Analysis of the orbital stability of periodic pendulum motions of a heavy rigid body with a fixed point under the Goryachev–Chaplygin condition

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Abstract. In this paper the motion of a rigid body with a fixed point in a uniform gravity field is considered. It is assumed that the main moments of inertia of the body correspond to the case of Goryachev–Chaplygin, i.e., they are in the ratio of 1:1:4. In contrast to the integrable case of Goryachev–Chaplygin, there are no restrictions imposed on the position of the center of mass of the body. The problem of the orbital stability of periodic pendulum motions of a body (oscillations and rotations) is investigated. The equations of perturbed motion were obtained and the problem of orbital stability was reduced to the stability problem of the equilibrium position of a second-order linear system with 2π-periodic coefficients, the right-hand sides of which depend on two parameters. Based on the analysis of the linearized system, it has been established that the rotations are orbitally unstable for all possible values of the parameters. Moreover, a diagram of stability of pendulum oscillations has been constructed, on which the regions of orbital instability (parametric resonance) and regions of orbital stability in the linear approximation are indicated. At small values of oscillations amplitudes, a nonlinear analysis was performed. Based on the analysis of the coefficients of the normalized Hamilton function using theorems of the KAM theory, rigorous conclusions on the orbital stability of pendulum oscillations with small amplitudes were obtained.

1 Introduction

In the dynamics of a rigid body periodic motions play a special role. Their study often allows one to draw important conclusions about the nature of the body's motion and qualitatively describe the structure of the phase space of the system. Both for theoretical mechanics and for its applications the problem of orbital stability of periodic pendulum motions of a heavy rigid body with a fixed point is of considerable interest. Modern methods of dynamical systems: the method of normal forms, the methods of KAM theory and the general theory of stability make it possible to obtain rigorous conclusions about the orbital stability of periodic solutions of this type.

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In the general case, the problem of stability of pendulum motions of a heavy rigid body with one fixed point contains four parameters. Therefore, it is rather difficult to obtain a complete solution to this problem. That is why a various number of special cases were investigated. In particular, the cases of Kovalevskaya, Goryachev–Chaplygin, Bobylev–Steklov and some cases of a dynamically symmetric body were considered. In the case of Kovalevskaya, the pendulum rotations of a rigid body are orbitally unstable, and the stability of pendulum oscillations depends on their amplitudes [1–5].

The case of Goryachev–Chaplygin was studied in [6–8]. In the case of Bobylev–Steklov, the problem has two parameters and, depending on the values of the parameters, both stability and instability might take place [9, 10]. For a dynamically symmetric rigid body, the case was investigated when body's center of mass lies in the equatorial plane of the ellipsoid of inertia [11]. A special case of a dynamically symmetric body was considered in [12] when the center of mass does not lie in the equatorial plane and certain conditions are imposed on the moments of inertia of the body ($A = C = 2B$). Moreover, the problem of orbital stability of periodic motions in linear approximation for Bobylev–Steklov case was considered in [13]. Also, pendulum motions of a rigid body carrying rotor were studied in [14].

2 Problem statement

Let us consider the motion of a rigid body of mass m around a fixed-point O in a uniform gravity field. To describe the motion of a body, we introduce a fixed coordinate system $OXYZ$, the OZ axis of which is vertically upward, and a moving coordinate system $Oxyz$, rigidly connected to the body, the axes of which are along the main axes of inertia of the body for point O. Furthermore, it is assumed that for the main moments of inertia A, B, C of the body relative to the axes Ox, Oy, Oz respectively equality $A = C = 4B$ holds.

We do not impose any additional conditions on the position of the center of mass. Let l be the distance from the center of mass of the body to the origin, α—the angle between the position vector of the center of mass and the equatorial plane of the ellipsoid of inertia. We assume without loss of generality that $0 \le \alpha \le \pi/2$. In this case, pendulum motions of the body relative to the axis of inertia lying in the equatorial plane of the ellipsoid of inertia are possible. It is worth noting that for $\alpha = 0$, the classical case of Goryachev–Chaplygin takes place. As we have mentioned, this case was carried out in [6, 15]. The aim of this work is to solve the problem of the orbital stability of the periodic pendulum motion of the body at $0 < \alpha \le \pi/2$.

