

Hopf bifurcation analysis in a stage-structure predator–prey model with two time delay

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Abstract. In this paper, we consider a predator-prey system with two time delays, which describes a prey–predator model with parental care for predators. The local stability of the positive equilibrium is analysed. By choosing the two time delays as the bifurcation parameter, the existence of Hopf bifurcation is studied. Numerical simulations show the positive equilibrium loses its stability via the Hopf bifurcation when the time delay increases beyond a threshold.

Keywords: Predator-prey model, Delay, Hopf bifurcation, Local stability.

1 Introduction

Predation relationship is one of the basic relationships of interactions among species that exist universally in nature, and it is also a major topic of population dynamics research. Partial differential system is an important tool to describe population ecology, in which the predator-prey model is one of the important models [1,15].

In the study of the predator-prey model, there are many factors that affect population dynamics, and it is impossible that every predator has the same ability to capture prey in reality. In nature, they all go through two stages of immaturity and maturity. Immature predators are raised by their parents. Therefore, the single-species growth model consisting of immature stage and mature stage has also received extensive attention in recent years [2-4,12-14]. Nutrient availability, whether from their parents or from prey, can enrich the model [5].

At the same time, with the deepening of research, people found that the predator-prey model could not accurately describe the time lag behavior of some research objects, that is, the time lag phenomenon. In the process of population development, the population development is not only related to the current state, but also related to the previous period of time due to the influence of factors such as pregnancy, regeneration, incubation and maturity of the population [8-11]. At this time, it is very important to add time delay factor into the population model. Therefore, this paper adopts a stage structure predator-prey model with parental care with time delay.

The main content of this paper is as follows. In the Section II, we give the model and analyze the model without time delay. In the section III, we discuss the stability of the

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equilibrium point and the Hopf bifurcation, considering various cases with added time delays. Section IV uses numerical simulation to verify the theoretical results according to the parameter conditions. Concluding remarks are contained in section V.

2 Existence and stability of equilibria

In this paper, we consider the stage-structure predator-prey model [6] with parental care

$$\begin{cases} \frac{du(t)}{dt} = u(t)(r - \alpha u(t)) - \beta u(t)v_2(t) \\ \frac{dv_1(t)}{dt} = k_1\beta u(t - \tau_1)v_2(t) \frac{v_2(t)}{\omega v_1(t) + v_2(t)} - m_1 v_1(t) - k_2\beta u(t)v_2(t) \frac{\omega v_1(t)}{\omega v_1(t) + v_2(t)} \\ \frac{dv_2(t)}{dt} = k_2\beta u(t)v_2(t) \frac{\omega v_1(t - \tau_2)}{\omega v_1(t - \tau_2) + v_2(t)} - m_2 v_2(t) \end{cases} \quad (2.1)$$

where u , v_1 and v_2 denote the densities of prey, immature predator and mature predator, respectively. r is the intrinsic grow rate of the prey. α is the density dependent coefficient of the prey. β is the attacking rate of the mature predator at the prey. ω measures the relative consumption ratio between one immature predator and one mature predator. k_1 is a conversion coefficient of the immature predator. k_2 is a proportional constant and this kind of transition rate was used by Roughgarden et al.[7] for an open marine population. m_1 and m_2 are the death of immature predator and mature predator. We always assume that adult predators forage the prey and provide parental care to their offspring and all the parameters in system (1.1) are assumed to be positive.

Using the following parameters transformation:

$$\begin{aligned} k_2\beta u/m_2 &\rightarrow u, \omega v_1 \rightarrow v_1, v_2 \rightarrow v_2, m_2 t \rightarrow t \\ k_2\beta u/m_2 &\rightarrow u, \omega v_1 \rightarrow v_1, v_2 \rightarrow v_2, m_2 t \rightarrow t \end{aligned}$$

the model (2.1) becomes

$$\begin{cases} \frac{du}{dt} = u(a - bu - cv_2) \\ \frac{dv_1}{dt} = \frac{ku(t - \tau_1)v_2^2}{v_1 + v_2} - mv_1 - \frac{\omega uv_1 v_2}{v_1 + v_2} \\ \frac{dv_2}{dt} = \frac{uv_1(t - \tau_2)v_2}{v_1(t - \tau_2) + v_2} - v_2 \end{cases} \quad (2.2)$$

where $a = r/m$, $b = \alpha/(k_2\beta)$, $c = \beta/m_2$, $k = wk_1/k_2$, $m = m_1/m_2$.

