Momentary finite element for elasticity 3D problems

Dmitry Chekmarev[1,*] and *Yasser* Abu Dawwas[1]

[1]Lobachevsky State University of Nizhny Novgorod, 23, Gagarina prospekt, Nizhny Novgorod, 603022, Russia

Abstract. A description of a new 8-node finite element in the form of a hexahedron is given for solving elasticity 3D problems. This finite element has the following features. This is a linear approximation of functions in the element, one point of integration and taking into account the moments of forces in the element. The finite element is based on "rare mesh" FEM schemes—finite element schemes in the form of n-dimensional cubes (square, cube, etc.) with templates in the form of inscribed simplexes (triangle, tetrahedron, etc.). Among the rare mesh schemes, schemes in 3-dimensional and 7-dimensional spaces are successful, in which the simplex can be arranged symmetrically with respect to the center of the n-dimensional cube. The rare mesh FEM schemes have not the hourglass instability due to the fact that the template of the finite element operator has the form of a simplex. Compared to traditional linear finite elements in the form of a simplex, rare mesh schemes are more economical and converge better, since they do not have the effect of overestimated shear stiffness. Moment FEM schemes are constructed by rare mesh schemes higher dimensional projection, respectively, on a two-dimensional or three-dimensional finite element mesh. The resulting finite elements are close to the known polylinear elements and surpass them in efficiency. The schemes contain parameters that allow you to control the convergence of numerical solutions. The possibility of applying this approach to the construction of numerical schemes for solving other problems of mathematical physics is discussed.

1 Introduction

When solving solid dynamic problems in combination with an explicit "cross" scheme, 4-node two-dimensional and 8-node three-dimensional finite elements with one integration point are widely used. In this case, as a rule, the problem of "hourglass instability" arises [1–3], associated with the incompleteness of the system of basic operators and leading to strong nonphysical distortions of the computational grid. To combat it, special means are used that require additional computational costs. Traditional schemes based on 2-dimensional triangular elements and 3-dimensional tetrahedral elements have poorer stability and slower convergence. One of the possible approaches to increasing the efficiency is the use of finite element schemes on rare meshes [4–7]. In rare mesh FEM schemes, the computational elements cover the problem area with regular intervals, which makes it possible to noticeably reduce the number of elements on grids of the same size and thereby reduce the amount of computations.

*e-mail: 4ekm@mm.unn.ru

The most successful rare mesh FEM scheme for solving 3D elasticity and plasticity problems is a scheme based on a hexahedral grid, in each cell (hexahedron) of which there is one linear element in the form of a central tetrahedron (figure 1), and the remaining 4 tetrahedrons are not involved in the calculations. In this case, in comparison with the traditional scheme of linear finite element on the same grids, the number of elements is reduced by a factor of 5, and the number of nodes is reduced by a factor of 2. With better efficiency, the rare mesh scheme has the second order of accuracy (no worse than that of schemes based on linear and bilinear finite elements). On a number of test problems, it also showed better convergence in comparison with the indicated schemes. As noted in [4], successful rare mesh schemes are obtained with a symmetric arrangement of a linear element (simplex) inside an n-dimensional cube. So, for $n = 2$ (a triangle in a square), the arrangement is asymmetric (figure 2), and for $n = 3$ (a tetrahedron in a cube), it is symmetric (the center of the tetrahedron coincides with the center of the cube (figure 1)). The next case of a lucky dimension is $n = 7$ (a regular 8-node simplex can be inscribed symmetrically into a 7-dimensional cube). When solving problems of "unsuccessful" dimension, effective FEM circuits can be obtained by projection a rare mesh circuit of a higher "successful" dimension. At the same time, it is natural to assume that the high quality of the circuit is preserved during projection. An approach to constructing new numerical FEM schemes by projection rare mesh schemes on a mesh of a lower dimension is described in [8].

