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Computational method for solving boundary value problems of mechanics deformable body using non-orthogonal functions

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Abstract. In this paper a complete system of non-orthogonal functions was built on the basis of orthogonal sines and cosines. It is shown that the known orthogonal systems of functions are a degenerate case of non-orthogonal systems of functions. It has been proven that the continuous function can be approximated non-orthogonal functions in such a way that one selected non-orthogonal function will not included in this amount. The boundary value problem of the elasticity theory has been considered for an inhomogeneous plate. A new method for solving the boundary value problem is developed for the fourth-order equation with variable coefficients. The proposed method is based on separating of the stress state of the plate, use of complete systems of non-orthogonal functions and a generalized quadratic form. Criteria have been established under which the constructed approximate solution coincides with the exact solution. This method has been adapted to solving a boundary value problem for high-order differential equations. The high accuracy of the method has been confirmed by numerical calculations.

1 Introduction

Currently, both continuous [1] and piecewise continuous [2, 3] systems of orthogonal functions are widely used. The use of modern computers allows us to develop new techniques for modeling functions and various processes in engineering. In [4], it is proposed to use non-orthogonal systems of functions to solve the boundary value problems of the elasticity theory. The development of computer techniques for non-orthogonal functions made it possible to propose a new analytical-numerical approach to solve the boundary value problems of the elasticity theory [5–7], as well as differential equations [8] using the least squares method [9, 10]. The computational method of solving boundary value problems of the mechanics of a deformable solids with variable elastic characteristics is considered in [11], where methods with use of non-orthogonal functions were proposed.

The non-orthogonal systems of functions naturally appear when constructing eigenfunctions of boundary value problems for differential equations [12]. They are also used to solve

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the boundary value problems of the elasticity theory [4]. However, up to date computational methods based on the non-orthogonal systems are not yet well developed for functions modelling, optimization of various processes, solving boundary value problems of mathematical physics, etc.

The purpose of research is to develop a new computer method to solve the boundary value problems involving both partial differential equations and ordinary differential equations and to devise analytical and computer tools for operating over the basis of the orthogonal sine-cosine families.

2 Approximation of continuous functions

Consider a complete non-orthogonal system of functions on the interval $[l_1, l_2]$. In this work, for its construction, we use the system of orthogonal functions (sinus and cosine), which will be considered on a smaller segment $[l_1, l_2]$ than the interval of their orthogonality $[A, B]$. An approximation of a continuous function $f(x)$ on the interval $[l_1, l_2]$ will be will be realized as a finite sum of the following form

$$f(x) \cong \sum_{k=0}^{M} c_k \varphi_k(x), \tag{1}$$

where c_k are unknown coefficients, $M = 2N$, N is a natural number, $\varphi_0(x) \equiv 1$, $\varphi_k(x) = \cos(k\omega x)$, $\varphi_{k+N}(x) = \sin(k\omega x)$ are the basic functions, $k = \overline{1, N}$; $\omega = 2\pi/(B - A)$, $[l_1, l_2] \subset [A, B]$, $l_2 - l_1 < 0.95(B - A)$. It is not difficult to verify that the selected system of functions $\{\varphi_k(x)\}$ will be complete and orthogonal in the interval $[A, B]$, and and complete but not orthogonal in the segment $x \in [l_1, l_2]$. Increasing the value N without bound in (1) leads to the conversion of the right hand side of equation (1) into a row.

The use of decomposition (1) by non-orthogonal functions allows us to satisfy the different values of the function at the edges of the segment $f(l_1) \neq f(l_2)$ without disrupting of the convergence conditions at points l_j, $j = \overline{1, 2}$. It requires a smaller quantity of number of the sum of the row (1) during approximation of functions with a given accuracy.

We formulate the important difference between non-orthogonal and orthogonal systems of functions.

Theorem 1. *A continuous function $f(x)$ is specified in the segment $[l_1, l_2]$ can be approximated by the sum of the row* (1) *in such a way that one arbitrarily selected function $\varphi_m(x)$ of the non-orthogonal basis* (1) *is not included in its representation.*

Proof. We have extended the function $f(x)$ over the segment $[A, B]$ and denoted it as $f_1(x)$. The function $f(x)$ can always be extended so that it remains continuous, and the integral over the segment $[A, B]$ of the function $f_1(x)\,\varphi_m(x)$ is equal to zero. According to [1], we construct the decomposition of a function $f_1(x)$ in the form of a series of orthogonal in segment $[A, B]$ of sinus and cosine, where equality have performed $c_m = 0$. Consider the function $f_1(x)$ on the segment $[l_1, l_2]$ and obtain its decomposition without a basic function $\varphi_m(x)$. The function $f_1(x)$ coincides with the function $f(x)$ on the segment $[l_1, l_2]$. So, we have received a decomposition of a function $f(x)$ in which there is no basic function $\varphi_m(x)$. End of the proof.

