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Optimization of the mean-square approximation procedures for iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional Wiener process based on Multiple Fourier–Legendre series

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Abstract. The article is devoted to approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 3 by the method of multiple Fourier–Legendre series. The mentioned stochastic integrals are part of strong numerical methods with convergence order 1.5 for Ito stochastic differential equations with multidimensional noncommutative noise. These numerical methods are based on the so-called Taylor–Ito and Taylor–Stratonovich expansions. We calculate the exact lengths of sequences of independent standard Gaussian random variables required for the mean-square approximation of iterated Stratonovich stochastic integrals. Thus, the computational cost for the implementation of numerical methods can be significantly reduced.

1 Strong Taylor–Ito and Taylor–Stratonovich numerical schemes with convergence order 1.5

The relevance of the problem of numerical integration of Ito stochastic differential equations (SDEs) is explained by a wide range of their applications related to the construction of adequate mathematical models of dynamical systems of various physical nature under random perturbations. Among these models, we note models in financial mathematics, biology, epidemiology, medicine, aerospace industry, hydrology, seismology, geophysics, genetics, electrodynamics, chemical kinetics [1–10]. Moreover, Ito SDEs are used to solve various mathematical problems such as signals filtering with random noises, stochastic optimal control, stochastic stability and bifurcation analysis, parameter estimation of stochastic systems [1, 2, 9]. In addition, exact solutions of Ito SDEs are known in rare cases, and besides, knowing the exact solution of Ito SDE does not always allow us to simulate it numerically without using special numerical methods.

One of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [1, 2, 9, 10]. The important feature of these expansions is iterated Ito and Stratonovich stochastic integrals with respect to components of the multi-dimensional Wiener process.

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The article continues the research [11] and is devoted to the development of an effective method of iterated Ito and Stratonovich stochastic integrals approximation based on generalized multiple Fourier series and proposed by the first author of the article in [10]. More precisely, we consider mean-square approximations of iterated Stratonovich stochastic integrals of multiplicities 1 to 3 with respect to components of the multi-dimensional Wiener process by the method of multiple Fourier–Legendre series. It should be noted that the iterated Stratonovich stochastic integrals are more convenient to use than the iterated Ito stochastic integrals due to simplicity of their approximations [1, 2, 9, 10, 12–22] (compare formulas (18) and (19)).

Among advantages of the method based on generalized multiple Fourier series [10, 12] over the methods [1, 2, 8, 9, 13–21] of mean-square approximation of iterated Ito and Stratonovich stochastic integrals we note the following. The well-known approach based on the Karhunen–Loeve expansion of the Brownian bridge process [1, 8] (also see [17]) leads to iterated application of the operation of limit transition. At the same time the operation of limit transition is implemented only once in theorem 1 (see below). This feature is more appropriate for the approximation. Moreover, the method [10, 12] allows one to calculate the exact lengths of sequences of independent standard Gaussian random variables individually for different iterated Ito or Stratonovich stochastic integrals. Thus, the computational cost for the implementation of numerical methods for Ito SDEs can be significantly reduced. For more details see chapter 6 of monograph [12].

Let (Ω, F, P) be a complete probability space, let $\{\mathrm{F}_t, t \in [0,T]\}$ be a nondecreasing right-continuous family of σ-algebras of F and let $\mathbf{w}_t$ be a standard m-dimensional Wiener process, which is F_t-measurable for any $t \in [0,T]$. We assume that the components $\mathbf{w}_t^{(i)}$ $(i = 1, \ldots, m)$ of this process are independent. Consider an Ito SDE in the integral form

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_\tau, \tau)\, d\tau + \sum_{i=1}^m \int_0^t B_i(\mathbf{x}_\tau, \tau)\, d\mathbf{w}_\tau^{(i)}, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \tag{1}$$

where $\omega \in \Omega$, $\mathbf{a}(\mathbf{x}, t)\colon\ \mathbf{R}^n \times [0,T] \to \mathbf{R}^n$, $B_i(\mathbf{x}, t)$ is the i-th column of $B(\mathbf{x}, t)\colon\ \mathbf{R}^n \times [0,T] \to \mathbf{R}^{n\times m}$, $\mathbf{x}_0$ is F_0-measurable, $\mathbf{M}|\mathbf{x}_0|^2 < \infty$ ($\mathbf{M}$ is an expectation), $\mathbf{x}_0$ and $\mathbf{w}_t - \mathbf{w}_0$ are independent $(t > 0)$.