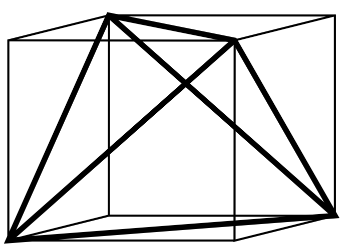

Figure 1. The cell of 3D rare mesh

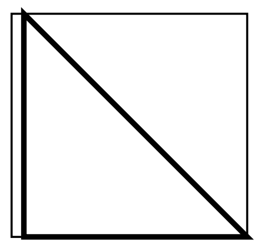

Figure 2. The cell of 2D rare mesh

This paper describes the implementation of the three-dimensional moment scheme proposed in [8]. Further presentation is based on the material [8].

2 2D FEM scheme for elasticity problems based on a 3D rare mesh scheme

Rare mesh FEM scheme for solution 3D dynamic elasticity problem on a uniform orthogonal mesh $x_i^1 = x_0^1 + h_1 i$, $x_j^2 = x_0^2 + h_2 j$, $x_k^3 = x_0^3 + h_3 k$; can be written in the following finite-difference form (see [4]):

$$(\lambda+\mu)\begin{vmatrix} D_{11}u_1 + D_{12}u_2 + D_{13}u_3 \\ D_{21}u_1 + D_{22}u_2 + D_{23}u_3 \\ D_{31}u_1 + D_{32}u_2 + D_{33}u_3 \end{vmatrix} + \mu D_\Delta \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} + \rho \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} = \rho D_{tt} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}. \tag{1}$$

This form is similar to the Lamé system of equations form

$$(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + \mu\Delta u + \rho F = \rho\frac{\partial^2 u}{\partial t^2}. \tag{2}$$

Here λ, μ are the Lame parameters, ρ—is the density of the medium, F—is the mass force, u—is the displacement field, D_{ij}, D_Δ—are the grid operators approximating the second derivatives and the Laplace operator. Operators are constructed as follows. Basic operators approximating the first partial derivatives in an element ($d_m^+ \approx \partial/\partial x^m$) are written as

$$
\begin{aligned}
(d_1^+ f)_{ijk} &= \frac{1}{2h_1}\left(f_{i+1,j+1,k+1} + f_{i+1,j,k} - f_{i,j+1,k} - f_{i,j,k+1}\right),\\
(d_2^+ f)_{ijk} &= \frac{1}{2h_2}\left(f_{i+1,j+1,k+1} + f_{i,j+1,k} - f_{i+1,j,k} - f_{i,j,k+1}\right),\\
(d_3^+ f)_{ijk} &= \frac{1}{2h_3}\left(f_{i+1,j+1,k+1} + f_{i,j,k+1} - f_{i,j+1,k} - f_{i+1,j,k}\right).
\end{aligned}
\tag{3}
$$

Operators

$$
\begin{aligned}
(d_1^- f)_{ijk} &= \frac{1}{2h_1}\left(-f_{i-1,j-1,k-1} - f_{i-1,j,k} + f_{i,j-1,k} + f_{i,j,k-1}\right),\\
(d_2^- f)_{ijk} &= \frac{1}{2h_2}\left(-f_{i-1,j-1,k-1} - f_{i,j-1,k} + f_{i-1,j,k} + f_{i,j,k-1}\right),\\
(d_3^- f)_{ijk} &= \frac{1}{2h_3}\left(-f_{i-1,j-1,k-1} - f_{i,j,k-1} + f_{i,j-1,k} + f_{i-1,j,k}\right)
\end{aligned}
\tag{4}
$$

are equal to the conjugate to (3), taken with the sign "−". We define the operators D_{ij} in terms of superpositions of operators (3) and (4): $D_{ij} = \frac{1}{2}\left(d_i^+ d_j^- + d_j^+ d_i^-\right)$, $D_\Delta = D_{11} + D_{22} + D_{33}$.