Corollary. *When satisfying the boundary conditions, depending on the physical nature of the problem being solved, we have an opportunity not to use one basic non-orthogonal function.*

The coefficients c_k are determined from the condition of the minimum functional, which characterizes the deviation between the approximation of the function (1) and its value [6, 7].

3 Solution of two-dimensional boundary value problems of the mechanics of the deformable solid body

Consider a two-dimensional boundary problem of the elasticity theory for a plate with variables Young's modulus and Poisson's ratio [13]. The median surface of the plate occupies a rectangular region $\Pi = \{(x, y) \in ([0, a] \times [-b, b])\}$, with a contour L on which the load is given. Stresses in the region Π are expressed through stresses function $F(x, y)$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \tau_{xy} = \tau = -\frac{\partial^2 F}{\partial x\, \partial y}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}. \tag{2}$$

The stresses function (2) satisfies the partial differential equation [13]

$$TF(x, y) \equiv \left\{ \Delta(\gamma\Delta) - q_y'' \frac{\partial^2}{\partial x^2} + 2q_{xy}'' \frac{\partial^2}{\partial y\, \partial x} - q_x'' \frac{\partial^2}{\partial y^2} \right\} F = 0, \tag{3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is two-dimensional Laplace operator; T is fourth-order two-dimensional operator defined by the relation (3), $\gamma(x, y) = \frac{1}{E(x, y)}$, $E(x, y)$ is the Young's modulus, $q = \frac{1}{2G} = \frac{1 + \nu}{E}$, $\nu(x, y)$ is the Poisson's ratio. We assume that the elastic constant of material E, ν are even with respect to the coordinate y and twice continuously differentiable functions in the region Π.

Let us consider in detail when $a \gg b$ and normal loads are given only on the transverse sides, so that the tangent loads in the angular points of the plate are zero. Consider even normal stresses, odd tangent stresses

$$\sigma_x(a_j, y) = \sigma_j(y), \quad \tau_{xy}(a_j, y) = \tau_j(y), \quad \tau_{xy}(a_j, \pm b) = 0, \quad j = \overline{1, 2}, \tag{4}$$

where $\sigma_j(y)$ are even normal, $\tau_j(y)$ are odd tangent loads, $a_1 = 0$, $a_2 = a$. For problem (4) we will find the basic stress state of plate Π [5]

$$\sigma_0 = \frac{1}{b} \int_0^b \sigma_1(y) = \frac{1}{b} \int_0^b \sigma_2(y),$$

so only one stress component $\sigma_x^0(x, y) = \sigma_0$ is not zero. Thus, we separate of the stress-strain state (SSS) of the plate by the sum of the basic SSS ($\sigma_0 = \sigma_x^0$) with one component of the forces, and the perturbed self-balanced state with zero basic vectors of forces and moments [5].

Given the condition $a \gg b$ the symmetric self-balanced problem will be divided into two tasks. Consider the first of these tasks in which only one side of the rectangle is loaded. We have wrote boundary conditions

$$\sigma_x(0, y) = \left.\frac{\partial^2 F}{\partial y^2}\right|_x = 0 = \sigma^1(y), \quad \tau_{xy}(0, y) = -\left.\frac{\partial^2 F}{\partial x\, \partial y}\right|_x = 0 = \tau_1(y), \tag{5}$$

$$\sigma_x(a, y) = 0, \quad \tau_{xy}(a, y) = 0, \quad y \in [0, b], \tag{6}$$

$$\sigma_y(x, b) = 0 \quad \tau_{yx}(x, b) = 0, \quad x \in [0, a], \tag{7}$$

where $\sigma^1(y) = \sigma_1(y) - \sigma_0$ is self-balanced load. On two longitudinal sides of the plate, zero boundary conditions (7) are presented. The Saint-Venant's principle implies that the self-balanced stress state of the plate will be rapidly decreased by removing from the transverse side $x = 0$. Therefore, the function of stresses that satisfies equations (3), (5)–(7) we can search with the symmetry of the stress state in such form:

$$F = E_0 \sum_{k=1}^{N} \left[(a_k + a_{k+N} y^2) \varphi_k + (d_k y + d_{k+N} y^3) \varphi_{k+N} \right] e^{-k\omega x}, \tag{8}$$

where a_k, d_k are unknown coefficients; N is natural number; $\varphi_k(x) = \cos(k\omega y)$, $\varphi_{k+N}(x) =$ $= \sin(k\omega y)$, $k = \overline{1, N}$, $\omega = 0.9\pi/b$.