Let us consider the following differential operators

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^n \mathbf{a}^{(i)}(\mathbf{x}, t)\frac{\partial}{\partial \mathbf{x}^{(i)}} + \frac{1}{2}\sum_{j=1}^m \sum_{l,i=1}^n B^{(lj)}(\mathbf{x}, t) B^{(ij)}(\mathbf{x}, t)\frac{\partial^2}{\partial \mathbf{x}^{(l)}\,\partial \mathbf{x}^{(i)}},\ \bar{L} = \frac{\partial}{\partial t} + \sum_{i=1}^n \bar{\mathbf{a}}^{(i)}(\mathbf{x}, t)\frac{\partial}{\partial \mathbf{x}^{(i)}}, \tag{2}$$

$$\bar{\mathbf{a}}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t) - \frac{1}{2}\sum_{j=1}^m G_0^{(j)} B_j(\mathbf{x}, t), \quad G_0^{(i)} = \sum_{j=1}^n B^{(ji)}(\mathbf{x}, t)\frac{\partial}{\partial \mathbf{x}^{(j)}}, \quad i = 1, \ldots, m, \tag{3}$$

where $\mathbf{a}^{(i)}(\mathbf{x}, t)$, $\bar{\mathbf{a}}^{(i)}(\mathbf{x}, t)$ are i-th components of $\mathbf{a}(\mathbf{x}, t)$, $\bar{\mathbf{a}}(\mathbf{x}, t)$ respectively and $B^{(ij)}(\mathbf{x}, t)$ is the ij-th element of $B(\mathbf{x}, t)$.

Consider the following iterated Ito and Stratonovich stochastic integrals [10–12]

$$I_{(l_1\ldots l_k)s,t}^{(i_1\ldots i_k)} = \int_t^s (t - t_k)^{l_k} \ldots \int_t^{t_2} (t - t_1)^{l_1} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{4}$$

$$I_{(l_1\ldots l_k)s,t}^{*(i_1\ldots i_k)} = \int_t^s (t - t_k)^{l_k} \ldots \int_t^{t_2} (t - t_1)^{l_1} \circ d\mathbf{w}_{t_1}^{(i_1)} \ldots \circ d\mathbf{w}_{t_k}^{(i_k)}, \tag{5}$$

where $\circ d\mathbf{w}_s^{(i_1)}$ is the Stratonovich differential, $l_1, \ldots, l_k = 0, 1, 2, \ldots$, and $i_1, \ldots, i_k = 1, \ldots, m$.

Assume that $\mathbf{a}(\mathbf{x},t)$ and $B(\mathbf{x},t)$ are enough smooth functions with respect to the variables $\mathbf{x}$ and t. Consider the partition $\{\tau_q\}_{q=0}^N$ such that $0=\tau_0<\tau_1<\ldots<\tau_N=T$, $\Delta_N=\max_{0\le q\le N-1}|\tau_{q+1}-\tau_q|$.

We will say [1] that a numerical scheme $\mathbf{y}_{\tau_q}$, $q=0,1,\ldots,N$ converges strongly with order $\gamma>0$ at time moment T to the process $\mathbf{x}_t$, $t\in[0,T]$ if there exist a constant $C>0$, which does not depend on Δ_N, and a $\delta>0$ such that $\mathbf{M}|\mathbf{x}_T-\mathbf{y}_T|\le C(\Delta_N)^\gamma$ for each $\Delta_N\in(0,\delta)$.

Consider the strong Taylor–Ito and Taylor–Stratonovich schemes for Ito SDE (1) with convergence order 1.5 [1, 8–10, 12]:

$$\mathbf{y}_{q+1}=\mathbf{y}_q+\sum_{i_1=1}^m B_{i_1}\hat{I}_{(0)\tau_{q+1},\tau_q}^{(i_1)}+\Delta\mathbf{a}+\sum_{i_1,i_2=1}^m G_0^{(i_1)}B_{i_2}\hat{I}_{(00)\tau_{q+1},\tau_q}^{(i_1i_2)}+$$
$$+\sum_{i_1=1}^m\left[G_0^{(i_1)}\mathbf{a}\left(\Delta\hat{I}_{(0)\tau_{q+1},\tau_q}^{(i_1)}+\hat{I}_{(1)\tau_{q+1},\tau_q}^{(i_1)}\right)-LB_{i_1}\hat{I}_{(1)\tau_{q+1},\tau_q}^{(i_1)}\right]+\sum_{i_1,i_2,i_3=1}^m G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{q+1},\tau_q}^{(i_1i_2i_3)}+\frac{\Delta^2}{2}L\mathbf{a}, \tag{6}$$