A numerical scheme for solving a two-dimensional elasticity problem (plane deformation) is obtained by projecting a three-dimensional one on a plane x_1Ox_2. Assuming that the three-dimensional computational domain has the form $\Omega \times [0, h_3]$, where Ω—area in R^2, take one layer of cells along the x^3 coordinate and impose the constraint on the solution $u_3 = 0$. In this case, the difference scheme (1) takes the form

$$
\begin{aligned}
(\lambda+\mu)(D_{11}u_1 + D_{12}u_2) + \mu(D_{11}u_1 + D_{22}u_1 + D_{33}u_1) + \rho F_1 &= \rho D_{tt}u_1,\\
(\lambda+\mu)(D_{21}u_1 + D_{22}u_2) + \mu(D_{11}u_2 + D_{22}u_2 + D_{33}u_2) + \rho F_2 &= \rho D_{tt}u_2.
\end{aligned}
\tag{5}
$$

In (5), the operators D_{ij} are obtained by projecting the operators considered above onto a two-dimensional grid, while the operators

$$
\begin{aligned}
(d_3^+ f)_{ij} &= \frac{1}{2h_3}\left(f_{i+1,j+1} + f_{i,j} - f_{i,j+1} - f_{i+1,j}\right),\\
(d_3^- f)_{ij} &= \frac{1}{2h_3}\left(f_{i-1,j-1} + f_{i,j} - f_{i,j-1} - f_{i-1,j}\right)
\end{aligned}
\tag{6}
$$

approximate, up to a factor, the operator of the second mixed derivative $\frac{\partial^2}{\partial x^1 \partial x^2}$.

The element size h_3 along the coordinate x^3 in the constructed numerical scheme turns into an adjustable parameter of the scheme. Thus, a one-parameter family of numerical FEM schemes for solving the plane elasticity problem was obtained. By adjusting the parameter h_3, the influence of the moment component in the element and thereby the convergence of numerical solutions can be regulated.

3 3D FEM scheme for elasticity problems based on a 7D rare mesh scheme

Generalizing this approach, we construct the 8-node finite element for solving 3D problems of elasticity theory, similar to the 8-node multilinear finite element.

3.1 Seven-dimensional rare mesh scheme

Among the vertices of a unit 7-dimensional cube, you can choose 8 vertices that form a regular simplex, the center of which coincides with the center of the cube. An example is a set of vertices

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\\0\\1\end{bmatrix}. \qquad (7)$$

All edges of a given simplex have the same length. Note also that the sets of the first three coordinates of the given vector system form the set of vertices of the unit cube in R^3.

Consider a hypothetical "theory of elasticity" in 7-dimensional space. We define the tensor of deformations in R^7 as a generalization of the Cauchy relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, \ldots, 7.$$

The relationship between stresses and strains is established on the basis of "Hooke's law":

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \quad i, j = 1, \ldots, 7.$$

As a result, the 7-dimensional equilibrium equations of the "elasticity theory" will be written in the form (2), where the gradient, divergence and the Laplace operator are defined, respectively, in R^7. Rare mesh FEM scheme based on a linear finite element on a uniform orthogonal mesh $x^1_{i_1} = x^1_0 + h_1 i_1, \ldots, x^7_{i_7} = x^7_0 + h_7 i_7$ will take a form similar to (1):

$$(\lambda+\mu)\begin{vmatrix}D_{11}u_1 + \ldots + D_{17}u_7\\ \ldots\\ D_{71}u_1 + \ldots + D_{77}u_7\end{vmatrix} + \mu D_\Delta \begin{vmatrix}u_1\\ \ldots\\ u_7\end{vmatrix} + \rho \begin{vmatrix}F_1\\ \ldots\\ F_7\end{vmatrix} = 0. \qquad (8)$$

To construct a three-dimensional scheme, we will proceed similarly to the case of a two-dimensional scheme considered above. Assuming at all nodes $u_4 = u_5 = u_6 = u_7 = 0$, we get the FEM scheme in the form

$$(\lambda+\mu)\begin{vmatrix}D_{11}u_1 + D_{12}u_2 + D_{13}u_3\\ D_{21}u_1 + D_{22}u_2 + D_{23}u_3\\ D_{31}u_1 + D_{32}u_2 + D_{33}u_3\end{vmatrix} + \mu D_\Delta \begin{vmatrix}u_1\\u_2\\u_3\end{vmatrix} + \mu(D_{44} + D_{55} + D_{66} + D_{77})\begin{vmatrix}u_1\\u_2\\u_3\end{vmatrix} + \rho\begin{vmatrix}F_1\\F_2\\F_3\end{vmatrix} = 0 \quad (9)$$

