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Algebraic wavenumber identification method in presence of uncertainty

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Abstract. This paper presents an algebraic wavenumber identification method to identify wavenumbers under stochastic conditions. Stochastic condition results from the introduction of small perturbation which is referred to the uncertainty of measurements points' coordinates caused by the operation faults or problems with experimental errors. The proposed method is compared with two popular alternatives, namely: inhomogeneous wave correlation method and inverse convolution method which are both capable of extracting the bending wavenumbers of a meta-structure. A good performance is observed for the identification of complex wavenumbers in presence of uncertainty. In addition, the proposed method needs to solve a linear problem, reducing the computational cost compared to the inhomogeneous wave correlation method.

1 Introduction

Complex structures, such as composite structures, meta-structures, or viscoelastic structures are of great interest in aerospace and civil engineering for their advantageous mechanical properties. For example, honeycomb sandwich composite structure displays low weight, high stiffness, and resistance to fatigue. Meta-structures can be designed to produce bandgaps which leads to reducing structural vibration.

The wave propagation characteristics of these structures can be analyzed by wavenumber identification as a function of frequency. The relationship between wavenumber and frequency is called the dispersion relation. In addition, the extracted wavenumber can be further used to estimate a structure's mechanical parameters, damping loss factors, and damage detection [1-2].

In the open literature, numerous inverse methods have been developed for wavenumber identification. These inverse methods are able to extract wavenumbers based on the frequency response of structures without prior knowledge of their mechanical properties. Such features make them more suited for industrial applications. According to the algorithmic principles, inverse methods can be divided into two families of methods: (i) nonlinear family methods and (ii) Prony linear family methods. In the nonlinear family methods, the Inhomogeneous Wave Correlation (IWC) [3-4] method and the McDaniel method [5-6] are often used. These two methods do not require periodic samples as input parameters and keep good robustness in stochastic conditions, but they require an expensive computational cost since they need to solve nonlinear issues. Moreover, the IWC is not able to provide accurate wavenumbers in the low frequency range due to the requirement of many wavelengths presented in the displacement field. Comparatively, linear Prony family methods, such as High Resolution Wavenumber Analysis (HRWA) [7] and INverse COnvolution MEthod (INCOME) [8], have the advantage of requiring a low computational cost and providing accurate wavenumbers in the wide frequency range, while they require that frequency response are measured by periodic sampling and they are sensitive to small perturbation of coordinates. The small perturbation problem is common in experiments and it leads to the mismatching problem between frequency response and each measuring point's coordinate. This problem could be caused by operation fault, grid distortion, or the problem of measuring equipment [9]. Unfortunately, there are few accessible papers addressing the problem of uncertainty propagation in the wavenumber identification process. For example, Lajili et al. [10] developed a stochastic wavenumber identification process by combing IWC-Variant (IWC-V) with Latin Hypercube Sampling (SHL) [11]. This method has been found to be robust to random variability of measurement points' coordinates, but it still suffers from the problem of IWC.

In order to overcome the aforementioned disadvantages of nonlinear family methods and linear Prony family methods, a new inverse method is

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proposed called Algebraic Wavenumber Identification (AWI) method. This paper is organized as follows: a brief introduction is provided in Section 2. In Section 3, the theory of AWI is introduced when only one wave propagates in the medium. The comparison of AWI, INCOME, and IWC for bending wavenumber identification in a meta-structure in presence of uncertainty is presented in Section 4. Finally, some conclusions are provided in Section 5.

2 Reminder of INCOME and IWC identification methods

2.1 Inhomogeneous wave correlation method

For the wavenumber identification of 1D structures, the Inhomogeneous Wave Correlation method compares the signal $u(x_n)$ which is frequency response at x_n point with the model of the inhomogeneous wave $\hat{o}(x_n,k)=e^{ikx_n}$. The IWC function is defined as:

$$IWC(x_n,k)=\frac{\left|\int u(x_n).\hat{o}^*(x_n,k)\,dx_n\right|}{\sqrt{\int|u(x_n)|^2\,dx_n.\int|\hat{o}(x_n,k)|^2\,dx_n}} \tag{1}$$

where symbol * is the complex conjugation. For each frequency, the wavenumber k can be estimated by searching the maximum of function IWC.

