Efficiency Improvement of Electro-Mechanical Systems

Pavol Oršanský and *Zuzana* Malacká

Department of Applied Mathematics, Faculty of Mechanical Engineering, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia

Abstract. A study was performed to investigate the effect of using Euler-Lagrange optimization decreasing energy demands for given electro-mechanical systems exploiting electric drives, which are typical for industrial and transportation applications e.g. robotic arm control, train movement control etc.

1 Introduction

Differential equations and systems of the differential equations describes many types of real physical systems of various disciplines of technical praxis. The theory of differential equations is one of the most important parts of modern applied mathematics. Differential equations can be divided into several types– classical ordinary, partial differential equations and functional differential equations 1. Apart from describing the properties of the equation itself, these classes of differential equations can help inform the choice of approach to a solution.

In this paper we discus some relations between differential equations, variational problem and energy optimal control of electric drives. The starting point for describing a drive control system is the is the differential equation modelling the motor and its driven load 2, 3, 4. The basic form of this equations is applicable to all types of electrical motors

$$J_r \dot{\omega} = M_e - M_L, \tag{1}$$

where J_r is moment of inertia to the shaft of electric motor,M_e is the electrical torque developed by the motor,M_L is the load torque acting on the rotor inertia to the rotor angular acceleration $\dot{\omega}$.

In paper 4, 5 the authors supposed the load torque in the form

$$M_L = A + B\omega + C\omega^2, \tag{2}$$

wherethe constants A, B and C represent Coulomb, viscous and aerodynamic friction terms.

2 Theoretical background

The cost function which we have to minimalised is defined by the relation

$$I = \frac{R_c}{k_t{}^2} \int_0^{T_m} M_e{}^2 \, \mathrm{d}t + \int_0^{T_m} M_L \, \omega \, \mathrm{d}t \ \rightarrow \text{min}, \tag{3}$$

where $R_c$is the stator resistance and k_t is the torque constant. The first part of (3) represents electrical losses in copper and the second part of (3) describes the mechanical-friction losses during the manoeuvre time T_m.
Now we obtain the corresponding Lagrange function

$$L = \frac{R_c}{k_t{}^2} M_e{}^2 + M_L\omega + \lambda_1(J_r\dot{\omega} - M_e + M_L) + \lambda_2(\dot{\varphi} - \omega). \tag{4}$$

By deriving this function according to the variable ω and taking into account the properties of the angular velocity we get

$$\frac{\partial L}{\partial \omega} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\omega}}\right) \ \Rightarrow \ M_L - \lambda_2 = \frac{\mathrm{d}}{\mathrm{d}t}(\lambda_1 J_r) = J_r\dot{\lambda}_1. \tag{5}$$

In a similar way for the variable φ

$$\frac{\partial L}{\partial \varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) \ \Rightarrow \ 0 = \frac{\mathrm{d}}{\mathrm{d}t}(\lambda_2) \ \Rightarrow \ \dot{\lambda}_2 = 0. \tag{6}$$

and the variable M_e

$$\frac{\partial L}{\partial M_e} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{M}_e}\right) \ \Rightarrow \ 0 = 2\frac{R_c}{k_t{}^2} M_e - \lambda_1 \ \Rightarrow \ M_e = \underbrace{\left(\frac{k_t{}^2}{2R_c}\right)}_{k_m}\lambda_1 = k_m\lambda_1. \tag{7}$$

Now from previous and from (1), also if we suppose $\dot{\varphi} = \omega$, we get system of four equations

$$J_r\dot{\lambda}_1 = M_L - \lambda_2, \tag{8}$$

$$\dot{\lambda}_2 = 0, \tag{9}$$

$$J_r\dot{\omega} = M_e - M_L = k_m\lambda_1 - M_L, \tag{10}$$

$$\dot{\varphi} = \omega. \tag{11}$$

In our case the we suppose that the load torque is constant, i.e.:$M_L = A = \text{const}$. So,the system is linear nonhomogeneous systems of differential equations in the form

$$\dot{\lambda}_1 = -\frac{\lambda_2}{J_r} + \frac{A}{J_r}, \tag{12}$$

$$\dot{\lambda}_2 = 0, \tag{13}$$

$$\dot{\omega} = \frac{k_m}{J_r}\lambda_1 - \frac{A}{J_r}, \tag{14}$$

$$\dot{\varphi} = \omega, \tag{15}$$

which can be written also matrix notation as

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\omega} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{J_r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_m}{J_r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \omega \\ \varphi \end{pmatrix} + \begin{pmatrix} \frac{A}{J_r} \\ 0 \\ -\frac{A}{J_r} \\ 0 \end{pmatrix} \ \Leftrightarrow \ \dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{B}. \tag{16}$$

The general solution of this system has got the form

$$\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Y}_p, \tag{17}$$

where$\boldsymbol{Y}$is the homogeneous solution of the system $\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X}$ and $\boldsymbol{Y}_p$ is some particular solution of the system (16).

