Numerical simulation of a heterogeneous liquid in a 2-D rectangular container

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Abstract. We study the sloshing of a heterogeneous liquid in a 2-D rectangular container in the presence of a horizontal dynamic excitation. The governing equations of the motion of the liquid are presented in the case of a liquid of low heterogeneity. The study of these equations shows the existence of a small zone of instability. A two-dimensional numerical method is carried out to simulate the mathematical model. The effect of liquid heterogeneity on the changing aspects of the free surface was studied. As shown in the results, the free surface profiles increase by increasing the heterogeneity parameter. The analysis of sloshing in the stable zone shows significant effects on the evolution of the free surface of the liquid, these effects depend on the coefficient of heterogeneity and frequency of excitation.

1 Introduction

Liquid sloshing is a wave motion that occurs in a partially filled container. This particular liquid motion was found to be influenced by the excitation forces, the shape of the container and the filling level [1]. The sloshing phenomenon is of great importance in maritime transport regarding the safety of sea transport of oil and liquefied natural gas [2]. Sloshing of liquids in partially filled containers, under the action of all possible external forces, constitutes a key topic of research activity because several problems might occur during transportation [3].

The phenomenon has been studied from several aspects, including the effect of the shape of the tank on sloshing, as well as the installation of baffles inside the tank. The sloshing of a homogeneous-Newtonian liquid in a partially filled container was the subject of numerous studies. But the case of a heterogeneous liquid which density varies linearly versus the depth has attracted yet little investigation. This topic has received some degree of interest in Essaouini et al. [4-5] and Capodanno et al. [6].

The motivation of the present work is to study the sloshing of an almost-homogeneous liquid from both an analytical and numerical perspective. The numerical aspect developed here has not been previously considered by researchers, as most of the performed studies were limited to purely analytical work. This paper is organized as follows. After the introduction given in the first section, the equations that govern the sloshing of almost heterogeneous liquid are recalled in the second section. The third section presents a numerical study, then results and discussions are given in section 4. Finally, the main conclusions are drawn.

2 Problem statement and the basic equations

2.1 Problem statement

Let us choose a stationary Cartesian coordinate system in the two-dimensional space $\mathbb{R}^2$, with the origin at the point O' and the axial vectors $\vec{x}'_k$ $(k=1,2)$. In addition to this absolute Cartesian coordinate system, a non-stationary (relative) coordinate system (Ox_1x_2) is introduced. Its origin is taken at a point O and its axes vectors $\vec{x}_k=\vec{x}'_k$ $(k=1,2)$ are attached to the rectangular container as shown in Fig. 1.

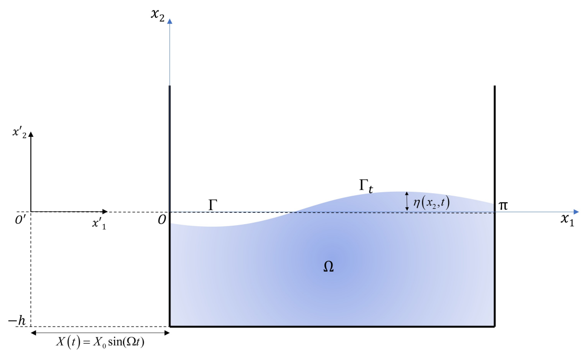

Fig. 1. Geometry of the 2D partially filled tank.

The rectangular container is supposed to be partially filled with a heavy ideal-heterogeneous liquid and the surface tension forces are ignored. Under these conditions, the free surface Γ of the heterogeneous

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liquid is horizontal and plane in the stable state, i.e., it is orthogonal to the acceleration of the gravitational field $\vec{g} = -g\vec{x}_2$. The pressure in the gas above the liquid is considered to vanish and the gas motion is ignored.