2.2 Inverse convolution method

The 1D version of INCOME is developed based on the classic Prony method which fits a signal to a weighted summation of w damped exponential model $u(x_n)=\sum_{r=1}^{w}A_r e^{-ik_r x_n}+B_r e^{ik_r x_n}$. It basically has three steps: the first step is to estimate the coefficients a_r of the characteristic polynomial $p=\sum_{r=0}^{2w}a_r X^r$ by the least-squares method as follows:

$$H^*Hx=\rho Qx \tag{2}$$

with

$$H=\begin{pmatrix} u(x_{2w+1})+u(x_1) & u(x_{2w})+u(x_2) & \cdots & u(x_{w+1}) \\ u(x_{2w+2})+u(x_2) & u(x_{2w+1})+u(x_3) & \cdots & u(x_{w+2}) \\ \vdots & \vdots & \ddots & \vdots \\ u(x_{N-2w})+u(x_N) & u(x_{N-2w+1})+u(x_{N-1}) & \cdots & u(x_{N-w}) \end{pmatrix}$$

$$Q=\begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & 2 & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where w is the number of waves and coefficients vector x is the eigenvector associated to the smallest eigenvalue ρ. The second step is that the propagation constants λ_r of $p=\prod_{r=1}^{w}(X-\lambda_r)(X-1/\lambda_r)$ can be estimated by finding the roots of characteristic polynomial and the final step is that the wavenumber is obtained by Bloch principle $k_r=i\ln(\lambda_r)/\Delta x$ with sampling interval Δx.

3 Algebraic wavenumber identification method

In this section, the main formula of AWI is introduced. AWI firstly uses the algebraic derivative method and Laplace transform to build a linear differential equation in the wavenumber domain, then the operational calculus transform is applied to each signal function and finally, the wavenumber is identified through the least-squares method. In this section, one wave propagating in the 1D structure is considered and it can also be extended to multiple waves cases.

A stochastic model of the 1D displacement field can be assumed when only one wave propagates in the medium as follows:

$$U(\tilde{x}_n)=Ae^{p\tilde{x}_n} \tag{3}$$

where A is unknown amplitudes and p is related to the unknown wavenumber k ($p=ik$). This paper aims to extract the wavenumber k. $\tilde{x}_n$ is the stochastic coordinate of the n^{th} measurement point and it can be obtained by introducing the random variable into each perfect measuring point's coordinate x_n. Thus, the coordinate of each measurement point can be expressed as:

$$\tilde{x}_n=x_n(1+\delta_n) \tag{4}$$

where δ_n is the small perturbation ratio which is a random variable obeying the uniform distribution. In reality, the small perturbation ratio of each measurement point's coordinate is unknown, therefore the corresponding displacement can be regarded as the displacement at the perfect coordinate.

In the wavenumber domain, the Laplace transform of Eq. (3) is given by:

$$S(k)=\frac{A}{k-p} \tag{5}$$

A characteristic polynomial is defined as:

$$\varphi(k)=k-p=\beta(1)+\beta(0)k \tag{6}$$

where $\beta(1)$ and $\beta(0)$ are coefficients of characteristic polynomial. Therefore, a differential equation can be established by taking the derivative of the product of Eq. (5) and Eq. (6) as:

$$\frac{d\left[S(k)\varphi(k)\right]}{dk}=0 \tag{7}$$

To eliminate derivation operation in the spatial domain, Eq. (7) is multiplied by k^2 and then operational calculus transform is applied to each of its terms. After arrangement, the following expression can be obtained:

$$\begin{aligned}&\beta(1)\int_0^x\int_0^{x_1} x_2 S(x_2)dx_2dx_1\\ &-\beta(0)\left[\int_0^x\int_0^{x_1} x_2 S(x_2)dx_2dx_1-\int_0^x x_1 S(x_1)dx_1\right]=0\end{aligned} \quad (8)$$

The integral of each term in Eq. (8) can be calculated by the trapezoidal integration method. Eq. (8) holds for each measurement point, thus the coefficients β(0) and β(1) of the characteristic polynomial, Eq. (6), can be identified by the least-squares method and finally the wavenumber can be obtained by:

$$k=-i\frac{\beta(1)}{\beta(0)} \quad (9)$$

It is noted that the integrals operate like a low filter [12], reducing the influence of large measurement error on wavenumber extraction. On the other hand, AWI treats the displacement field as a continuous function, thus it is not limited to periodic sampling.