$$\mathbf{y}_{q+1}=\mathbf{y}_q+\sum_{i_1=1}^m B_{i_1}\hat{I}_{(0)\tau_{q+1},\tau_q}^{*(i_1)}+\Delta\bar{\mathbf{a}}+\sum_{i_1,i_2=1}^m G_0^{(i_1)}B_{i_2}\hat{I}_{(00)\tau_{q+1},\tau_q}^{*(i_1i_2)}+$$
$$+\sum_{i_1=1}^m\left[G_0^{(i_1)}\bar{\mathbf{a}}\left(\Delta\hat{I}_{(0)\tau_{q+1},\tau_q}^{*(i_1)}+\hat{I}_{(1)\tau_{q+1},\tau_q}^{*(i_1)}\right)-\overline{L}B_{i_1}\hat{I}_{(1)\tau_{q+1},\tau_q}^{*(i_1)}\right]+\sum_{i_1,i_2,i_3=1}^m G_0^{(i_1)}G_0^{(i_2)}B_{i_3}\hat{I}_{(000)\tau_{q+1},\tau_q}^{*(i_1i_2i_3)}+\frac{\Delta^2}{2}L\mathbf{a}, \tag{7}$$

where the functions B_i, $\mathbf{a}$, $G_0^{(i)}B_j$, $\ldots$ are calculated at the point $(\mathbf{y}_q,\tau_q)$, $\mathbf{y}_q$ is an approximation of solution to Ito SDE (1) at moment τ_q, $\hat{I}_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{(i_1\ldots i_k)}$, $\hat{I}_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{*(i_1\ldots i_k)}$ are approximations of the iterated Ito and Stratonovich stochastic integrals $I_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{(i_1\ldots i_k)}$, $I_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{*(i_1\ldots i_k)}$ respectively, and L, $\overline{L}$, $\bar{\mathbf{a}}$, $G_0^{(i)}$, $i=1,\ldots,m$ are defined by equalities (2) and (3). Note that the first four terms on the right-hand side of scheme (6) correspond to the strong Milstein scheme [8] with convergence order 1.0. Among the standard conditions ensuring the convergence of schemes (6) and (7), we note the conditions for approximation of iterated Ito and Stratonovich stochastic integrals [1]:

$$\mathbf{M}\left(I_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{(i_1\ldots i_k)}-\hat{I}_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{(i_1\ldots i_k)}\right)^2\le C\Delta^4, \tag{8}$$

$$\mathbf{M}\left(I_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{*(i_1\ldots i_k)}-\hat{I}_{(l_1\ldots l_k)\tau_{q+1},\tau_q}^{*(i_1\ldots i_k)}\right)^2\le \overline{C}\Delta^4, \tag{9}$$

where $q=0,1,\ldots,N-1$, constants C and $\overline{C}$ are independent of Δ.

2 Method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series

Let us consider the effective method of expansion and mean-square approximation of iterated Ito stochastic integrals [10, 12] (other approaches can be found in [1, 2, 8, 9, 13–21]):

$$J[\psi^{(k)}]_{T,t}=\int_t^T\psi_k(t_k)\ldots\int_t^{t_2}\psi_1(t_1)\,d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{10}$$

where $i_1,\ldots,i_k=0,1,\ldots,m$, every $\psi_l(\tau)$ ($l=1,\ldots,k$) is a continuous non-random function on $[t,T]$, $\mathbf{w}_\tau^{(i)}$ ($i=1,\ldots,m$) are independent standard Wiener processes, and $\mathbf{w}_\tau^{(0)}=\tau$.

Let $K(t_1,\ldots,t_k)=\psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}_{\{t_1<\ldots<t_k\}}$ for $t_1,\ldots,t_k\in[t,T]$ $(k\ge 2)$ and $K(t_1)\equiv\psi_1(t_1)$ for $t_1\in[t,T]$, where $\mathbf{1}_A$ is the indicator of the set A. Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$. It is well known that the generalized multiple Fourier series of $K(t_1,\ldots,t_k)\in L_2([t,T]^k)$ is converging to $K(t_1,\ldots,t_k)$ in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty}\left\|K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right\|=0,$$

where

$$C_{j_k\ldots j_1}=\int_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)\,dt_1\ldots dt_k \tag{11}$$

is the Fourier coefficient and $\|\cdot\|$ is the $L_2([t,T]^k)$-norm.