We substitute the function of stresses (8) in relations (2) and we obtain an obvious type of components stresses

$$\begin{gathered}
\sigma_x = E_0 \sum_{k=1}^{N} \Big\{ -k^2\omega^2 a_k \varphi_k + a_{k+N} \left[(2 - k^2\omega^2 y^2) \varphi_k - 4k\omega y \varphi_{k+N} \right] + \\
+ (2k\omega\varphi_k - k^2\omega^2 y \varphi_{k+N}) d_k + d_{k+N} \left[6y\varphi_{k+N} + 6y^2 k\omega \cos(k\omega y) - k^2\omega^2 \varphi_{k+N} \right] \Big\} e^{-k\omega x}, \\
\sigma_y = E_0 \sum_{k=0}^{N} k^2\omega^2 \left[(a_k + a_{k+N} y^2) \varphi_k + (d_k y + d_{k+N} y^3) \varphi_{k+N} \right] e^{-k\omega x}, \\
\tau = \sum_{k=0}^{N} k\omega \Big\{ -k\omega a_k \varphi_{k+N} + \left[2y\varphi_k - k\omega y^2 \varphi_{k+N} \right] a_{k+N} + d_k (k\omega y \varphi_k + \varphi_{k+N}) + \\
+ d_{k+N} \left[3y^2 \varphi_{k+N} + k\omega y^3 \varphi_k \right] \Big\} e^{-k\omega x}.
\end{gathered} \tag{9}$$

The expression of the stress function (8) has been chosen so that it had a part that coincides with the expression of the stress function for a homogeneous isotropic material [5].

We introduce the norm [14] in the space of two-dimensional of two-dimensional functions $F(x, y)$specified in the rectangular region Π

$$\|F(x, y)\| = \sqrt{\int_0^a \int_0^b [F(x, y)]^2 \, dx \, dy}. \tag{10}$$

It is known [14] that if the function $F(x, y)$ is a continuous function and $\|F(x, y)\| = 0$, then $F(x, y) = 0$.

Note that unknown coefficients a_k, d_k of stress function (8) must satisfy the equation (3) in the region Π, which will be present in the following form:

$$|TF(x, y)| = \left| \sum_{k=1}^{N} \sum_{n=0}^{1} \{ T_{k+nN}(x, y) \, a_{k+nN} + S_{k+nN}(x, y) \, d_{k+nN} \} \right| \to 0, \tag{11}$$

where

$$\begin{gathered}
T_k(x, y) = 2\chi_{k,0}^{0,1} - \chi_{k,0}^{0}, \quad T_{k+N}(x, y) = 2\psi_{k,1}^{0} - 4k\omega\psi_{k,2}^{1} + 2\chi_{k,0}^{2,1} - \chi_{k,0}^{2}, \\
S_k = 2k\omega\psi_{k,1}^{0} + 2\chi_{k,1}^{1,1} - \chi_{k,1}^{1}, \quad S_{k+N} = 6k\omega\psi_{k,1}^{2} + 6\psi_{k,2}^{1} + 2\chi_{k,1}^{3,1} - \chi_{k,1}^{3}, \\
\psi_{k,1}^{m}(x, y) = (\Delta\gamma) y^m \cos(k\omega y) e^{-k\omega x}, \quad \psi_{k,2}^{m}(x, y) = (\Delta\gamma) \, y^m \sin(k\omega y) \, e^{-k\omega x}, \\
\chi_{k,n}^{m} = \left(\frac{\partial^2 g}{\partial y^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} \frac{\partial^2}{\partial y^2} \right) y^m \varphi_{k+nN} \, e^{-k\omega x}, \\
\chi_{k,n}^{m,1} = \frac{\partial^2 g}{\partial x \, \partial y} \frac{\partial^2}{\partial x \, \partial y} y^m \varphi_{k+nN} e^{-k\omega x}, \quad n = \overline{0, 1}, \quad k = \overline{1, N}.
\end{gathered} \tag{12}$$