The geometry of the container as well as the definition of the domain and boundaries are illustrated in Fig. 1. The container is assumed to be subjected to a horizontal excitation having the following form:

$$X(t) = X_0 \sin(\omega t) \quad (1)$$

where X_0 and ω denote the amplitude and frequency.

2.2 Basic equations

Let's denote by $\vec{u}(x,t)$ the "relative" displacement of the particle at the equilibrium state that occupies the position (x_1, x_2) at the instant t, and $P(x,t)$ the pressure at that point.

The liquid inside the container is assumed to be almost homogeneous with a density in the equilibrium state that takes the form:

$$\rho_0(x_2) = \rho\,(1-\beta x_2) \quad (2)$$

where ρ and β are positive constants, and β being sufficiently small.
Taking the atmospheric pressure $p_a = 0$, the equilibrium pressure that settles in the liquid is: $P_0(x_2) = -\rho g x_2$. The dynamic pressure $p(x,t)$ in the fluid verifies:

$$P(x,t) = P_0(x_2) + p(x,t) = -\rho g x_2 + p \quad (3)$$

The linearized Euler equation takes then the form:

$$\rho\left(\ddot{\vec{u}} + \ddot{X}\vec{x}_1\right) = -\overrightarrow{grad}\,p - \rho\beta g u_2 \vec{x}_2 \ \text{ in } \Omega \quad (4)$$

One should add the following fluid incompressibility condition:

$$div(\vec{u}) = 0 \ \text{ in } \Omega \quad (5)$$

and kinematics conditions:

$$\begin{cases} u_1 = 0 \ \text{ for } \ x_1 = 0;\ x_1 = \pi \\ u_2 = 0 \ \text{ for } \ x_2 = -h \end{cases} \quad (6)$$

From $P_{|\Gamma_t} = 0$, the dynamic condition on the free surface Γ_t, position of Γ at time t, can be written as:

$$P_{|\Gamma} = \rho g u_{2|\Gamma} \quad (7)$$

2.3 Analytical solutions to the problem

Looking for a harmonic solution of Eq. (4), $\vec{u} = \vec{U}(x_1,x_2)\sin(\omega t)$, $p = P(x_1,x_2)\sin(\omega t)$, yields to:

$$\begin{cases} \dfrac{\partial p}{\partial x_1} = \rho\omega^2 (U_1 + X_0) \\ \dfrac{\partial p}{\partial x_2} = \rho(\omega^2 - \beta g)\mathrm{U}_2 \end{cases} \quad (8)$$

Considering now Eq.(5), i.e., $\dfrac{\partial(U_1 + X_0)}{\partial x_1} + \dfrac{\partial U_2}{\partial x_2} = 0$, one obtains from Eq. (8), when $\omega^2 \neq \beta g$:

$$\frac{1}{\omega^2}\frac{\partial^2 P}{\partial x_1^2} + \frac{1}{\omega^2 - \beta g}\frac{\partial^2 P}{\partial x_2^2} = 0 \quad (9)$$

By letting $Q = P - \rho\omega^2 X_0 x_1$, one can verify that this quantity verifies the same equation than Eq. (9):

$$\frac{1}{\omega^2}\frac{\partial^2 Q}{\partial x_1^2} + \frac{1}{\omega^2 - \beta g}\frac{\partial^2 Q}{\partial x_2^2} = 0 \quad (10)$$

Using Eq. (3), Eq. (4) and the kinematics conditions given in Eq. (6), one obtains after some basic calculus:

$$\begin{cases} \dfrac{\partial Q}{\partial x_1} = 0 \text{ for } x_1 = 0,\ x_1 = \pi \\ \dfrac{\partial Q}{\partial x_2} = 0 \text{ for } x_2 = -h \end{cases} \quad (11)$$