Theorem 1 [10], [12]**.** Suppose that every $\psi_l(\tau)$ $(l=1,\ldots,k)$ is a continuous non-random function on $[t,T]$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_2([t,T])$. Then

$$\mathbf{M}\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}\right)^2\le k!\left(I_k-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}^2\right)\to 0 \tag{12}$$

if $p_1,\ldots,p_k\to\infty$, where $I_k^{1/2}$ is the $L_2([t,T]^k)$-norm of $K(t_1,\ldots,t_k)$,

$$J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}=\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in G_k}\phi_{j_1}(\tau_{l_1})\,\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\,\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right),$$

$i_1,\ldots,i_k=1,\ldots,m$ for $T-t\in(0,\infty)$ and $i_1,\ldots,i_k=0,1,\ldots,m$ for $T-t\in(0,1)$, $J[\psi^{(k)}]_{T,t}$ is defined by formula (10), l.i.m. is a limit in the mean-square sense, $\zeta_j^{(i)}=\int_t^T\phi_j(s)\,d\mathbf{w}_s^{(i)}$ $(i=0,1,\ldots,m)$ are i.i.d. $N(0,1)$-r.v.'s for various i or j if $i\ne 0$, $C_{j_k\ldots j_1}$ is defined by formula (11), $G_k=H_k\backslash L_k$, $H_k=\{(l_1,\ldots,l_k)\colon l_1,\ldots,l_k=0,1,\ldots,N-1\}$, $L_k=\{(l_1,\ldots,l_k)\colon l_1,\ldots,l_k=0,1,\ldots,N-1;\,l_g\ne l_r\,(g\ne r);\,g,r=1,\ldots,k\}$, $\Delta\mathbf{w}_{\tau_q}^{(i)}=\mathbf{w}_{\tau_{q+1}}^{(i)}-\mathbf{w}_{\tau_q}^{(i)}$, $\{\tau_q\}_{q=0}^{N}$ is a partition of $[t,T]$ such that $t=\tau_0<\ldots<\tau_N=T$, $\max_{0\le q\le N-1}(\tau_{q+1}-\tau_q)\to 0$ if $N\to\infty$.

Note that a number of generalizations and modifications of theorem 1 can be found in [12].

Let $J[\psi^{(k)}]_{T,t}^{p}$ be $J[\psi^{(k)}]_{T,t}^{p_1,\ldots,p_k}$ for $p_1=\ldots=p_k=p$ and let $E_k^p=\mathbf{M}\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{p}\right)^2$. Combining estimates (8) and (12) for $p_1=\ldots=p_k=p$ and $k=3$, we obtain

$$3!\left(I_3-\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}^2\right)\le C(T-t)^4. \tag{13}$$

It is not difficult to see that the multiplier factor 3! on the left-hand side of inequality (13) leads to a significant increase of computational costs for approximation of iterated Ito stochastic integrals. The mentioned problem can be overcome if we calculate the mean-square approximation error E_k^p exactly.

Theorem 2 [12]**.** Suppose that the conditions of theorem 1 are fulfilled. Then

$$E_k^p=I_k-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\mathbf{M}\left(J[\psi^{(k)}]_{T,t}\sum_{(j_1,\ldots,j_k)}\int_t^T\phi_{j_k}(t_k)\ldots\int_t^{t_2}\phi_{j_1}(t_1)\,d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right), \tag{14}$$

where $i_1,\ldots,i_k = 1,\ldots,m$; expression $\sum_{(j_1,\ldots,j_k)}$ means the sum with respect to all possible permutations $(j_1,\ldots,j_k)$. At the same time if j_r swapped with j_q in the permutation $(j_1,\ldots,j_k)$, then i_r swapped with i_q in the permutation $(i_1,\ldots,i_k)$; another notations are the same as in theorem 1.