The eigenvalues λ_i of the matrix $\boldsymbol{A}$are the roots of the characteristic equation

$$\det(\boldsymbol{A} - \lambda\boldsymbol{E}) = 0. \tag{18}$$

We obtain $\lambda_{1,2,3,4} = 0$, which is an eigenvalue of the matrix $\boldsymbol{A}$with the multiplicity $k = 4$.

If we consider the system of linearly independent vectors

$$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \tag{19}$$

Which satisfy the following relations

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v}_1 &= \lambda\boldsymbol{v}_1, \\ \boldsymbol{A}\boldsymbol{v}_j &= \lambda\boldsymbol{v}_j - \boldsymbol{v}_{j-1}, \end{aligned} \tag{20}$$

for $j = 2,3,4$, then the functions

$$\boldsymbol{x}_i = \boldsymbol{w}_i(t) \cdot \mathrm{e}^{\lambda t} = \boldsymbol{w}_i(t), \tag{21}$$

for $i = 1,2,3,4$, where

$$\begin{aligned} \boldsymbol{w}_1(t) &= \boldsymbol{v}_1, \\ \boldsymbol{w}_2(t) &= t\boldsymbol{v}_1 + \boldsymbol{v}_2, \\ \boldsymbol{w}_3(t) &= \frac{t^2}{2}\boldsymbol{v}_1 + t\boldsymbol{v}_2 + \boldsymbol{v}_3, \\ \boldsymbol{w}_4(t) &= \frac{t^3}{6}\boldsymbol{v}_1 + \frac{t^2}{2}\boldsymbol{v}_2 + t\boldsymbol{v}_3 + \boldsymbol{v}_4, \end{aligned} \tag{22}$$

are linearly independent solutions of the homogeneous system $\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X}$. The vectors $\mathbf{v}_i$ with the properties (21) are called generalized eigenvectors of the matrix $\boldsymbol{A}$. To obtain these generalized eigenvectors we use so-called Weyr's theory of matrices [6]. After this procedure we get these generalized vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-k_m}{J_r{}^2} \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{-k_m}{J_r{}^2} \\ 0 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} \frac{-1}{J_r} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

and the homogeneous solution of the system can be written in the form

$$\boldsymbol{Y}=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \omega\\ \varphi\end{pmatrix}=C_1\begin{pmatrix}0\\0\\0\\ \frac{-k_m}{J_r{}^2}\end{pmatrix}+C_2\left(t\begin{pmatrix}0\\0\\0\\ \frac{-k_m}{J_r{}^2}\end{pmatrix}+\begin{pmatrix}0\\0\\ \frac{-k_m}{J_r{}^2}\\0\end{pmatrix}\right)$$
$$+C_3\left(\frac{t^2}{2}\begin{pmatrix}0\\0\\0\\ \frac{-k_m}{J_r{}^2}\end{pmatrix}+t\begin{pmatrix}0\\0\\ \frac{-k_m}{J_r{}^2}\\0\end{pmatrix}+\begin{pmatrix}\frac{-1}{J_r}\\0\\0\\0\end{pmatrix}\right) \tag{24}$$
$$+C_4\left(\frac{t^3}{6}\begin{pmatrix}0\\0\\0\\ \frac{-k_m}{J_r{}^2}\end{pmatrix}+\frac{t^2}{2}\begin{pmatrix}0\\0\\ \frac{-k_m}{J_r{}^2}\\0\end{pmatrix}+t\begin{pmatrix}\frac{-1}{J_r}\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right),$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are real constants.

And also is it easy to proof that the function

$$\boldsymbol{Y}_p=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \omega\\ \varphi\end{pmatrix}=\begin{pmatrix}\frac{At}{J_r}\\ 0\\ \frac{Ak_m}{2J_r{}^2}t^2-\frac{At}{J_r}\\ \frac{Ak_m}{6J_r{}^2}t^3-\frac{At^2}{2J_r}\end{pmatrix} \tag{25}$$

Is the particular solution of nonhomogeneous system (16).