after projection into R^3. In (9), the operators $D_{ij} = \frac{1}{2}\left(d^+_i d^-_j + d^+_j d^-_i\right)$, $D_\Delta = D_{11} + D_{22} + D_{33}$ expressed in terms of basic operators $d^+_1, d^+_2, d^+_3, d^+_4, d^+_5, d^+_6, d^+_7$ and their conjugates (with a— sign) $d^-_1, d^-_2, d^-_3, d^-_4, d^-_5, d^-_6, d^-_7$ defined on 8-node templates. Moreover, the operators d^+_1, d^+_2, d^+_3 and d^-_1, d^-_2, d^-_3 approximate the first derivatives with respect to the corresponding coordinates, and the operators $d^+_4, d^+_5, d^+_6, d^+_7$ and $d^-_4, d^-_5, d^-_6, d^-_7$ approximate, up to a positive factor, the differentiation operators $\frac{\partial^2}{\partial x^1 \partial x^2}, \frac{\partial^2}{\partial x^1 \partial x^3}, \frac{\partial^2}{\partial x^2 \partial x^3}, \frac{\partial^3}{\partial x^1 \partial x^2 \partial x^3}$. Similar approximations of the moment components take place in the traditional scheme of a multilinear 8-node finite element (see, for example, [9]).

Thus, a 4-parameter family of numerical FEM schemes for solving a 3D elasticity problem was obtained. The parameters h_4, h_5, h_6, h_7 can be adjusted for changing the influence of the moments in the element.

4 Implementation of a 3D moment FEM scheme

The implementation of a moment element can be performed in the traditional FEM technique using vector and matrix notation, building basis functions, mapping finite elements to a standard cubic element, etc. [9–12]. It is more convenient and economical to make the implementation in a form similar to Wilkins' scheme [13], because stresses, strains and Jacobian of the mapping are constants in the element. The derivatives in an arbitrary hexahedral element are calculated by the formula

$$\frac{\partial f}{\partial x^i} \approx d_i^+ f = \frac{\det V_i}{\det V},$$

where the matrix V has the form, $V = \begin{bmatrix} 1 & x_1^1 & x_1^2 & x_1^3 & 0 & 0 & 0 & 0 \\ 1 & x_2^1 & x_2^2 & x_2^3 & 0 & h_5 & h_6 & h_7 \\ 1 & x_3^1 & x_3^2 & x_3^3 & h_4 & 0 & h_6 & h_7 \\ 1 & x_4^1 & x_4^2 & x_4^3 & h_4 & h_5 & 0 & 0 \\ 1 & x_5^1 & x_5^2 & x_5^3 & h_4 & h_5 & 0 & h_7 \\ 1 & x_6^1 & x_6^2 & x_6^3 & h_4 & 0 & h_6 & 0 \\ 1 & x_7^1 & x_7^2 & x_7^3 & 0 & h_5 & h_6 & 0 \\ 1 & x_8^1 & x_8^2 & x_8^3 & 0 & 0 & 0 & h_7 \end{bmatrix}$, and matrices V_i are obtained from the matrix V by replacing the column of the corresponding coordinate with the column of function values, for example $V_1 = \begin{bmatrix} 1 & f_1 & x_1^2 & x_1^3 & 0 & 0 & 0 & 0 \\ 1 & f_2 & x_2^2 & x_2^3 & 0 & h_5 & h_6 & h_7 \\ 1 & f_3 & x_3^2 & x_3^3 & h_4 & 0 & h_6 & h_7 \\ 1 & f_4 & x_4^2 & x_4^3 & h_4 & h_5 & 0 & 0 \\ 1 & f_5 & x_5^2 & x_5^3 & h_4 & h_5 & 0 & h_7 \\ 1 & f_6 & x_6^2 & x_6^3 & h_4 & 0 & h_6 & 0 \\ 1 & f_7 & x_7^2 & x_7^3 & 0 & h_5 & h_6 & 0 \\ 1 & f_8 & x_8^2 & x_8^3 & 0 & 0 & 0 & h_7 \end{bmatrix}$ etc.