For the further consideration, we note that

$$\mathbf{M}\left(J[\psi^{(k)}]_{T,t}\int_t^T \phi_{j_k}(t_k)\ldots\int_t^{t_2}\phi_{j_1}(t_1)\,d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right) = C_{j_k\ldots j_1}. \tag{15}$$

Theorem 3 [12]**.** Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of third multiplicity $I^{*(i_1i_2i_3)}_{(000)T,t}$ the following expansion

$$I^{*(i_1i_2i_3)}_{(000)T,t} = \underset{p_1,p_2,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C^{000}_{j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}\quad (i_1,i_2,i_3=1,\ldots,m) \tag{16}$$

that converges in the mean-square sense is valid, where

$$C^{000}_{j_3j_2j_1} = \int_t^T \phi_{j_3}(t_3)\int_t^{t_3}\phi_{j_2}(t_2)\int_t^{t_2}\phi_{j_1}(t_1)\,dt_1\,dt_2\,dt_3; \tag{17}$$

another notations are the same as in theorem 1.

3 Approximation of iterated Ito and Stratonovich stochastic integrals using Legendre polynomials

Using theorems 1 and 3 and the complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$, we obtain the following formulas for numerical modeling of iterated Ito and Stratonovich stochastic integrals from numerical schemes (6) and (7) [10, 12]

$$I^{(i_1)}_{(0)T,t} = \sqrt{T-t}\,\zeta^{(i_1)}_0,\quad I^{(i_1)}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2}\left(\zeta^{(i_1)}_0+\frac{1}{\sqrt{3}}\zeta^{(i_1)}_1\right),$$

$$I^{(i_1i_2)q}_{(00)T,t} = \frac{T-t}{2}\left(\zeta^{(i_1)}_0\zeta^{(i_2)}_0+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta^{(i_1)}_{i-1}\zeta^{(i_2)}_i-\zeta^{(i_1)}_i\zeta^{(i_2)}_{i-1}\right)-\mathbf{1}_{\{i_1=i_2\}}\right),$$

$$I^{(i_1i_2i_3)q_1}_{(000)T,t} = \sum_{j_1,j_2,j_3=0}^{q_1} C^{000}_{j_3j_2j_1}\Big(\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta^{(i_3)}_{j_3}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta^{(i_1)}_{j_1}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta^{(i_2)}_{j_2}\Big), \tag{18}$$

$$I^{*(i_1)}_{(0)T,t} = \sqrt{T-t}\,\zeta^{(i_1)}_0,\; I^{*(i_1)}_{(1)T,t} = -\frac{(T-t)^{3/2}}{2}\left(\zeta^{(i_1)}_0+\frac{1}{\sqrt{3}}\zeta^{(i_1)}_1\right),$$

$$I^{*(i_1i_2)q}_{(00)T,t} = \frac{T-t}{2}\left(\zeta^{(i_1)}_0\zeta^{(i_2)}_0+\sum_{i=1}^{q}\frac{1}{\sqrt{4i^2-1}}\left(\zeta^{(i_1)}_{i-1}\zeta^{(i_2)}_i-\zeta^{(i_1)}_i\zeta^{(i_2)}_{i-1}\right)\right),$$

$$I^{*(i_1i_2i_3)q_1}_{(000)T,t} = \sum_{j_1,j_2,j_3=0}^{q_1} C^{000}_{j_3j_2j_1}\zeta^{(i_1)}_{j_1}\zeta^{(i_2)}_{j_2}\zeta^{(i_3)}_{j_3}, \tag{19}$$

where $\mathbf{1}_A$ is the indicator of the set A; $i_1,\ldots,i_3 = 1,\ldots,m$,

$$C^{000}_{j_3j_2j_1} = \frac{1}{8}L_{j_1j_2j_3}(T-t)^{3/2}\bar{C}^{000}_{j_3j_2j_1},\quad L_{j_1\ldots j_k} = \left(\prod_{l=1}^{k}(2j_l+1)\right)^{1/2},$$

$$\bar{C}^{000}_{j_3j_2j_1} = \int_{-1}^{1}P_{j_3}(z)\int_{-1}^{z}P_{j_2}(y)\int_{-1}^{y}P_{j_1}(x)\,dx\,dy\,dz,$$

$P_j(x)$ is the Legendre polynomial; other notations are the same as in theorem 1.

Let $E_k^{(l_1\ldots l_k)p}$ be the left-hand side of equality (14) for inetgral (4) and $E_k^{*(l_1\ldots l_k)p}$ be the analogous value for integral (5).

4 Main results

This section is devoted to the optimization of approximation procedures for iterated Stratonovich stochastic integrals, i.e. we discuss how to essentially minimize the values q and q_1 from the previous section for iterated Stratonovich stochastic integrals.