4 Application of AWI to a 1D meta-structure

In this section a comparison is made between the three methods including AWI, INCOME and IWC. This was performed for bending wavenumber extraction in a 1D meta-structure which is a steel beam equipped with spatially distributed small-scale resonators.

Fig. 1. shows the schematic representation of the 1D meta-structure. The steel beam has the dimension $80\times3\times1\,cm$. The mechanical properties of the steel host beam and resonators are listed in Table 1 and Table 2, respectively. The sampling interval is chosen as $1cm$, resulting in 80 measurement points. The total mass of the resonators is 2% of the mass of the host steel beam. The small perturbation ratio is considered as 20% in this section.

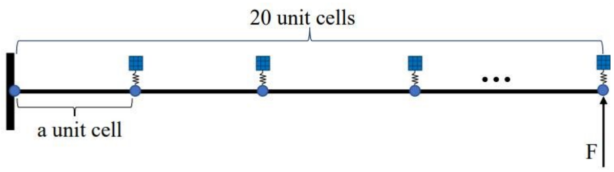

Fig. 1. Schematic representation of the resonator.

Table 1. Characteristics of steel host beam

Yang's modulus	Density	Poisson ratio	Damping
210 GPa	7900 kg/m^3	0.3	0.8%

Fig. 2 shows the displacement at 400 Hz, from which the measurements' error caused by small perturbation problem can be observed.

Table 2. Characteristics of resonators

Number	Mass ratio	Resonance frequency	Damping
20	2%	400 Hz	0.5%

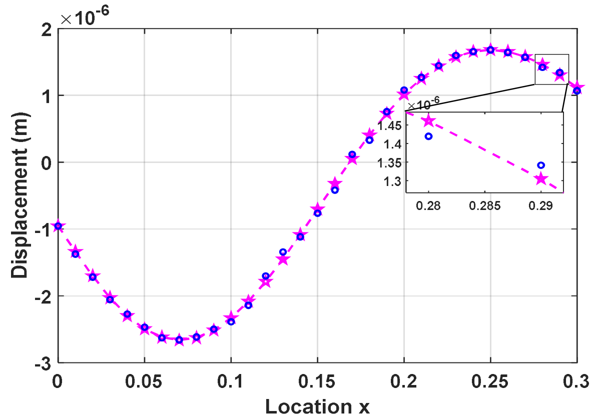

Fig. 2. Displacements at 400 Hz under small perturbation condition ($\delta_n = 20\%$). Small perturbed samples (○) and perfect samples (--★--).

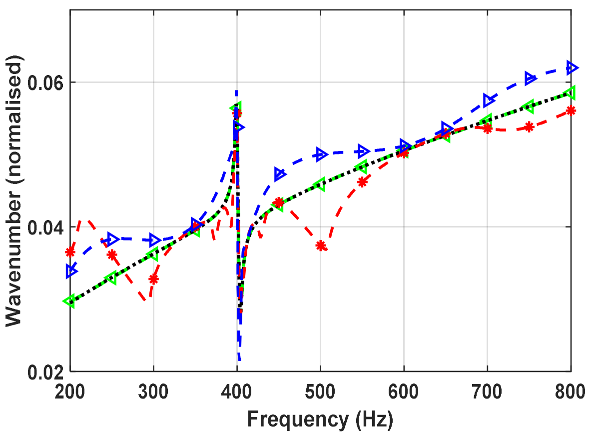

Fig. 3. The real part of wavenumbers obtained by INCOME (--*--), IWC (--▷--) and AWI (--◁--) under small perturbation condition ($\delta_n = 20\%$). Numerical solution (reference) (···).