The model (2.1) has a unique positive equilibrium if and only if

$$a > b, k(a - b)^2 > b[\omega(a - b) + bm] \quad (2.3)$$

We denote the unique positive equilibrium as $\tilde{u} = (u^*, v_1^*, v_2^*)$, where

$$\begin{cases} u^* = \frac{a + cv_1^*}{b + cv_1^*} \\ v_1^* = \frac{-[2bm + \omega(a - b)] + \{[2bm + \omega(a - b)]^2 - 4m\{b[bm + \omega(a - b)] - k(a - b)^2\}\}^{\frac{1}{2}}}{2mc} \\ v_2^* = \frac{(a - b)v_1^*}{b + cv_1^*} \end{cases} \quad (2.4)$$

$$J = \begin{pmatrix} -bu^* & 0 & -cu^* \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} & \frac{(k + \omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} \end{pmatrix} \quad (2.5)$$

The characteristic equation of J : $\det(J - \lambda I_3) = 0$, marked as: $\Delta(\lambda, 0)$.

$$\Delta(\lambda, 0) = \begin{vmatrix} -bu^* - \lambda & 0 & -cu^* \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} - \frac{(k + \omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} - \lambda & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} - \lambda \end{vmatrix}$$

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then the characteristic equation is simplified as:

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 - (a_{13}a_{31} + a_{23}a_{32} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33})\lambda - a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} = 0$$

Then

$$\Delta 1 = a_1 = -(a_{11} + a_{22} + a_{33}) > 0$$

$$\Delta 2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_3a_0 = (a_{11} + a_{22} + a_{33})(a_{13}a_{31} + a_{23}a_{32} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33}) - (-a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}) > 0$$

$$\Delta 3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} = \Delta 2 \times a_3 > 0$$

So $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.

The following result was given in paper [6].

Lemma 2.1. If $m < 1$, then the positive constant equilibrium $\tilde{u} = (u^*, v_1^*, v_2^*)$ of (2.1) is locally asymptotically stable.

3 Stability and Hopf bifurcation

In this section, we will discuss existence of a Hopf bifurcation at the positive equilibrium in three cases.

Case 1. $\tau_1 \neq 0, \tau_2 = 0$

The linearization of system around the equilibrium \tilde{u} can be expressed by

$$\frac{d\tilde{u}}{dt} = J_1u^* + J_2u^*(\tau) \quad (3.1)$$

where:

$$J_1 = \begin{pmatrix} -bu^* & 0 & -cu^* \\ 0 & -\frac{(k+\omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} -bu^* & 0 & -cu^* \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} e^{-\lambda\tau} & \frac{(k+\omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} \end{pmatrix}$$

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then the associated characteristic equation of (3.1) is

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 + B_4e^{-\lambda\tau} = 0 \quad (3.2)$$

where:

$$B_1 = -a_{11} - a_{22} - a_{33}$$

$$B_2 = a_{11} * a_{22} + a_{11} * a_{33} - a_{13} * a_{31} + a_{22} * a_{33} - a_{23} * a_{32}$$

$$B_3 = -a_{11} * a_{22} * a_{33} + a_{11} * a_{23} * a_{32} + a_{13} * a_{22} * a_{31}$$

$$B_4 = -a_{13} * a_{21} * a_{32}$$

For $\tau > 0$, if $\lambda = i\omega (\omega > 0)$ is a root of (3.2), then we have

$$\begin{cases} \omega^3 - B_2\omega + B_4\sin \omega\tau = 0 \\ B_1\omega^2 - B_3 - B_4\cos \omega\tau = 0 \end{cases} \quad (3.3)$$

Adding up the squares of both equations, we obtain

$$\omega^6 + (B_1^2 - 2B_2)\omega^4 + (B_2^2 - 2B_1B_3)\omega^2 + B_3^2 - B_4^2 = 0 \quad (3.4)$$

Denote $z = \omega^2$, then (3.4) becomes

$$z^3 + pz^2 + qz + r = 0 \quad (3.5)$$

where $p = B_1^2 - 2B_2, q = B_2^2 - 2B_1B_3, r = B_3^2 - B_4^2$

Lemma 3.1. if $B_3^2 - B_4^2 > 0$, then equation (3.5) has at least one positive real root.