Ignoring temporarily Eq. (8), solution of Eq. (10), subjected to the boundary condition Eq. (11) can be obtained by the method of separation of variables as:

$$Q(x_1,x_2) = Y_1(x_1)\cdot Y_2(x_2) \quad (12)$$

Substituting Eq. (12) into Eq. (10), and considering Eq. (11), one obtains:

$$Y_1(x_1) = \cos n x_1,\ (n = 1,2...) \quad (13)$$

$$Y_2'' + n^2\left(\frac{\beta g}{\omega^2} - 1\right)Y_2 = 0,\ (n = 1,2...) \quad (14)$$

Let's introduce the following quantity

$$\alpha = \sqrt{\frac{\beta g}{\omega^2} - 1} \quad (15)$$

with α a real number if $\omega^2 < \beta g$, and pure imaginary if $\omega^2 > \beta g$. Then, the solution of Eq. (14) takes the form: $Y_2(x_2) = Ae^{in\alpha x_2} + Be^{-in\alpha x_2}$, with $Y_2'(-h) = 0$. This last condition yields $A = Be^{2in\alpha h}$ and

$$Y_2(x_2) = Be^{i\alpha h}\left\{e^{in\alpha(x_2+h)} + e^{-in\alpha(x_2+h)}\right\} \quad (16)$$

Now, to take into account Eq. (8), the solution in searched as a series of the following form:

$$Q(x_1,x_2) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n x_1)\left(e^{in\alpha(x_2+h)} + e^{-in\alpha(x_2+h)}\right) \quad (17)$$

where the coefficients A_n are still to be determined. The dynamic condition, Eq. (8), on the free surface Γ_Γ writes: $Q(x_1,0)+\rho\omega^2 X_0 x_1 = \dfrac{g}{\omega^2-\beta g}\dfrac{\partial^2 Q(x_1,0)}{\partial x_2^2}$ or

$$\sum_{n=1}^{\infty} A_n \cos(nx_1)\Big\{\Big(e^{in\alpha(x_2+h)}+e^{-in\alpha(x_2+h)}\Big) - \frac{in\alpha g}{\omega^2}\Big(e^{in\alpha h}-e^{-in\alpha h}\Big)\Big\} = -A_0-\rho\omega^2 X_0 x_1 \tag{18}$$

Substituting the development in Fourier series of the function x_1, $x_1=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)x_1}{(2n+1)^2}$, into Eq. (18), one obtains the explicit expressions of the coefficients A_n. Replacing them into Eq. (17) yields:

$$Q(x_1,x_2) = -\frac{\pi}{2}\rho\omega^2 X_0 + \frac{4}{\pi}\rho\omega^2 X_0\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\times \frac{\cos[(2n+1)x_1]\cdot\cos[(2n+1)h(x_2+h)]}{\cos[(2n+1)\alpha h]+\dfrac{\alpha g(2n+1)}{\omega^2-\beta g}\sin[(2n+1)\alpha h]} \tag{19}$$

From the definition of Q and Eq. (4), one obtains

$$\begin{cases} \rho\omega^2 U_1 = \dfrac{\partial Q}{\partial x_1} \\ \rho\left(\omega^2-\beta g\right)U_2 = \dfrac{\partial Q}{\partial x_2} \end{cases} \tag{20}$$

So, the amplitude of the dynamic pressure can be expressed as:

$$p(x_1,x_2) = \rho\omega^2 X_0\left(x_1-\frac{\pi}{2}\right)+\frac{4}{\pi}\rho\omega^2 X_0\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\times \frac{\cos[(2n+1)x_1]\cdot\cos\left[(2n+1)h\left(x_2+h\right)\right]}{\cos[(2n+1)\alpha h]+\dfrac{\alpha g(2n+1)}{\omega^2-\beta g}\sin[(2n+1)\alpha h]} \tag{21}$$