Equation (11) defines the continuous function in the region Π. We minimize the deviations of the function (11) from zero. To do this, we substitute expressions (12) and after using the norm (10) in the region Π we get a non-negative quadratic form

$$||TF(x,y)||^2 = \int_0^a \int_0^b \left[\sum_{k=1}^{M} c_k T_k(x,y) \right]^2 dx\,dy = \sum_{k,j=1}^{M} c_k c_j B_{kj} \to 0, \tag{13}$$

where $M = 4N$, $T_{k+2N} = S_k$; $c_k = a_k$, $c_{k+2N} = d_k$, $k = \overline{0,2N}$ are variables and $B_{kj} = \int_0^a \int_0^b T_k T_j\,dx\,dy$, $k = \overline{1,M}$ are the coefficients of the quadratic form. The given method allows us to reduce of numerical solution of the equation (3) to an estimation of the minimum of the quadratic form (13).

By means of selecting the exponentially reduced presentation of the stress state, the conditions (6) will be approximately satisfied. It remains to satisfy the boundary conditions (5), (7). After substitution the stresses (9) in them they are reduced to a compact form

$$\sum_{k=1}^{M} c_k A_{m,k}(\gamma_m) = P_m(\gamma_m), \quad \gamma_m \in [0, \alpha_m], \quad m = \overline{1,K}, \tag{14}$$

where

$$K = 4, \quad P_1 = \sigma^1(y), \quad P_2 = \tau_1(y), \quad P_j = 0, \quad j = \overline{3,4}, \quad \alpha_1 = \alpha_2 = b, \quad \alpha_3 = \alpha_4 = a,$$
$$A_{1,k} = (2k\omega\cos(k\omega y) - k^2\omega^2 y\sin(k\omega y)), \quad A_{2,k} = -k^2\omega^2\sin(k\omega y),$$
$$A_{3,k} = k^2\omega^2\cos(k\omega b)\,e^{-k\omega x}, \quad A_{4,k} = -k^2\omega^2\sin(k\omega b)\,e^{-k\omega x}, \quad k = \overline{1,N};$$
$$A_{1,k} = [(2 - k^2\omega^2 y^2)\cos(k\omega y) - 4k\omega y\sin(k\omega y)], \quad A_{2,k} = k\omega[2y\cos(k\omega y) - k\omega y^2\sin(k\omega y)],$$
$$A_{3,k} = b^2k^2\omega^2\cos(k\omega b)\,e^{-k\omega x}, \quad A_{4,k} = k\omega[2b\cos(k\omega b) - k\omega b^2\sin(k\omega b)]\,e^{-k\omega x},$$
$$k = \overline{N+1,2N}; \quad A_{2,k} = -k^2\omega^2\sin(k\omega y), \quad A_{1,k} = (2k\omega\cos(k\omega y) - k^2\omega^2 y\sin(k\omega y)),$$
$$A_{3,k} = k^2\omega^2 b\sin(k\omega b)\,e^{-k\omega x}, \quad A_{4,k} = -k^2\omega^2\sin(k\omega b)\,e^{-k\omega x}, \quad k = \overline{2N+1,3N};$$
$$A_{1,k} = [6y\sin(k\omega y) + 6y^2k\omega\cos(k\omega y) - k^2\omega^2\sin(k\omega y)],$$
$$A_{2,k} = k\omega[3y^2\sin(k\omega y) + k\omega y^3\cos(k\omega y)], \quad A_{3,k} = k^2\omega^2 b^3\sin(k\omega b)\,e^{-k\omega x},$$
$$A_{4,k} = k\omega[3b^2\sin(k\omega b) + k\omega b^3\cos(k\omega b)]\,e^{-k\omega x}, \quad k = \overline{3N+1,4N}.$$

We denote the left-hand sides of equation (14) by

$$f_{m,N}(\gamma_m) = \sum_{k=1}^{M} c_k A_{m,k}(\gamma_m), \quad m = \overline{1,K}. \tag{15}$$

To solve equations (14), we had used the analytical-numerical method developed in [5, 6]. The method is based on the convergence of all expressions in the left parts of equations (14), (15) to the loads $P_m(\gamma_m)$. We have written these expressions

$$\left| \sum_{k=1}^{M} c_k A_{m,k}(\gamma_m) - P_m(\gamma_m) \right| \to 0, \quad \gamma_m \in [0, \alpha_m], \quad m = \overline{1,K}. \tag{16}$$