The position of the rigid body we define by using the Euler angles ψ, θ, φ. The equations of motion of a body can be written in form of canonical equations with the Hamiltonian

$$H = \frac{1}{2}\left(\frac{(p_\theta \cos\varphi - p_\varphi \cot\theta \sin\varphi)^2}{A} + \frac{(p_\theta \sin\varphi + p_\varphi \cot\theta \cos\varphi)^2}{B} + \frac{p_\varphi^2}{C}\right) + mgl\sin\theta\sin(\varphi+\alpha), \tag{1}$$

where p_ψ, p_θ, p_φ are momenta corresponding to the Euler angles. The angle ψ is a cyclic coordinate, that is why $p_\psi = \text{const}$. In what follows, we put $p_\psi = 0$.

The equations of motion have a particular solution describing the motion such that the Oz axis keeps a constant horizontal position, and the body performs pendulum motions relative to this axis. Depending on the initial conditions, pendulum motions can be periodic motions (oscillations or rotations about this axis), or they can be asymptomatic motions approaching an unstable equilibrium position. Since pendulum periodic motions are unstable in the sense of Lyapunov, the problem of the orbital stability of these motions is of interest.

To begin with let us introduce the dimensionless time $\tau = \mu t$, $\mu^2 = mgl/C$. In addition to that, to describe the behavior of a body in the neighborhood of its periodic pendulum motions, it is convenient to carry out the following dimensionless coordinates

$$\varphi = q_1 - \alpha + \frac{3\pi}{2}, \quad \theta = q_2 + \frac{\pi}{2}, \quad p_1 = \frac{p_\varphi}{C\mu}, \quad p_2 = \frac{p_\theta}{C\mu}. \tag{2}$$

In the new variables the Hamiltonian of the problem takes the form

$$H = \frac{1}{2}\Big[\big(p_2 \sin(q_1 - \alpha) - p_1 \tan q_2 \cos(q_1 - \alpha)\big)^2 + \\ + 4\big(p_2 \cos(q_1 - \alpha) + p_1 \tan q_2 \sin(q_1 - \alpha)\big)^2 + p_1^2\Big] - \cos q_2 \cos q_1. \tag{3}$$

On the pendulum motions the equality $q_2 = p_2 = 0$ holds and evolution of variables q_1, p_1 is determined by the canonical system with the Hamiltonian

$$H_0 = \frac{1}{2}p_1^2 - \cos q_1. \tag{4}$$

The type of pendulum motions depends on the value of the constant of the energy integral $H_0 = h$: if $|h| < 1$, then pendulum oscillations take place, if $|h| > 1$, then pendulum rotations take place.

3 Local variables and isoenergetic reduction

In order to study the orbital stability of pendulum periodic motions we follow the method described in [7]. In accordance of this method, we introduce local variables in a neighborhood of an unperturbed periodic motion. To this end we perform the following canonical change of variables

$$q_1 = f + \frac{\sin f}{V^2}\eta - \frac{\sin f}{2V^4}\eta^2 + O(\eta^3), \quad p_1 = g + \frac{g}{V^2}\eta - \frac{g\cos f}{2V^4}\eta^2 + O(\eta^3), \tag{5}$$

where $V^2 = g^2 + \sin^2 f$.

Depending on the type of periodic motion, the functions f and g are introduced in different ways: in case of oscillations

$$f = \arcsin[k_1 sn(\xi, k_1)], \quad g = 2k_1 cn(\xi, k_1), \quad k_1^2 = \frac{h+1}{2}, \tag{6}$$

and in case of rotations

$$f = 2am(\xi, k_2), \quad g = 2k_2^{-1} dn(\xi, k_2), \quad k_2^2 = \frac{2}{h+1}, \tag{7}$$

In these cases, the periods are equal to $2\pi/\omega$, where in the case of oscillations $\omega = \pi/2K(k_1)$, in the case of rotations $\omega = \pi/k_2K(k_2)$. Here we use the generally accepted notation for elliptic functions and integrals.