Suppose the equation (3.5) has at least one positive root, which is general, and the assumption (3.5) has three positive roots, which are recorded as z_1, z_2, z_3 .

The equation (3.5) also has three positive roots

$$\omega_k = \sqrt{v_k} \quad k = 1, 2, 3$$

$\omega = \omega_j$, the root of order (3.4) could be obtained:

$$\tau_j^{(n)} = \frac{\arccos(h(\omega_j))}{\omega_j} + \frac{2n\pi}{\omega_j}, n = 1, 2, \dots \quad (3.6)$$

where:

$$h(\omega_j) = \frac{B_1\omega_j^2 - B_3}{B_4}, \tau^* = \tau_{j_0}^{(0)} = \min_{j=1,2,3} \{ \tau_j^{(0)} \}, \omega^* = \omega_{j_0}$$

Lemma 3.2. Suppose $h'(z) \neq 0$. If $\tau = \tau^*$, then $\frac{d(\text{Re } \lambda)}{d\tau} \neq 0$

Proof. Sorting equations (2.11) can be obtained

$$e^{-\lambda\tau} = \frac{-\lambda^3 - B_1\lambda^2 - B_2\lambda - B_3}{B_4} \quad (3.7)$$

Differentiating the above equations with respect to τ , we have

$$\lambda B_4 e^{-\lambda\tau} = (3\lambda^2 + 2B_1\lambda + B_2 - B_4\tau e^{-\lambda\tau}) \frac{d\lambda}{d\tau} \quad (3.8)$$

Sort it out

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2B_1\lambda + B_2}{B_4\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} \quad (3.9)$$

From (3.7) and (3.9) we have:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2B_1\lambda + B_2}{\lambda(-\lambda^3 - B_1\lambda^2 - B_2\lambda - B_3)} - \frac{\tau}{\lambda} \quad (3.10)$$

By substituting $\lambda = \omega i, \omega = \omega^*$

$$\text{Re} \left\{ \left(\frac{d\lambda}{d\tau}\right)^{-1} \right\} = \frac{3\omega^6 + (2B_1^2 - 4B_2)\omega^4 + (B_2^2 - 2B_1B_3)\omega^2}{\omega^8 + (B_1 - 2B_2)\omega^6 + (B_2^2 - 2B_1B_3)\omega^4 + B_3^2\omega^2} \quad (3.11)$$

As $z^* = \omega^*$ and $h'(z) \neq 0$, we have

$$\operatorname{Re} \left\{ \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\} = \frac{\omega^{*2} h'(z^*)}{\omega^{*2} h(z^*) + B_4^2 \omega^{*2}} \neq 0 \quad (3.12)$$

This completes the Proof.

By the lemma 3.1 and 3.2, the transversality condition is satisfied. In addition the equation $\Delta(i\omega, \tau) = 0$ has a pair of simple purely imaginary roots $\pm i\omega^*$ at τ^* . Therefore we have the following theorem.

Theorem 3.3. Suppose that (2.3) holds, then we can obtain Hopf bifurcation of the system which holds at $\tau = \tau^*$

Case 2. $\tau_1 = 0, \tau_2 \neq 0$

The proof is similar to the **Case 1**.