An effective analytic and numerical methodology has been developed, which allowed us to simultaneously minimize all K expressions (16) in the norms $L_2[0, \alpha_m]$ and quadratic form (13) [11]. The given method allows us to have been reduced numerical solution of equations (13), (16) to the definition of a minimum of the generalized quadratic form

$$||TF(x,y)||^2 + \sum_{m=1}^{4} ||f_{m,N}(\gamma) - P_m(\gamma)||_m^2 = \sum_{k,j=1}^{M} c_k c_j W_{kj} - 2\sum_{k=1}^{M} c_k V_k + P^2, \qquad (17)$$

where $||f(\gamma)||_m = \sqrt{\int_0^{\alpha_m} f^2(\gamma)\,d\gamma}$ are norms $L_2[0,\alpha_m]$, $P^2 = \sum_{m=1}^{2} ||P_m||_m^2$, $m = \overline{1,K}$,

$$W_{k,j} = B_{k,j} + \int_0^{\alpha_m} \sum_{m=1}^{4} A_{m,k}(\gamma_m) A_{m,j}(\gamma_m)\,d\gamma_m, \quad V_k = \int_0^{\alpha_m} \sum_{m=1}^{2} A_{1,k}(\gamma_m) P_m(\gamma_m)\,d\gamma_m,$$
$$W_{k,j} = W_{j,k}, \quad k,j = \overline{1,M}.$$

The minimum of a generalized quadratic form (17) for the specified N is denoted $\Lambda(N)$, and unknown variables c_k on which it is achieved, we denote c_k^N. Variables c_k^N are defined from the condition of its minimum. The function of the stresses is defined by the found coefficients c_k^N is denoted $F_N(x,y)$.

4 Solutions of the differential equations with continuous coefficients

Consider the boundary value problem for the equation 2K-order:

$$T_K y \equiv \sum_{k=0}^{2K} f_k(x) y^{(k)}(x) = 0; \quad y^{(k)}(l_j) = u_k^j, \quad k = \overline{0,K-1}, \quad j = \overline{1,2}, \quad x \in [l_1,l_2], \qquad (18)$$

here T_K is differential operator. Let us represent an approximate solution of problem (18) as the amount of finite series (1)

$$y_N(x) = \sum_{n=1}^{K} \sum_{k=1}^{2N} c_{k+2N(n-1)} x^{n-1} \varphi_k(x). \qquad (19)$$

The coefficients c_k are determined after the solution substitution (19) in conditions (18). In the one-dimensional case the relation (17) is simplified to the following generalized quadratic form:

$$\int_{l_1}^{l_2} [T_1 y_N(x)]^2 dx + \sum_{k=0}^{K-1} \sum_{j=1}^{2} (y^{(k)}(l_j) - u_k^j)^2 = \sum_{k,j=1}^{M} c_k c_j W_{kj} - 2\sum_{k=1}^{M} c_k V_k + P^2, \qquad (20)$$

where $M = 2NK$.

Two-point edge task has been solved for the method testing

$$T_1 y \equiv xy'' + y' + xy = 0, \quad x \in [l_1,l_2], \quad y(l_1) = y_1, \quad y(l_2) = y_2, \qquad (21)$$

where $0 < l_1 < l_2$, T_1 is the operator for Bessel's equation of order zero. The coefficients of form (20) for equation (21) are found in the analytical form [8] and calculated by the computer. A numerical experiment was carried out: $l_1 = 1$, $l_2 = 12$, $y_1 = 3$, $y_2 = 5$.

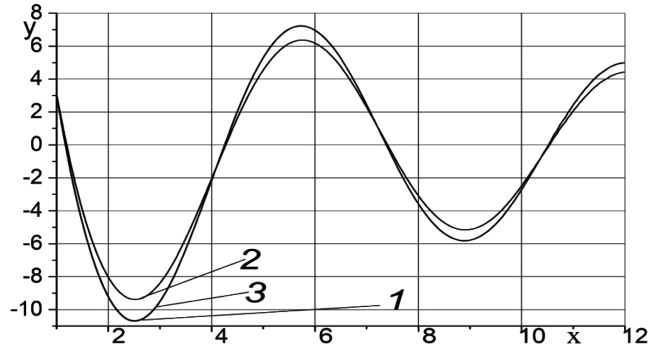

Figure 1. Graphs of solutions: 1 is accurate solution; 2, 3 are approximate solutions: $N = 10$, $N = 16$

The simulation results are shown in figure 1. The approximate solution of the equation (21) at $N = 10$ is not essentially different from the exact solution, and at $N = 16$ coincides with the exact solution with the error of less than 10^{-3}.