From theorem 2 we obtain (equality (15)) [11, 12]

$$E_2^{(00)p} = E_2^{*(00)p} = \frac{(T-t)^2}{2}\left(\frac{1}{2} - \sum_{i=1}^{p}\frac{1}{4i^2-1}\right), \quad i_1 \neq i_2, \tag{20}$$

$$E_3^{(000)p} = E_3^{*(000)p} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p} L_{j_1j_2j_3}^2 (\bar{C}_{j_3j_2j_1}^{000})^2\right), \quad i_1 \neq i_2, \quad i_1 \neq i_3, \quad i_2 \neq i_3, \tag{21}$$

$$E_3^{(000)p} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p} L_{j_1j_2j_3}^2 \left(\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2 + \bar{C}_{j_3j_1j_2}^{000}\bar{C}_{j_3j_2j_1}^{000}\right)\right), \quad i_1 = i_2 \neq i_3, \tag{22}$$

$$E_3^{(000)p} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p} L_{j_1j_2j_3}^2 \left(\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2 + \bar{C}_{j_2j_3j_1}^{000}\bar{C}_{j_3j_2j_1}^{000}\right)\right), \quad i_1 \neq i_2 = i_3, \tag{23}$$

$$E_3^{(000)p} = (T-t)^3\left(\frac{1}{6} - \frac{1}{64}\sum_{j_1,j_2,j_3=0}^{p} L_{j_1j_2j_3}^2 \left(\left(\bar{C}_{j_3j_2j_1}^{000}\right)^2 + \bar{C}_{j_3j_2j_1}^{000}\bar{C}_{j_1j_2j_3}^{000}\right)\right), \quad i_1 = i_3 \neq i_2. \tag{24}$$

Obviously, conditions (20)–(24) do not contain the multiplier factors 2!, 3! in contrast to estimate (12). However, the number of the mentioned conditions is quite large, which is inconvenient for practice. In [11] we proposed and confirmed the hypothesis that all formulas (20)–(24) can be replaced by formulas (20) and (21) in which we can suppose that $i_1,\ldots,i_3 = 1,\ldots,m$. At that we have no noticeable loss of the mean-square approximation accuracy of iterated Ito stochastic integrals. In this article, we propose and confirm similar hypothesis for iterated Stratonovich stochastic integrals.

It should be noted that unlike the method based on theorem 1, existing approaches to the mean-square approximation of iterated stochastic integrals [1, 2, 8, 9, 13–21] do not allow to choose different numbers p for approximations of different iterated stochastic integrals with multiplicities $k = 2, 3, \ldots$ Moreover, the noted approaches exclude the possibility for obtaining of approximate and exact expressions similar to formulas (12), (14). The detailed comparison of theorem 1 with methods from [1, 2, 8, 9, 13–21] is given in chapter 6 of monograph [12].

Consider the analogs of formulas (22)–(24) for iterated Stratonovich stochastic integrals.

Theorem 4. Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([t,T])$. Then, the following relations are hold

$$E_3^{*(000)p} = E_3^{(000)p} + (T-t)^3\left(\frac{1}{12} - \frac{1}{16}\sum_{j_1=0}^{p} L_{j_1}^2\left(\bar{C}_{0j_1j_1}^{000} + \bar{C}_{1j_1j_1}^{000}\right) + \frac{1}{64}\sum_{j_3=0}^{p} L_{j_3}^2\left(\sum_{j_1=0}^{p} L_{j_1}^2 \bar{C}_{j_3j_1j_1}^{000}\right)^2\right),$$
$$i_1 = i_2 \neq i_3, \tag{25}$$

$$E_3^{*(000)p} = E_3^{(000)p} + (T-t)^3 \left(\frac{1}{12} - \frac{1}{16} \sum_{j_3=0}^{p} L_{j_3}^2 \left(\bar{C}_{j_3 j_3 0}^{000} - \bar{C}_{j_3 j_3 1}^{000} \right) + \frac{1}{64} \sum_{j_1=0}^{p} L_{j_1}^2 \left(\sum_{j_3=0}^{p} L_{j_3}^2 \bar{C}_{j_3 j_3 j_1}^{000} \right)^2 \right),$$

$$i_1 \neq i_2 = i_3, \tag{26}$$

$$E_3^{*(000)p} = E_3^{(000)p} + \frac{(T-t)^3}{64} \sum_{j_2=0}^{p} L_{j_2}^2 \left(\sum_{j_1=0}^{p} L_{j_1}^2 \bar{C}_{j_1 j_2 j_1}^{000} \right)^2, \quad i_1 = i_3 \neq i_2. \tag{27}$$