In this case, the strain tensor takes the form:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15} & \varepsilon_{16} & \varepsilon_{17} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} & \varepsilon_{25} & \varepsilon_{26} & \varepsilon_{27} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & \varepsilon_{34} & \varepsilon_{35} & \varepsilon_{36} & \varepsilon_{37} \end{bmatrix}, \quad \varepsilon_{ij} = \begin{cases} \frac{1}{2}\left(d_i^+ u_j + d_j^+ u_i\right), & j \leq 3, \\ \frac{1}{2} d_j^+ u_i, & j > 3 \end{cases}$$

and the stress tensor—respectively

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} \end{bmatrix}.$$

The last 4 columns correspond to three bending moments and one torque moment in the finite element.

Further, the integrals of the components of the stress tensor determine the forces applied to the nodes of the element, as a result of which the equations of motion of the nodes are formed.

5 Numerical Results

Let us consider two problems of vibrations of a square bar under the action of an instantaneously applied pressure. A beam with dimensions $L = 10$ cm, $H_1 = 1$ cm, $H_2 = 1$ cm with rigidly embedded ends, loaded with instantly applied pressure 0,17 GPa, uniformly distributed over the site $4 < x^1 < 6$ (figure 3). Properties of the material: $\rho = 7.8$ g/cm^3, $E = 210$ GPa, $\nu = 0.3$.

The problem was solved on two grids: $40 \times 4 \times 4$ elements and $80 \times 8 \times 8$ elements. The second (asymmetric) problem was solved in a similar setting, but the loaded area was half of that considered in the first problem (figure 4).

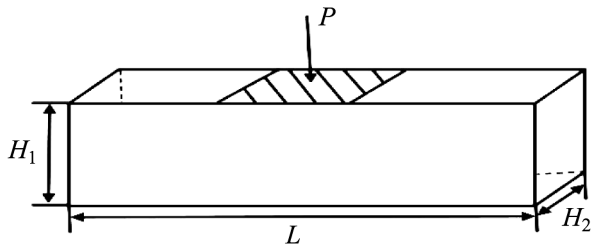

Figure 3. Symmetric problem

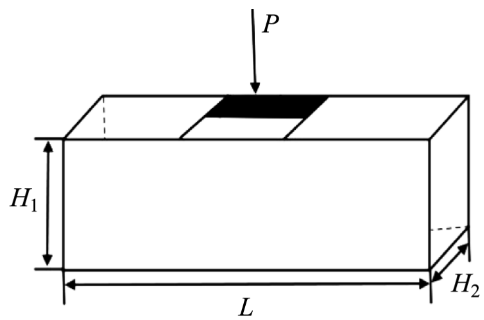

Figure 4. Asymmetric problem

For comparison, the same problem were solved using the rare mesh scheme and Wilkins' scheme.

The symmetrical problem was previously solved using the Wilkins scheme and the rare mesh scheme. The solution results are given in [6]. The solution based on the moment scheme is shown in figure 5. It show the graphs of the velocity υ_3 in the center of the face opposite to the loaded one, depending on time. The graphs show the results of calculations up to 747 μs. Solid lines correspond to calculations on a grid of $40 \times 4 \times 4$ elements, dashed lines correspond to calculations on a grid of $80 \times 8 \times 8$ elements. Note that all three schemes showed approximately the same results. The slowest convergence was observed for the Wilkins scheme.

Figures 6–8 show similar results of solving the asymmetric problem.

Numerical results show good qualities of the rare mesh and moment schemes for both problems and their better convergence in comparison with the Wilkins scheme. Note the difference between the numerical solutions of the symmetric and asymmetric problems. The hourglass instability of the Wilkins scheme can be realized to varying degrees depending on the grid size, boundary conditions, and other factors. Therefore, it is not observed in the symmetric problem. Wilkins scheme on an asymmetric problem shows completely unsatisfactory