Letting $\eta = u_{2|\Gamma}$, Eq. (8) leads to the amplitude of free surface elevation as:

$$\eta(x_1) = \frac{\omega^2}{g}X_0\left(x_1-\frac{\pi}{2}\right)+\frac{4\omega^2 X_0}{\pi g}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\times \frac{\cos[(2n+1)x_1]\cdot\cos\left[(2n+1)h\left(x_2+h\right)\right]}{\cos[(2n+1)\alpha h]+\dfrac{\alpha g(2n+1)}{\omega^2-\beta g}\sin[(2n+1)\alpha h]} \tag{22}$$

3 Numerical study

Numerical simulation is a good and effective way to study general motion of fluids. In this study, use was made of COMSOL Multiphysics software to conduct simulations of a heterogeneous liquid contained in a two-dimensional partially filled rectangular tank, with length L and still liquid depth h. A Cartesian coordinate system (Ox_1x_2) is attached to the tank as indicated in Fig. 1.

Irrotational and incompressible heterogeneous fluid motion is considered under the action of a horizontal harmonic excitation. In simulations, the parameter β monitors the heterogeneity level of the liquid such that: $\rho_0(x_2)=\rho(1-\beta x_2)$ where $\rho=1000\ kg/m^3$. On the other hand, the parameter ω fixes the frequency of horizontal excitation of the liquid: $\ddot{X}(t)=-X_0\omega^2\sin(\omega t)$. The extent of the effect of these two variables on the free surface of the heterogeneous liquid was analyzed parametrically.

3.1 Model validation

The numerical model is validated by comparing simulation results with those of a 2D numerical model considered by Lin et al. [7]. The selected tank dimensions were length $L=0.9m$ and liquid's depth $h=0.6m$. The tank was horizontally excited with a sinusoidal acceleration for which the displacement amplitude is $X_0=0.002\ m$ and $\omega=5.5\ rad/s$.

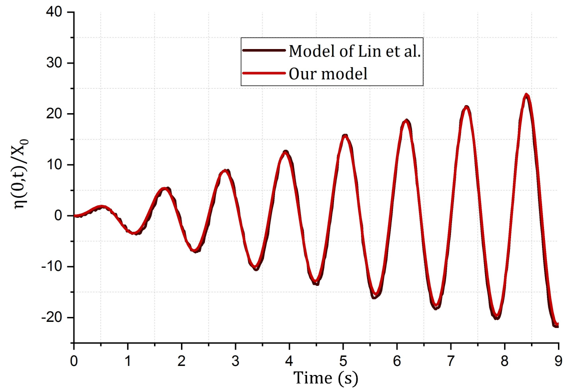

Fig. 2. Comparison of the free surface profiles as obtained by the numerical model developed under COMSOL and that in [7]

Fig. 2 presents a comparative analysis of the free surface elevation of the liquid as obtained by the present numerical model that that of Lin et al. [7]. The results show the same tendency, with almost no errors being observed. This enables to assess accuracy of the numerical modeling performed under COMSOL.

4 Results and discussions

In this section, the sloshing wave profiles are simulated for a 2D rectangular tank excited horizontally by a harmonic acceleration according to Eq. (1). The chosen dimensions of the container, see Fg. 1, are: length $L=\pi$ and liquid depth $h=1\ m$. The numerical analysis was conducted using COMSOL Multiphysics to analyze the influence of varying liquid density on sloshing. The varying parameters are the frequency of excitation ω and the heterogeneity parameter β. The results obtained for a heterogeneous liquid are compared to those of a homogeneous liquid for which $\beta = 0$.

4.1 Evolution profiles of the free surface

The following section presents the results of simulations that were performed in order to anlayze the influence of liquid density on the characteristics of the free surface profiles for an almost homogeneous liquid. The fluid density varies according to the x_2-axis of the tank by means of parameters β taken in the interval $[0, 0.5]$. The frequency was fixed at $\omega = 2.5\ rad/s$.