5 Convergence criteria for the constructed solution

Lemma [7]**.** *The function* $\Lambda(N)$ *is nonnegative and not increasing.*

If we assume that $\Lambda(N) = 0$, then all separate terms that are included in the left part of the expression (17) will be zero. Consequently, all equations (13), (16) will be satisfied and the initial task is solved.

We assume that the proposed computational process is stable [6]. The following theorem is faithful.

Theorem 2. *If there exists N such that* $\Lambda(N) < \varepsilon^2/4$ *for any* $\varepsilon > 0$*, then the stresses written through function* $F(x, y) = \lim_{N\to\infty} F_N(x, y)$ *exactly satisfy the conditions* (5)*,* (7)*, as well as equation* (3)*.*

Proof. Consider the sequence of the small positive numbers ε_N convergent to zero. This sequence corresponds to the natural number sequence $N \to \infty$. Now due to Lemma and theorem 2 assumption there exist N and $\varepsilon_N \le \varepsilon$ for any $\varepsilon > 0$ so that the relation

$$\Lambda(N + k) \le \Lambda(N) < \frac{\varepsilon_N^2}{4}$$

are satisfied for any natural number k. Hence if we tend, then the limit $\Lambda(N)$ turns into

$$\lim_{N\to\infty} \Lambda(N) = 0.$$

Let us show that decrease of ε_N and relative increase of N lead to the following: The stresses constructed with the coefficients c_k^N and functions (9) meet conditions (14) with the given error ε in the metrics of the spaces $L_2[0, \alpha_m]$. Indeed, condition (17) and the triangle inequality ([14, p. 378]) imply for functions $f_{m,N}(\gamma_m)$ since

$$\|f_{m,j} - f_{m,k}\|_m \le \|f_{m,j} - P_m\|_m + \|f_{m,k} - P_m\|_m \le 2\sqrt{\Lambda(N)} < \varepsilon_N \le \varepsilon, \quad m = \overline{1, K}$$

for arbitrary $k, j \ge N$, the function sequences $f_{m,N}(\gamma_m)$ are Cauchy sequences in the metrics of $L_2[0, \alpha_m]$. So, there are limits of functions sequences $f_{m,N}(\gamma_m)$ that we denote by

$$f_m(\gamma_m) = \lim_{N\to\infty} f_{m,N}(\gamma_m), \quad \gamma_m \in [0, \alpha_m], \quad m = \overline{1, K}. \tag{22}$$

According to (17), we write down an assessment of satisfaction of conditions (13) and boundary conditions (14) in the metric $L_2[0, \alpha_m]$

$$\|TF_N(x, y)\| \leq \sqrt{\Lambda(N)} < \varepsilon_N/2,$$
$$\|f_{m,N}(\gamma_m) - P_m\|_m \leq \sqrt{\Lambda(N)} < \varepsilon_N/2, \quad m = \overline{1, K}. \tag{23}$$

We tend to the limit $\varepsilon_N \to 0$ in inequalities (23) and we obtain that stress function $F(x, y)$ satisfies the conditions (3), and the functions $f_m(\gamma_m)$ accurately satisfy the boundary conditions (14) in the metric of spaces L_2

$$\|TF(x, y)\| = 0, \quad \|f_m(\gamma_m) - P_m\|_m = 0, \quad m = \overline{1, K}. \tag{24}$$

Since all functions in conditions (24) are continuous, then according to [14], we will have

$$TF(x, y) = 0, \quad f_m(\gamma_m) = P_m(\gamma_m), \quad m = \overline{1, K}. \tag{25}$$

End of the proof of Theorem 2.

6 Conclusion

It has been proven that the boundary conditions can be satisfied without using one basic non-orthogonal function. In numerical solutions, the stress state of the plate has been separated by the main state and the perturbed state, which decreases with distance from the loaded edges. An effective way to solve boundary value problems in a rectangular plate with variable elastic characteristics is proposed. The proposed approach replaces the satisfaction of the boundary conditions and the solution of the partial differential equation, by the search a minimum of a generalized quadratic form. The generalized quadratic form is introduced, which is not linear, but its minimum tends to zero and determines the convergence and accuracy of the solution. It has been proven that the accuracy of the approximate solution of the partial differential equations with variable coefficients is estimated by one number, namely, a minimum of a generalized quadratic form. It has been established that a slight amount of non-orthogonal functions very well approximates the exact solution of the differential equation. Calculations confirmed the high accuracy of the proposed method.

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