3 Reached results

So finally, we must find the unknown coefficients $C_1, C_2, C_3, C_4 \in \mathbb{R}$ using the boundary conditions.

$$\begin{aligned}\omega(0)&=0,\\ \omega(T_m)&=0,\\ \varphi(0)&=0,\\ \varphi(T_m)&=2\pi,\end{aligned} \tag{26}$$

for system

$$\lambda_1=-C_3\frac{1}{J_r}-C_4\frac{t}{J_r}+\frac{At}{J_r} \quad\Leftrightarrow\quad M_e=k_m\lambda_1=-C_3\frac{k_m}{J_r}-C_4\frac{k_m t}{J_r}+\frac{k_m At}{J_r}$$
$$\lambda_2=0,$$
$$\omega=-C_2\frac{k_m}{J_r{}^2}-C_3 t\frac{k_m}{J_r{}^2}-C_4 t^2\frac{k_m}{2J_r{}^2}+t^2\frac{Ak_m}{2J_r{}^2}-\frac{At}{J_r},$$
$$\varphi=-C_1\frac{k_m}{J_r{}^2}-C_2 t\frac{k_m}{J_r{}^2}-C_3 t^2\frac{k_m}{2J_r{}^2}-C_4 t^3\frac{k_m}{6J_r{}^2}+t^3\frac{Ak_m}{6J_r{}^2}-\frac{At^2}{2J_r{}^2},$$

where $J_r=0.3$ kg m^2, $k_m=\frac{k_t{}^2}{2R_c}$ and $k_t=2.7$ N m A^{-1}, $R_c=0.11\ \Omega A=0.2$ N m. The solution is $C_1=0, C_2=0, C_3=-0.1042, C_4=0.4048$.

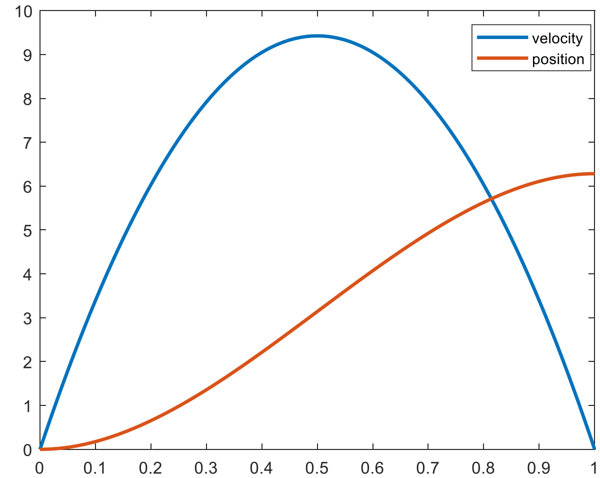

Fig. 1. Position and angular speed of the system. (The *x*-axis represents the time in seconds and the *y*-axis represents the position in radians, respectively angular velocityin radians per second).

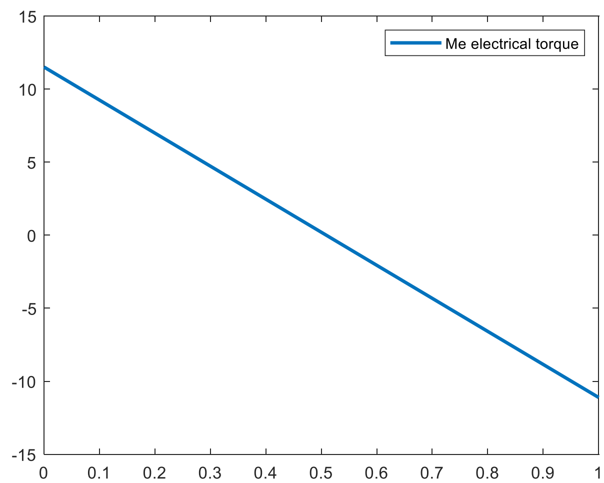

Fig. 2. M_e electrical torque (The *x*-axis represents the time in seconds).

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References

1. H. Šamajová, B. Ftorek, E. Špániková Asymptotic character of non-oscillatory solutions to functional differential system. Electronic Journal of Qualitative Theory of Differential Equations, 42, 1–13 (2015).
2. S.J. Dodds, G. Sooriyakumar, R. Perryman Minimum Energy Forced Dynamics Control of PMSM. Proc. of the 3rd IASME-WSEAS Int. Conference on Energy & Environment, Cambridge, United Kingdom, p. 173-179.
3. M.A. Sheta, V. Agarwal, P.S.V. Nataraj A new energy optimal control scheme for a separately excited dc motor based incremental motion. International Journal of Automation and Computing, 267-276 (2006).
4. S.J. Dodds, G. Sooriyakumar, R. Perryman Minimum Sliding Mode Minimum Energy Position Controller for Permanent Magnet Synchronous Motor Drives. WSEAS Trans. on Systems and Control, 3, 4, 299-309 (2008.)
5. J. Vittek, P. Butko, B. Ftorek, P. Makyš, L. Gorel Energy Near-Optimal Control Strategies for Industrial and Traction Drives with a.c. Motors. Mathematical Problems in Engineering Volume 2017, Article ID 1857186 (2017).
6. J. Vittek, B. Ftorek, P. Butko, T. Fedor Two Approaches Comparison to Energy Optimal Position Control with Constant and Linear Torques Proceedings ACEMP-OPTIM 2015, International conference Antalya, Turkey (2015).
7. O. Borůvka Foundations of matrix theory Praha: Academia (1971).