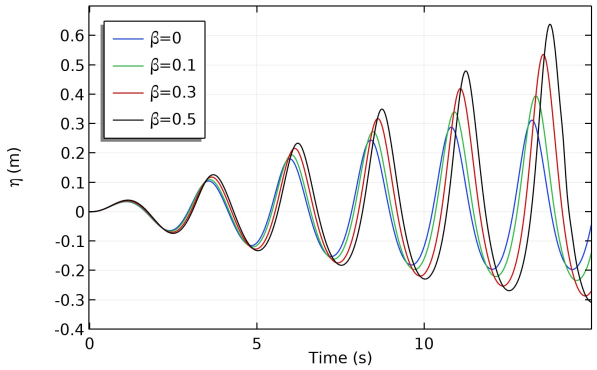

Fig. 3. Free surface elevation at the left side of the container as function of parameter β, for $\omega = 2.5\ rad/s$.

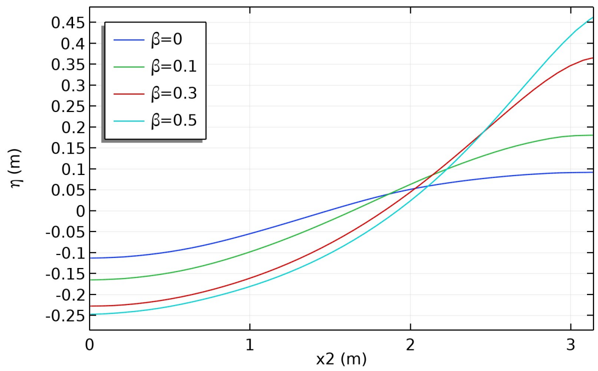

Fig. 4. Wave profiles for different values of β parameter at $t = 11.24s$, with $\omega = 2.5\ rad/s$.

Fig. 3 and Fig. 4 show that the elevation of the free surface of the heterogeneous liquid in a rectangular tank, subjected to horizontal excitation, increases more for large values of the heterogeneity coefficient β. The elevation tends towards a large limit for all the tested values of β, with $\omega^2 < \beta g$. This observed behavior is similar to a resonance phenomenon.

4.2 Effect of the frequency on sloshing

Next, the effect of the frequency on the free surface profiles of the almost homogeneous liquid by changing the external excitations is studied by varying the parameter ω. The density parameter was fixed at $\beta = 0.1$. Fig. 5 gives the obtained results.

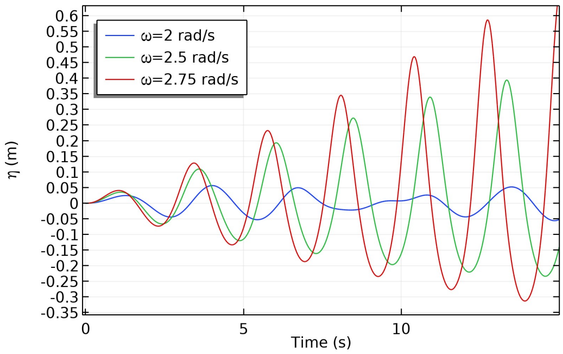

Fig. 5. Comparison of wave profiles for different values of the frequency, with $\beta = 0.1$.

Fig. 5 shows that the elevation of the free surface increases with the increase of the frequency of the excitation. The observed liquid behavior is very affected by the excitation as the frequency increases. Here again, resonance can be observed for all the tested values verifying $\omega^2 > \beta g$.

5 Conclusions

In this work, the sloshing of an ideal heterogeneous liquid in a partially filled rectangular tank was studied when the system is subjected to a horizontal harmonic base acceleration. Taking into account the heterogeneity of the liquid, as represented by means of the density parameter, was found to give way to new physical effects that are different than those observed for a homogeneous liquid. Mainly, the amplitude of motion of the free surface of the liquid increases with heterogeneity and frequency as long as $\omega^2 < \beta g$ or $\omega^2 > \beta g$. This instability occurs for all the values of frequency of excitation and density parameter that were tested numerically in the present study.

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