Proof. Consider the following three cases $i_1 = i_2 \neq i_3$, $i_1 \neq i_2 = i_3$, $i_1 = i_3 \neq i_2$. First, consider the case $i_1 = i_2 \neq i_3$. From the standard relations between Stratonovich and Ito stochastic integrals [1] and formulas (18), (19) we obtain

$$E_3^{*(000)p} = \mathbf{M} \left(I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)p} + \frac{1}{2} \int_t^T \int_t^\tau ds\, d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1, j_3=0}^{p} C_{j_3 j_1 j_1}^{000} \zeta_{j_3}^{(i_3)} \right)^2. \tag{28}$$

According to formulas (1.39), (1.83) [12], the quantity $I_{(000)T,t}^{(i_1 i_2 i_3)} - I_{(000)T,t}^{(i_1 i_2 i_3)p}$ includes only iterated Ito stochastic integrals of multiplicity 3. At the same time, the quantity

$$\frac{1}{2} \int_t^T \int_t^\tau ds\, d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1, j_3=0}^{p} C_{j_3 j_1 j_1}^{000} \zeta_{j_3}^{(i_3)}$$

contains only iterated Ito stochastic integrals of multiplicity 1. This means that from (28) we get

$$E_3^{*(000)p} = E_3^{(000)p} + \mathbf{M} \left(\frac{1}{2} \int_t^T (\tau - t)\, d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1, j_3=0}^{p} C_{j_3 j_1 j_1}^{000} \zeta_{j_3}^{(i_3)} \right)^2. \tag{29}$$

Theorem 2 implies that

$$E_3^{(000)p} = \frac{(T-t)^3}{6} - \sum_{j_1, j_2, j_3=0}^{p} \left(C_{j_3 j_2 j_1}^{000} \right)^2 - \sum_{j_1, j_2, j_3=0}^{p} C_{j_3 j_1 j_2}^{000} C_{j_3 j_2 j_1}^{000}, \quad i_1 = i_2 \neq i_3. \tag{30}$$

We have

$$\mathbf{M} \left(\frac{1}{2} \int_t^T (\tau - t)\, d\mathbf{f}_\tau^{(i_3)} - \sum_{j_1, j_3=0}^{p} C_{j_3 j_1 j_1}^{000} \zeta_{j_3}^{(i_3)} \right)^2 =$$

$$= \frac{1}{4} \int_t^T (\tau - t)^2 d\tau - \sum_{j_1, j_3=0}^{p} C_{j_3 j_1 j_1}^{000} \int_t^T (\tau - t)\, \phi_{j_3}(\tau)\, d\tau + \sum_{j_3=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1}^{000} \right)^2, \tag{31}$$

where $\phi_{j_3}(\tau)$ is the Legendre polynomial. Moreover, we obtain

$$\int_t^T (\tau - t)\, \phi_{j_3}(\tau)\, d\tau = \frac{(T-t)^{3/2}}{2} \begin{cases} 1, & j_3 = 0, \\ 1/\sqrt{3}, & j_3 = 1, \\ 0, & j_3 \geq 2. \end{cases} \tag{32}$$

Combining (29)–(32), we get

$$E_3^{*(000)p} = \frac{(T-t)^3}{4} - \sum_{j_1, j_2, j_3=0}^{p} \left(C_{j_3 j_2 j_1}^{000} \right)^2 - \sum_{j_1, j_2, j_3=0}^{p} C_{j_3 j_1 j_2}^{000} C_{j_3 j_2 j_1}^{000} -$$

$$- \frac{(T-t)^{3/2}}{2} \sum_{j_1=0}^{p} \left(C_{0 j_1 j_1}^{000} + \frac{1}{\sqrt{3}} C_{1 j_1 j_1}^{000} \right) + \sum_{j_3=0}^{p} \left(\sum_{j_1=0}^{p} C_{j_3 j_1 j_1}^{000} \right)^2, \tag{33}$$

where $i_1 = i_2 \neq i_3$. The case $i_1 = i_2 \neq i_3$ is proved. The cases $i_1 \neq i_2 = i_3$ and $i_1 = i_3 \neq i_2$ can be considered by analogy to the case $i_1 = i_2 \neq i_3$. Theorem 4 is proved.