Let us carry out one more canonical change of variables by the formulas

$$\xi = \frac{1}{\omega}w, \quad \eta = \omega r. \tag{8}$$

The expansion of the Hamilton function in a power series in a neighborhood $q_2 = p_2 = \eta = 0$ has the form

$$\Gamma = \Gamma_2 + \Gamma_4 + \dots \tag{9}$$

where

$$\Gamma_2 = \omega r + \sum_{i+j=2} \varphi_{ij} q_2^i p_2^j, \quad \Gamma_4 = \omega^2 \chi(w) r^2 + \omega r \sum_{i+j=2} \psi_{ij} q_2^i p_2^j + \sum_{i+j=4} \varphi_{ij} q_2^i p_2^j, \tag{10}$$

The coefficients in (10) 2π-periodically depend on w. They can be calculated by means following formulas

$$\begin{gathered}
\chi(w) = \frac{(\cos f - 1)(\sin^2 f - g^2)}{2V^4}, \\
\psi_{20} = \frac{6g^2\cos(2f-2\alpha) - 3g^2\cos(f-2\alpha) + 3g^2\cos(3f-2\alpha) - 10g^2 - \cos 2f - 1}{4V^2}, \\
\psi_{11} = \frac{-3g\sin(2f-2\alpha) - 3g\sin(3f-2\alpha) + 3g\sin(f-2\alpha)}{2V^2}, \\
\psi_{02} = \frac{3\cos(f-2\alpha) - 3\cos(3f-2\alpha)}{4V^2}, \quad \varphi_{20} = \frac{-3g^2\cos(2f-2\alpha) + 2\cos f + 5g^2}{4}, \\
\varphi_{11} = \frac{3g\sin(2f-2\alpha)}{2}, \quad \varphi_{02} = \frac{3\cos(2f-2\alpha)+5}{4}, \\
\varphi_{40} = \frac{-12g^2\cos(2f-2\alpha) - \cos f + 20g^2}{24}, \\
\varphi_{31} = \frac{g\sin(2f-2\alpha)}{2}, \quad \varphi_{13} = \varphi_{22} = \varphi_{04} = 0,
\end{gathered} \tag{11}$$

where f, g are 2π-periodic functions of the variable w. Their explicit and analytic form is obtained by substituting (8) into (6) and (7), respectively. Note that the moduli of the elliptic integral, k_1 and k_2, depend on h (see (6) and (7)). Thus, coefficients (11) are 2π-periodic functions of the variable w and analytically depend on parameters h, α.

Let us consider the motion on zero isoenergetic level $\Gamma = 0$. The evolution of the variables q_2, p_2 at the level $\Gamma = 0$ can be described by means of a reduced canonical system (Whittaker equations)

$$\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}, \tag{12}$$

where w plays the role of a new independent variable. The Hamiltonian K is obtained as a solution to the equation $\Gamma = 0$ with the respect to r. For small r Hamiltonian K can be represented as a power series expansion of q_2, p_2. Taking formulas (9)–(10) into account, we have the following expansion of the Hamilton function K in a power series of q_2, p_2

$$K = K_2 + K_4 + \ldots, \tag{13}$$

$$K_2 = \frac{1}{\omega} \sum_{i+j=2} \varphi_{ij} q_2^i p_2^j, \tag{13.1}$$

$$K_4 = \frac{1}{\omega}\left[\chi(w)\left(\sum_{i+j=2} \varphi_{ij} q_2^i p_2^j\right)^2 + \left(\sum_{i+j=2} \psi_{ij} q_2^i p_2^j\right)\left(\sum_{i+j=2} \varphi_{ij} q_2^i p_2^j\right) + \sum_{i+j=4} \varphi_{ij} q_2^i p_2^j.\right], \tag{13.2}$$

The problem of orbital stability of periodic motions of the rigid body is equivalent to the problem of stability of the trivial equilibrium position of system (12).

4 Linear stability analysis

First, we study the stability problem in linear approximation. To this end we consider linear canonical system with the Hamiltonian K_2

$$\frac{dq_2}{dw} = \varphi_{11}q_2 + 2\varphi_{02}p_2, \quad \frac{dp_2}{dw} = -2\varphi_{20}q_2 - \varphi_{11}p_2. \tag{14}$$

The conclusion about the stability of the linear system (14) can be made on the basis of the analysis of the roots of its characteristic equation

$$\rho^2 - 2\kappa\rho + 1 = 0, \tag{15}$$

where κ is the sum of the diagonal elements of the monodromy matrix of the system (14).