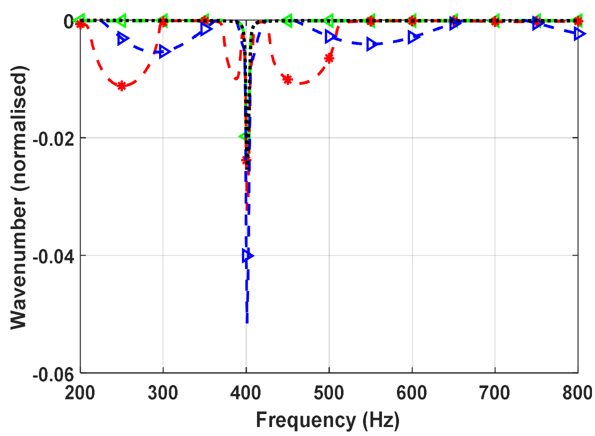

Fig. 4. The imaginary part of wavenumbers obtained by INCOME (--*--), IWC (--▷--) and AWI (--◁--) under small perturbation condition ($\delta_n = 20\%$). Numerical solution (reference) (···).

Fig. 3 and Fig. 4 are real part of wavenumbers and imaginary part of wavenumbers extracted by INCOME, IWC and AWI, respectively. From Figs. (2) to (4), three observations can be made: First, AWI is in very good agreement with numerical predictions. This can be explained by the fact that the introduction of multiple integrals in the framework of AWI yields some insensitivity to small perturbations. Second, INCOME is not able to provide the accurate wavenumber in this case. This sensitivity to errors is an intrinsic feature of Prony. Third, dispersion curve obtained by IWC shows big fluctuations between 200Hz and 800Hz. This is because the IWC requires that the displacement field contains many wavelengths, while the obtained frequency responses are not able to provide enough wavelengths in the range of frequencies considered in this case. In addition, as can be seen in Fig. 4, the band gap can be identified accurately by the result of AWI and it extends from 370 Hz to 430 Hz, however, it is hard to identify band gap from the result of IWC and INCOME in this case.

The computational time is only dependent on the number of measurements. Thus, 10 groups of perfect measurements points from 10 to 100 are firstly obtained from wave shape and then the computational time is averaged on 30 frequency points for each group measurement point. Fig. 5 shows the comparison result of the computational time. This figure illustrates that INCOME and AWI have a clear reduction of computational time compared with IWC because they require solving a linear problem without a nonlinear iterative optimization process. Among them, the AWI is 5 to 12 times faster than IWC.

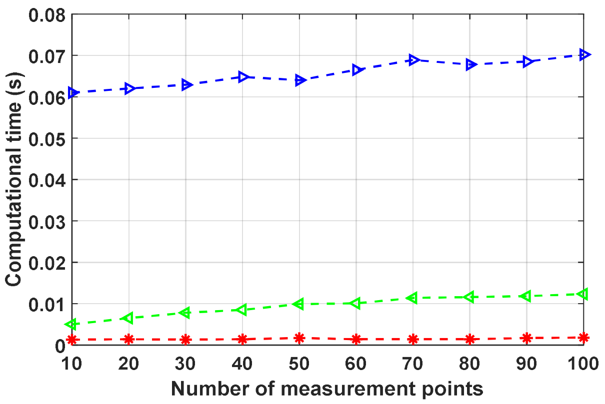

Fig. 5. Computational time between IWC, INCOME and AWI versus the number of measurement points. INCOME (--*--), IWC (--▷--) and AWI (--◁--).

5 Conclusions

In this work, an algebraic wavenumber identification method (AWI) to extract complex wavenumbers of 1D structure, in presence of uncertainty which is caused by the random variability of measurement points' coordinates, was presented. This method basically overcomes the disadvantages of the two popular inverse methods IWC and INCOME. In order to validate the effectiveness of AWI, AWI was compared with IWC and INCOME in the case of a meta-structure. One can notice that the proposed method is not limited to periodic samples since the displacement field is treated as a continuous function instead of a collection of discrete signals. Furthermore, this method is not sensitive to small perturbations that may be caused by the random variability of geometric measurement points' coordinates, because of the introduction of multiple integrals. Besides, AWI requires solving a linear problem, leading to a clear reduction of computation time compared with nonlinear family methods such as IWC.

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