Case 3. $\tau_1 = \tau_2 \neq 0$

The linearization of system (2.1) around the equilibrium u^* can be expressed by

$$\frac{dx}{dt} = J_1 x + J_2 x(\tau) \quad (3.13)$$

Where

$$J_1 = \begin{pmatrix} -bu^* & 0 & -cu^* \\ 0 & -\frac{(k+\omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ 0 & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} & 0 & 0 \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & 0 & 0 \end{pmatrix}$$

The coefficient matrix of a linearized system can be expressed as:

$$J = \begin{pmatrix} -bu^* & 0 & -cu^* \\ \frac{v_2^*(kv_2^* - \omega v_1^*)}{v_1^* + v_2^*} e^{-\lambda\tau} & -\frac{(k+\omega)u^*v_2^* + m(v_1^* + v_2^*)^2}{(v_1^* + v_2^*)^2} & \frac{u^*(2kv_1^*v_2^* + kv_2^{*2} - \omega v_1^{*2})}{(v_1^* + v_2^*)^2} \\ \frac{v_1^*v_2^*}{v_1^* + v_2^*} & \frac{u^*v_2^{*2}}{(v_1^* + v_2^*)^2} & -\frac{u^*v_1^*v_2^*}{(v_1^* + v_2^*)^2} \end{pmatrix}$$

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then the associated characteristic equation of (3.13) is

$$\lambda^3 + B_1\lambda^2 + (B_2 + B_4e^{-\lambda\tau})\lambda + B_3 + B_5e^{-\lambda\tau} \quad (3.14)$$

where:

$$B_1 = -a_{11} - a_{22} - a_{33}$$

$$B_2 = a_{11} * a_{22} + a_{11} * a_{33} - a_{13} * a_{31} + a_{22} * a_{33} - a_{23} * a_{32}$$

$$B_3 = -a_{11} * a_{22} * a_{33} + a_{11} * a_{23} * a_{32} + a_{13} * a_{22} * a_{31}$$

$$B_4 = -a_{13} * a_{21} * a_{32}$$

For $\tau > 0$, if $\lambda = i\omega (\omega > 0)$ is a root of (3.14), then we have

$$\begin{cases} \omega^3 - B_2\omega - B_4\cos \omega\tau + B_5\sin \omega\tau = 0 \\ B_1\omega^2 - B_3 - B_4\sin \omega\tau - B_5\cos \omega\tau = 0 \end{cases} \quad (3.15)$$

Adding up the squares of both equations, we obtain:

$$\omega^6 + (B_1^2 - 2B_2)\omega^4 + (B_2^2 - 2B_1B_3 - B_4^2)\omega^2 + B_3^2 - B_5^2 = 0 \quad (3.16)$$

Denote $z = \omega^2$, then (3.16) becomes

$$z^3 + pz^2 + qz + r = 0 \quad (3.17)$$

where $p = B_1^2 - 2B_2, q = B_2^2 - 2B_1B_3 - B_4^2, r = B_3^2 - B_5^2$

Suppose the equation (3.17) has at least one positive root, which is general, and the assumption (3.19) has three positive roots, which are recorded as z_1, z_2, z_3 .

The equation (3.16) also has three positive roots

$$\omega_k = \sqrt{v_k} \quad k = 1, 2, 3$$

$\omega = \omega_j$, the root of order (3.14) could be obtained:

$$\tau_j^{(n)} = \frac{\arccos(h(\omega_j))}{\omega_j} + \frac{2n\pi}{\omega_j}, n = 1, 2, \dots \quad (3.18)$$

where:

$$h(\omega_j) = \frac{B_4\omega_j^4 + (B_1B_5 - B_2B_4)\omega_j^2 + B_3B_5}{B_4^2\omega_j^2 + B_5^2}$$

$$\tau^* = \tau_{j0}^{(0)} = \min_{j=1,2,3} \{ \tau_j^{(0)} \}, \omega^* = \omega_{j0}$$

Lemma 3.2 Suppose $h'(z) \neq 0$. If $\tau = \tau^*$, then $\frac{d(\text{Re } \lambda)}{d\tau} \neq 0$

Proof. Sorting equations (3.15) can be obtained

$$e^{-\lambda\tau} = \frac{-\lambda^3 - B_1\lambda^2 - B_2\lambda - B_3}{B_4\lambda + B_5} \quad (3.19)$$

Differentiating the above equations with respect to τ , we have

$$\lambda^2 B_4 e^{-\lambda\tau} + \lambda B_5 e^{-\lambda\tau} = (3\lambda^2 + 2B_1\lambda + B_2 - B_4\lambda\tau e^{-\lambda\tau} - B_5\lambda e^{-\lambda\tau}) \frac{d\lambda}{d\tau} \quad (3.20)$$