Let $q(\alpha)$ be numbers p from formulas (20), (21), (25)–(27), where α are numbers of these formulas. Let

$$E_2^{*p} \leq (T-t)^4, E_3^{*p} \leq (T-t)^4, \tag{34}$$

where E_2^{*p}, E_3^{*p} mean the left-hand sides of equalities (20), (21), (25)–(27) respectively.

Let us show by numerical experiments that in most situations the following inequalities are fulfilled (under condition (34)): $q(21) \geq q(25)$, $q(21) \geq q(26)$, $q(21) \geq q(27)$, where all numbers in these inequalities are minimal natural numbers satisfying to condition (34). In tables 1–3 we can see the results of numerical experiments. In table 3 numbers $p(21)$, $p(25)$–$p(27)$ means numbers p corresponding to formulas (21), (25)–(27) respectively. These results confirm the hypothesis proposed in this paper.

Table 1. Condition (34)

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$q(21)$	12	16	28	36	47	50
$q(25)$	6	8	14	18	23	25
$q(26)$	6	8	14	18	23	25
$q(27)$	12	16	28	37	47	51

Table 2. Strong scheme with order 1.5. Condition (34)

$T-t$	2^{-1}	2^{-3}	2^{-5}	2^{-8}
$q(20)$	1	8	128	8192
$q(21)$	0	1	4	32
$q(25)$	0	0	2	16
$q(26)$	0	0	2	16
$q(27)$	0	1	4	33

Table 3. Values $E_3^{*(000)p} \cdot (T-t)^{-3} \overset{\text{def}}{=} E_p^*$

$T-t$	0.011	0.008	0.0045	0.0035	0.0027	0.0025
$p = p(21)$	12	16	28	36	47	50
E_p^*	0.010154	0.007681	0.004433	0.003456	0.002652	0.002494
$p = p(25)$	12	16	28	36	47	50
E_p^*	0.005102	0.003855	0.002221	0.001731	0.001328	0.001248
$p = p(26)$	12	16	28	36	47	50
E_p^*	0.005102	0.003855	0.002221	0.001731	0.001328	0.001248
$p = p(27)$	12	16	28	36	47	50
E_p^*	0.010407	0.007845	0.004500	0.003501	0.002680	0.002519

Let q_1 be minimal natural number satisfying to $E_3^{*(000)q_1} \leq (T-t)^4$ (the left-hand side of these inequality is defined by formula (21)). Let p_1 be minimal natural number satisfying to $3!\, E_3^{*(000)p_1} \leq (T-t)^4$, where the value $E_3^{*(000)p_1}$ is defined by formula (21).

Table 4. Comparison of numbers q_1 and p_1

$T-t$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
q_1	0	0	1	2	4	8
p_1	1	3	6	12	24	48

In table 4 we can see the numerical comparison of numbers q_1 and p_1. Obviously, excluding of the multiplier factor 3! essentially (in many times) reduces the calculation costs for the mean-square approximations of iterated Stratonovich stochastic integrals. Note that in this paper we use the exactly calculated Fourier–Legendre coefficients using the Python programming language [23].

5 Conclusion

As we mentioned above, existing approaches [1, 2, 8, 9, 13–21] to the mean-square approximation of iterated stochastic integrals do not allow to choose different numbers p (see theorem 2) for approximations of different iterated stochastic integrals with multiplicity $k = 2, 3, \ldots$ and exclude the possibility for obtaining of approximate and exact expressions similar to formulas (12) and (14). This leads to unnecessary terms usage in the expansions of iterated Ito and Stratonovich stochastic integrals and, as a consequence, to essential increase of computational costs for the implementation of numerical methods for Ito SDEs. In this article, we have optimized method based on theorems 1–3, which makes it possible to correctly choose the lengths of sequences of standard Gaussian random variables required for the approximation of iterated Stratonovich stochastic integrals. Thus, the computational costs for the implementation of numerical methods for Ito SDEs based on the Taylor–Stratonovich expansion are significantly reduced. The analogous optimization for iterated Ito stochastic integrals is carried out in [11].

On the base of the obtained results we can recommend the following conditions for correct choosing the minimal natural numbers q and q_1: $E_2^{*(00)q} \le \overline{C}(T-t)^4$, $E_3^{*(000)q_1} \le \overline{C}(T-t)^4$ (for the strong scheme (7) with order 1.5). Here the left-hand sides of the above inequalities are defined by relations (20) and (21), correspondingly; $\overline{C}$ is a constant from condition (9).

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