Let us recall that coefficients of the linear system (14) depend on parameters h and α. Hence, the coefficient κ is a function of the parameters h and α. Based on the general theory of stability of A.M. Lyapunov, linear system is stable if all roots of characteristic equation are simple and have modulus equal to one. If there is at least one root with modulus not equal to one, then linear system is unstable. The instability of a linear system implies instability in a complete nonlinear system with Hamiltonian (13). The latter means the orbital instability of the unperturbed periodic motion. If $|\kappa| > 1$, then the characteristic equation has a root with modulus greater than one; therefore, linear system (14) is unstable. If $|\kappa| < 1$, then the trivial equilibrium position of the linear system is stable, i.e., unperturbed periodic motion is orbitally stable in the linear approximation.

The question on orbital stability of pendulum rotations ($h > 1$) was solved on the basis of linear analysis. To this end, by integrating the system (14) from 0 to 2π the coefficient κ was calculated. The calculations were performed for all parameters values chosen from the intervals $1 < h < 100$, $0 < \alpha \leq \pi/2$ with tolerance 0.01. It appeared that in the above-mentioned range of parameter values the inequality $|\kappa| > 1$ holds, hence, the linear system is unstable. It yields orbital instability of pendulum rotations of the rigid body.

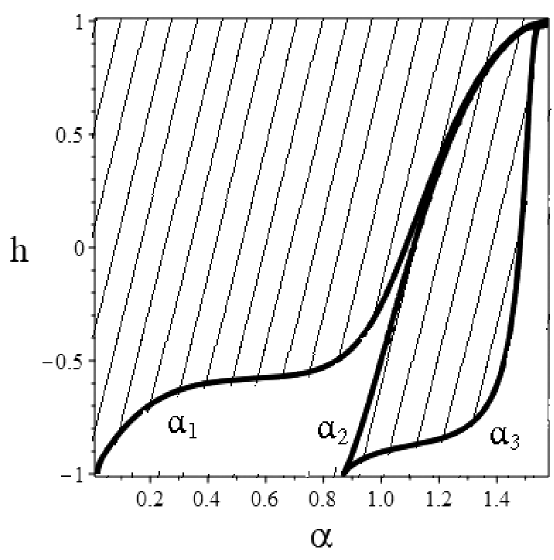

Figure 1. The diagram of orbital stability of pendulum oscillations

In the case of pendulum oscillations ($|h| < 1$) linear analysis showed that either inequality $|\kappa| > 1$ or the inequality $|\kappa| < 1$ holds, that is, depending on parameter values, the oscillations of a rigid body can be either orbital stable or orbital unstable. Based on numerical calculations in the plane of the parameters, a diagram of the orbital stability of pendulum oscillations was obtained (see figure 1). The regions of instability are shaded. In the unshaded areas, the pendulum oscillations are stable in the linear approximation. By α_i ($i = 1, 2, 3$) the boundaries of instability regions are denoted. At $\alpha = 0$ the equation (15) has multiple root $\rho = 1$. In [15] it was shown that in this case the so-called identical resonance takes place. The rigorous study of orbital stability in this case has performed in [6].

5 Investigation of stability for the case of small amplitude oscillations

In this section we suppose that amplitudes of oscillations are sufficiently small. In this case we can introduce a small parameter and perform rigorous analytical study by using small parameter method and the KAM theory. As a small parameter, it is convenient to choose the modulus of the elliptic integral k. At the first stage we study the linear system with the Hamiltonian K_2 whose series expansion in powers of k has the form

$$K_2 = \frac{1}{2}q_2^2 + \frac{1}{2}p_2^2(1 + 3\cos^2\alpha) + (-3\cos w \sin 2\alpha p_2 q_2 + 3p_2^2 \sin w \sin 2\alpha \cos\alpha)k + O(k^2). \quad (16)$$

For $k = 0$, the linear system is autonomous and describes harmonic oscillations with a frequency $\Omega_0 = \sqrt{1 + 3\cos^2\alpha}$. If $\Omega_0 \neq N/2$, $N \in Z$, then for sufficiently small k stability takes place in the linear approximation. If $\Omega_0 \neq N/2$, $N \in Z$, then at $k \ll 1$ the phenomenon of parametric resonance is possible, leading to instability. In our problem, parametric resonance takes place in two cases: $\Omega_0 \approx 2$ ($N = 4$) and $\Omega_0 \approx 3/2$ ($N = 3$). The regions of parametric resonance in the plane of parameters (α, h) emanate at $k = 0$ from the points $\alpha = 0$ and $\alpha = \arccos(5/\sqrt{12})$.