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2B_1\lambda + B_2 + B_4 e^{-\lambda\tau} - B_4\lambda\tau e^{-\lambda\tau} - B_5\tau e^{-\lambda\tau}}{B_4\lambda^2 e^{-\lambda\tau} + B_5\lambda e^{-\lambda\tau}} \\ &= \frac{3\lambda^2 + 2B_1\lambda + B_2}{\lambda(B_4\lambda + B_5)e^{-\lambda\tau}} + \frac{B_4 e^{-\lambda\tau}}{(B_4\lambda^2 + B_5\lambda)e^{-\lambda\tau}} - \frac{\tau(B_4\lambda + B_5)e^{-\lambda\tau}}{\lambda(B_4\lambda + B_5)e^{-\lambda\tau}} \end{aligned} \quad (3.21)$$

From (3.19) and (3.21) we have:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2B_1\lambda + B_2}{-\lambda(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3)} + \frac{B_4}{B_4\lambda^2 + B_5\lambda} - \frac{\tau}{\lambda} \quad (3.22)$$

By substituting $\lambda = i\omega, \omega = \omega^*$

$$\begin{aligned} \text{Re} \left\{ \left(\frac{d\lambda}{d\tau}\right)^{-1} \right\} &= \frac{1}{\omega} \left(\frac{(B_2 - 3\omega^2)(B_2\omega - \omega^3)}{(B_2\omega - \omega^3)^2 + (B_1\omega^2 - B_3)^2} - \frac{B_4^2\omega^3}{B_4^2\omega^3 + B_5^2\omega^2} \right) \\ &= \frac{1}{\omega^2} \left\{ 1 + \frac{B_4\omega^8 + B_5^2\omega^6 + (2B_2B_5^2 + 2B_1B_3B_4)\omega^4}{[(B_2\omega - \omega^3)^2 + (B_1\omega^2 - B_3)^2](B_4^2\omega^2 + B_5^2)} - \frac{B_2^2B_4^2\omega^4 + B_3^2B_4^2\omega^2 + B_3^2B_5^2}{[(B_2\omega - \omega^3)^2 + (B_1\omega^2 - B_3)^2](B_4^2\omega^2 + B_5^2)} \right\} \end{aligned} \quad (3.23)$$

Obviously $1 - \frac{B_2^2B_4^2\omega^4 + B_3^2B_4^2\omega^2 + B_3^2B_5^2}{[(B_2\omega - \omega^3)^2 + (B_1\omega^2 - B_3)^2](B_4^2\omega^2 + B_5^2)} > 0$.

So the equation $\Delta(i\omega, \tau) = 0$ has a pair of simple purely imaginary roots $\pm i\omega^*$ at τ^* . So we can obtain Hopf bifurcation of the system which holds at $\tau = \tau^*$.

4 Numerical simulation

In this section, we carry out numerical simulation to demonstrate the analytical results. We use simple Euler method with a time step of $\Delta t = 0.01$.

Moreover, this set of parameters gives a unique positive equilibrium:

$$\tilde{u} = (u^*, v_1^*, v_2^*) = (0.336, 0.368, 2.082)$$

Of which

$$a = 45, b = \frac{5}{12}, c = 20, k = \frac{4}{15}, m = 1, \omega = 0.8$$

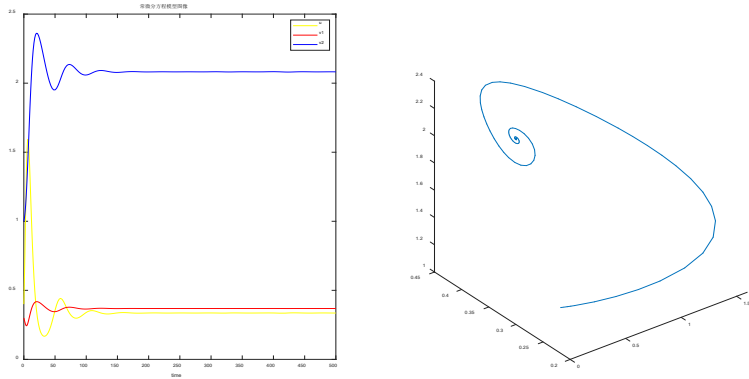


Fig. 6.1 Population dynamics (left) and phase portraits (right) for differential equation model.

