary 3), and assign $W := W \cup \{v\}$, otherwise $v \notin \Psi(Y)$. Finally, assign $V := U \cup W$ and go to step 3.

Step 3. If $U = \emptyset$, then the process ends and in this case $W = \Psi(Y)$ (according to (12)). Otherwise, go to step 2.

4 Results

Thus, three practical methods for finding all main quasi-homogeneous forms of a polynomial are described (1st is the easiest and 3rd is the most difficult). However, with a large number of terms in the polynomial, the 3rd one is most preferable. Let us show by examples the advantage of the 2nd method in comparison with the 1st, and also the 3rd in comparison with the 2nd.

Example 2. (1) Consider the polynomial $p(x, y, z) = 3xy + 5xz - yz + 7xyz + 4x^2yz-$ $-6xy^2z + 2xyz^2$, where $N_p = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2)\}$.

Let us consider the case when it is required to select all main A-quasi-homogeneous forms of $p(x, y, z)$ for $A \in \mathbb{N}^n$. Then, in the case of applying the method 1, one would have to enumerate $2^{|N_p|} - 1 = 2^7 - 1 = 127$ possible cases, in each of which the corresponding LPP would be solved. In the case of using method 2, $\Omega(N_p) = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, and therefore it suffices to consider $2^{|\Omega(N_p)|} - 1 = 2^3 - 1 = 7$ cases.

(2) Consider the polynomial $p(x, y) = 3y^{10} + 5x^2y^9 - x^3y^8 + 7x^4y^7 + 4x^5y^5 - 6x^6y^3 + 2x^7$, where $N_p = \{(0, 10), (2, 9), (3, 8), (4, 7), (5, 5), (6, 3), (7, 0)\}$. Let us consider the case when it is required to select all main A-quasi-homogeneous forms of $p(x, y)$ for $A \in \mathbb{N}^n$. Then, in the case of applying the method 2, $\Omega(N_p) = N_p$, and therefore one would have toenumerate $2^{|\Omega(N_p)|} - 1 = 2^7 - 1 = 127$ possible cases. In the case of using method 3, we have: $\bar{k} = (7, 10)$, $\Omega(N_p) \backslash riCo\{N_p \cup \{\bar{k}\}\} = \{(0, 10), (7, 0)\}$, and therefore consideration of no more than 3 cases will be required, in each of which we obtain one of the appropriate quasi-homogeneous forms: $3y^{10}$ for $A = (2, 1)$, $2x^7$ for $A = (1, 4)$ and $3y^{10} + 2x^7$ for $A = (10, 7)$.

5 Conclusion

We consider the problem of identifying the main quasi-homogeneous forms of a polynomial (quasi-homogeneous form of a polynomial is a generalization of the concept of its homogeneous form), which are the sums of the terms of the polynomial belonging to some face of the Newton polytope of this polynomial.

Finding these forms is necessary for solving many practical problems, in particular, for construction of necessary and sufficient conditions of an extremum for polynomials and power series [1, 2], as well as checking matrices for D-stability arising in the study of ecosystem stability [6, 7].

For various practical applications, it is sometimes required to select all the main quasi-homogeneous forms. Meanwhile, when solving some problems, it is enough to use some part of them (for example, “southwest” forms—in the case of two variables). In the latter case, it is possible to consider not only polynomials, but also power series (i.e., analytic functions). In accordance with this, two variants of methods (algorithms) are considered.

On the basis of the proposed methods, practically realizable algorithms are described that use a significant reduction in the number of options to be enumerated, in comparison with the methods of simple enumeration.

A practically realizable rather economical algorithm for solving an auxiliary problem is described—finding the set of corner points of the polytope CoY, where Y is a finite set from $\mathbb{R}^n$.

References

[1] V.N. Nefedov, *Estimation of the error in convex polynomial optimization problems*, Comput. Math. and Math. Phys., **30(1)** (1990)

[2] V.N. Nefedov, *Necessary and sufficient conditions for an extremum in analytical optimization problems*, Electronic journal “Trudy MAI”, **33** (2009)

[3] S.G. Gindikin, *Energy estimates associated with Newton’s polyhedron*, Trudy Moskov. mat. soc., **101** (1974)

[4] A.D. Bruno, *Power geometry in algebraic and differential equations* (Nauka, Fizmatlit, Moscow, 1998)

[5] L.R. Volevich, S.G. Gindikin, *Newton polytope method in the theory of partial differential equations* (URSS, Moscow, 2002)

[6] A.G. Khovanskiy, *Polyhedra and algebra*, Proceedings of ISA RAS, **38** (2008)

[7] G.V. Kanovei, D.O. Logofet, *Relations, properties and invariant transformations of D- and aD-stable matrices*, Vestn. Moscow Un-ty. Ser. 1. Mat., Mekh., **6** (2001)

[8] D.O. Logofet, *Stronger-than-Lyapunov notions of matrix stability, or how "flowers" help solve problems in mathematical ecology*, Linear Algebra Appl., **398** (2005)

[9] V.N. Nefedov, *On one method for studying a polynomial for definiteness in positive orthant*, Electronic journal "Trudy MAI", **22** (2006)

[10] E.G. Belousov, *Introduction to convex analysis and integer programming* (Publishing house of Moscow University, Moscow, 1977)

[11] I.I. Eremin, N.N. Astafiev, *Introduction to the theory of linear and convex programming* (Nauka, Moscow, 1976)

[12] A. Brondsted, *An introduction to convex polytopes* (Springer-Verlag, New York, 1983)