At $k \ll 1$ the boundaries α_i ($i = 1, 2, 3$) of parametric resonance regions can be obtained in a form of convergent power series in small parameter k

$$\alpha_i = \alpha_{i0} + k\alpha_{i1} + \ldots \quad (17)$$

In order to obtain the coefficients of the above series we use the technique developed in [16]. In accordance with this technique, we have to perform a linear canonical change of variables $(q_2, p_2) \to (X, Y)$, which brings the Hamiltonian K_2 into the simplest form (the so-called normal form). In new variables the normalized Hamiltonian K_2 does not depends on independent variable w and takes the following form

$$K_2 = k_{20}X^2 + k_{11}XY + k_{02}Y^2, \quad (18)$$

where k_{02}, k_{20}, k_{11} analytically depend on k and α. To construct the above-mentioned linear change of variables the Depri–Hori method [6] can be used. It also allows to calculate the expressions for coefficients k_{02}, k_{20}, k_{11} in a form of series in powers k.

Simple analysis of the canonical system with the Hamiltonian (16) shows that it is stable if $k_{11}^2 < 4k_{20}k_{02}$, otherwise if $k_{11}^2 > 4k_{20}k_{02}$ it is unstable. Thus, the boundaries of the parametric resonance region are determined by following equation

$$k_{11}^2 = 4k_{20}k_{02}. \quad (19)$$

Now we recall that k_{02}, k_{20}, k_{11} depend on k and α. By substituting (17) into (18) and equating the terms of at equal power of k in right- and left-hand sides of (18) we will get equations for coefficients of series (17).

Our calculations based on this approach have shown that in the case of resonance $\Omega_0 \approx 2$, the boundary of the parametric resonance region is determined by the following asymptotic formula

$$\alpha_1 = \frac{\sqrt{3}}{2} k_1^2 + O(k_1^3), \tag{20}$$

In the case of resonance $\Omega_0 \approx 3/2$ we have

$$\alpha_2 = \arccos \sqrt{\frac{5}{12} + \frac{27\sqrt{35}}{160} k_1^2 + \frac{315}{128} k_1^3 + O(k_1^4)}, \quad \alpha_3 = \arccos \sqrt{\frac{5}{12} + \frac{27\sqrt{35}}{160} k_1^2 - \frac{315}{128} k_1^3 + O(k_1^4)}, \tag{21}$$

Outside the regions of parametric resonance, the linear system is stable. However, it does not mean the stability of the original nonlinear system. Hence, to obtain rigorous conclusions about orbital stability for parameter values outside the regions of parametric resonance a nonlinear analysis is required.

We start nonlinear study of stability from the simplification of the quadratic part of the Hamilton function. Outside the regions of parametric resonance, we can construct a canonical change of variables

$$q_2 = a_{11}(w)x + a_{12}y, \quad p_2 = a_{21}(w)x + a_{22}(w)y, \tag{22}$$

to bring the Hamiltonian (16) into the following normal form, which does not depend on w

$$K_2 = \frac{1}{2}\Omega(x^2 + y^2), \tag{23}$$

The coefficients $a_{11}(w)$, $a_{12}(w)$, $a_{21}(w)$, $a_{22}(w)$ are 2π-periodic functions of the variable w and are convergent series in the small parameter k. Ω is an analytic function of k and α. As above we used the Depri–Hori method to obtain the power series expansion of Ω with respect to k. Our calculations have shown that $\Omega = \Omega_0 + \Omega_1 k^2 + O(k^3)$, where

$$\Omega_1 = \frac{(144\Omega_0^2 + 288)\cos^4\alpha + (-144\Omega_0^2 - 288)\cos^2\alpha - 16\Omega_0^6 + 48\Omega_0^4 + 69\Omega_0^2 - 20}{16\Omega_0^3 - 4\Omega_0}. \tag{24}$$

Now it is convenient to pass to the canonical polar coordinates by the formulas

$$x = \sqrt{2\rho}\sin\vartheta, \quad y = \sqrt{2\rho}\cos\vartheta, \tag{25}$$

The Hamiltonian K reads

$$K = \Omega\rho + G_4(\vartheta, w)\rho^2 + O(\rho^3), \tag{26}$$

The expression for $G_4(\vartheta, w)$ is omitted here due to the cumbersomeness. Rigorous conclusions on stability of the trivial solution of the system with Hamiltonian (26) can be obtained on basis of the KAM theory. For this purpose, let us perform further simplification of the Hamiltonian $K$and reduce it to a normal form up to terms of the order R^2.