Figure 6.1 shows the equilibrium is locally asymptotically stable (left), and the density phase diagram simulation of prey, immature predator and mature predator and the dynamic curve of their density over time (right).

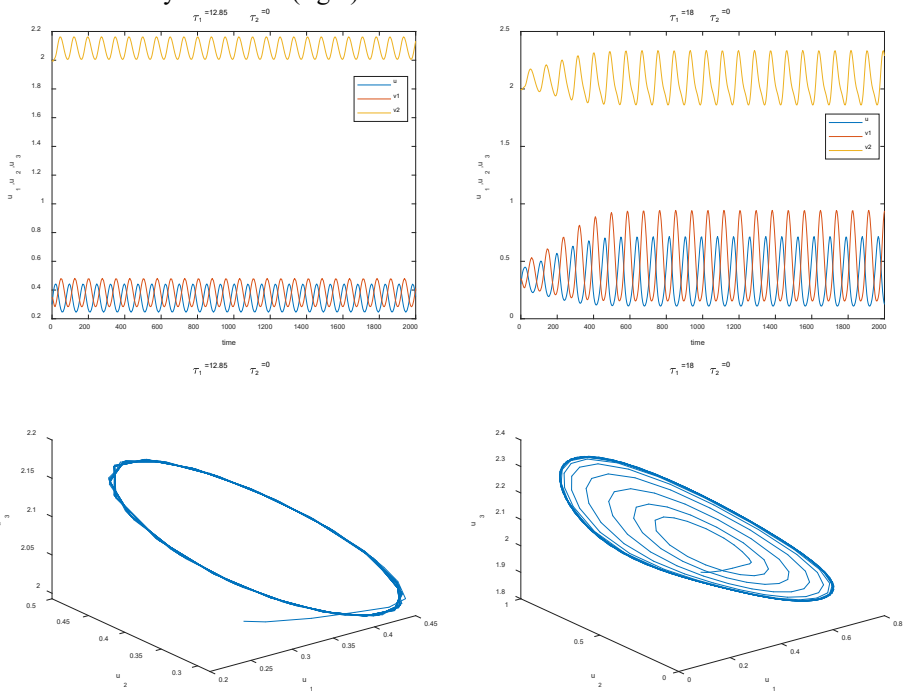


Fig. 6.2 Population dynamics (top) and phase portraits (bottom) for the temporal model at $\tau = 12.85$ (left) and $\tau = 18$ (right).

Figure 6.2 shows that in the case $\tau_1 \neq 0, \tau_2 = 0$, the equilibrium is locally stable when $\tau < \tau^*$ (12.85), the limit cycle appears when $\tau = \tau^*$ and unstable when $\tau > \tau^*$.