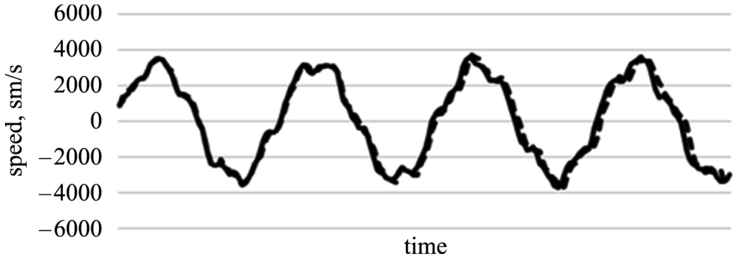

Figure 5. Symmetric problem. Momentary scheme

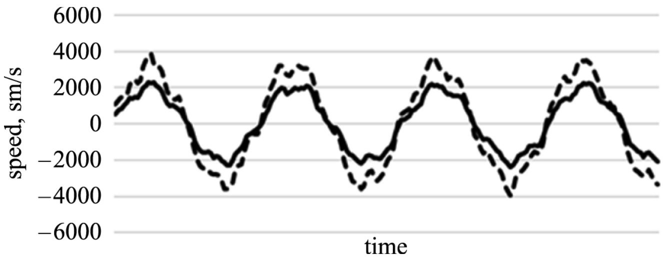

Figure 6. Asymmetric problem. Rare mesh scheme

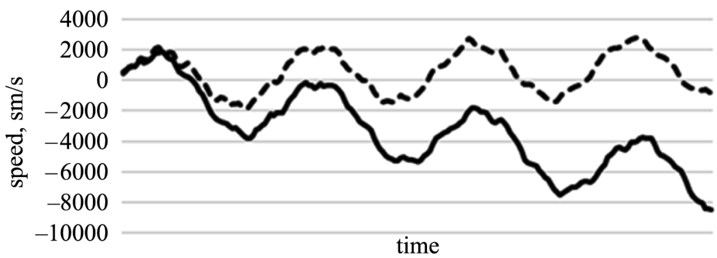

Figure 7. Asymmetric problem. Wilkins scheme

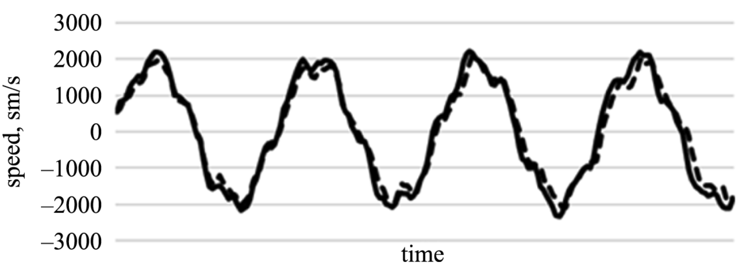

Figure 8. Asymmetric problem. Momentary scheme

results, which is due to the hourglass instability. Note that in the Wilkins scheme, no means were used to combat this type of instability. According to the results of solving both test problems, the moment scheme showed the best convergence.

6 Conclusion

A new moment FEM scheme is implemented for solving 3D dynamic elasticity problems. It is free from hourglass instability effect. This scheme is very economical since it has one integration point. The circuit has adjustable parameters.

The creation of new effective numerical methods for solving problems of mathematical physics is relevant both for the present and for the future due to the obvious trend of an ever wider and more detailed application of mathematical modeling in various fields of science. In particular, the discontinuous Galerkin method is currently being actively developed [14]. The finite element considered in the paper is a development of the concept of rare mesh finite element schemes. This concept appears to be very promising. Note that rare mesh schemes are similar in some aspects to the discontinuous Galerkin method, since they ensure the continuity of physical fields only on the edges of finite elements. Rare mesh schemes for solving 3D elasticity and plasticity problems have been theoretically studied in detail. They were verified and validated on model problems, compared with numerical solutions obtained using traditional FEM schemes and experimental data. The conducted studies have shown that rare mesh schemes are not inferior to traditional schemes in terms of accuracy and exceed them in efficiency, in particular, in speed - several times. Note that the concept of rare mesh schemes, as well as the moment schemes built on their basis, are based on purely geometric ideas that are in no way connected with the physical features of the simulated processes. Therefore, there is every reason to believe that schemes built within the framework of this concept, including moment schemes, can be used not only in the mechanics of a deformable solid body, but also in fluid and gas mechanics, electromagnetic theory and other areas of mathematical physics. Schemes based on this approach can be applied to solving problems of dimension more than three (if any).

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