If there are no third and fourth order resonances in the system, that is, $\Omega \neq n/3$, $n \in Z$ and $\Omega \neq n/4$, $n \in Z$, respectively, then the Hamilton function (26) can be reduced to a simpler form using a close to identity, analytic in k, canonical change of variables $(\vartheta, \rho) \to (\varphi, R)$

$$\Phi = \Omega R + c_2 R^2 + O(R^3), \quad c_2 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} G_4(\vartheta, w)\, dw\, d\vartheta. \tag{27}$$

Calculations have shown that

$$c_2 = -\frac{27\sin^2\alpha(2\cos^2\alpha + 1)(\cos^2\alpha + 1)}{8(4\cos^2\alpha + 1)(3\cos^2\alpha + 1)} + O(k_1). \tag{28}$$

The equation $c_2 = 0$ has no solutions in the interval $0 < \alpha \leq \pi/2$. Hence $c_2 \neq 0$, then by the Arnold–Moser theorem [17], the equilibrium point of system (12) is Lyapunov stable. It yields the orbital stability of pendulum oscillations with small amplitudes in non-resonant case.

To complete the study of orbital stability of the pendulum oscillations with small amplitudes we have to consider the cases of the third- and fourth-order resonances. Since the original Hamilton function does not contain third-order terms with respect to variables q_2, p_2, third-order resonances do not appear in the system. The fourth-order resonances are possible for $\Omega = n/4$, $n \in Z$. In our problem they occur only for two values of n: $n = 5$, $n = 7$. The resonance values of the parameters corresponding to the resonances $\Omega = 5/4$ and $\Omega = 7/4$ are respectively determined by such relations

$$\alpha = \arccos\frac{\sqrt{3}}{4} + \frac{3\sqrt{39}}{7}k_1^2 + O(k_1^3), \quad \alpha = \arccos\frac{\sqrt{11}}{4} + \frac{3\sqrt{55}}{55}k_1^2 + O(k_1^3). \tag{29}$$

In the case of fourth-order resonances the Hamiltonian (26) can be reduced to the form

$$\Phi = \Omega R + (c_2 + a_4\cos(4\vartheta - nw) + b_4\sin(4\vartheta - nw))\,R^2 + O(R^3), \tag{30}$$

where

$$a_4 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_4(\vartheta, w)\cos(4\vartheta - nw)\,dw\,d\vartheta,$$
$$b_4 = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_4(\vartheta, w)\sin(4\vartheta - nw)\,dw\,d\vartheta.$$

Our calculations have shown that in the case of $\Omega = 7/4$ the following coefficients are

$$c_2 = -\frac{4617}{3136}k_1 + O(k_1^3) \quad a_4 = O(k_1^3), \quad b_4 = O(k_1^3). \tag{31}$$

In the case of $\Omega = 5/4$, the calculated coefficients are

$$c_2 = -\frac{73359}{11200}k_1 + O(k_1^3), \quad a_4 = O(k_1^3), \quad b_4 = O(k_1^3). \tag{32}$$

In accordance with Markeev's theorem [17] the stability condition has a form

$$|c_2| > \sqrt{a_4^2 + b_4^2}. \tag{33}$$

It can be seen that in both cases, this condition is fulfilled, that is, there is Lyapunov stability for system (12), which implies the conclusion about the orbital stability of the original nonlinear system. Therefore, outside the regions of parametric resonance, the oscillations are orbitally stable.

6 Conclusion

In this paper, a complete study of the orbital stability of pendulum periodic motions of a heavy rigid body with a fixed point was carried out under the assumption when the main moments of inertia of the body satisfy the equality $A = C = 4B$ (Goryachev–Chaplygin condition). On the basis of linear analysis, it was proved that pendulum rotations are orbitally unstable. Pendulum oscillations can be either orbitally stable or orbitally unstable depending on parameters values. The result of orbital stability study in linear approximation have been presented in stability diagram for $0 < \alpha \leq \pi/2$. The domains of parametric resonances leading to instability were shown. Outside these domains the pendulum oscillations are orbitally stable in linear approximation. This result is in a good agreement with the results of [15] where the case $\alpha = 0$ was considered. At small amplitudes of oscillations, the study was carried out analytically and rigorous conclusions on orbital stability have been obtained. In particular, it was shown that outside the regions of parametric resonance, the oscillations are orbitally stable.

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