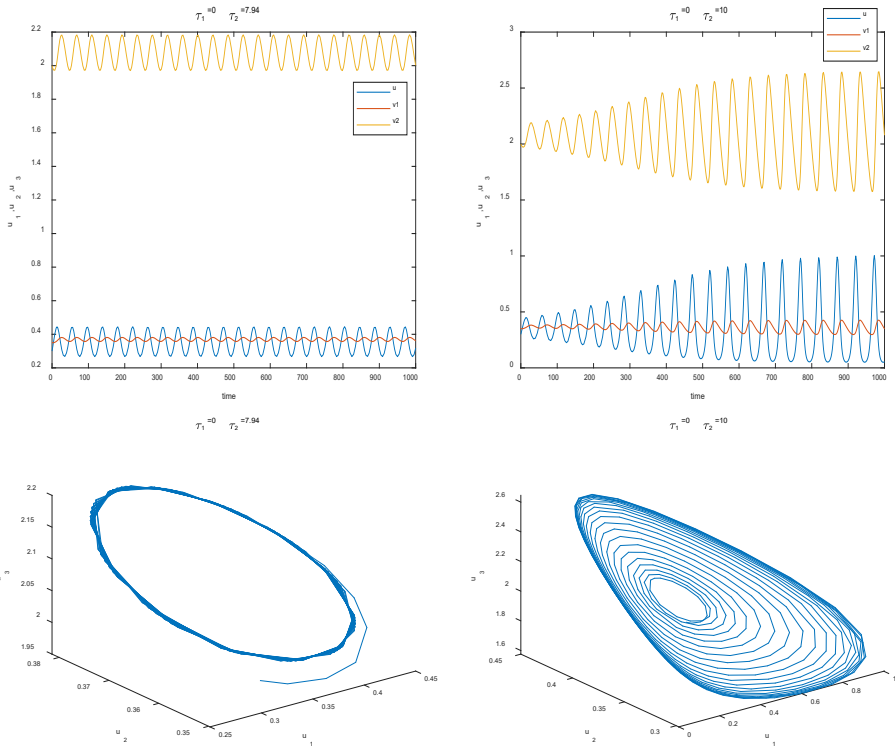
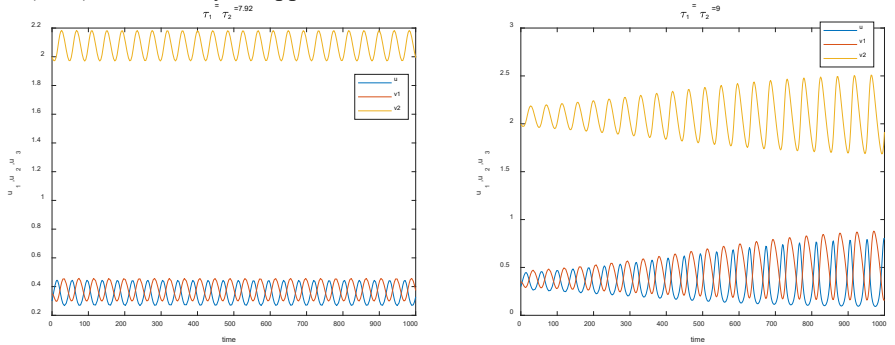


Fig. 6.3 Population dynamics (top) and phase portraits (bottom) for the temporal model at $\tau = 7.94$ (left) and $\tau = 10$ (right).

Figure 6.3 shows that in the case $\tau_1 = 0, \tau_2 \neq 0$, the equilibrium is locally stable when $\tau < \tau^*$ (7.94), the limit cycle appears when $\tau = \tau^*$ and unstable when $\tau > \tau^*$.



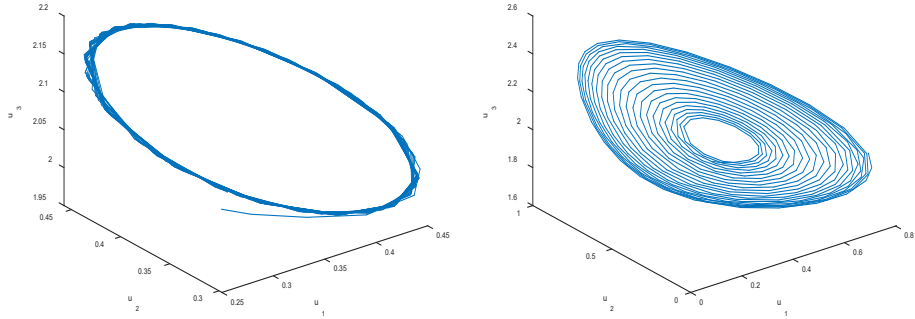


Fig. 6.4 Population dynamics (top) and phase portraits (bottom) for the temporal model at $\tau = 7.92$ (left) and $\tau = 9$ (right).

Figure 6.4 shows that in the case $\tau_1 = \tau_2 \neq 0$, the equilibrium is locally stable when $\tau < \tau^*$ (7.92), the limit cycle appears when $\tau = \tau^*$ and unstable when $\tau > \tau^*$.

5 Conclusion

In this paper, we study a strongly coupled nonlinear reaction diffusion system arising from a prey–predator model with parental care for predators. Sufficient condition which ensures the stability of equilibrium and the existence of Hopf bifurcation are obtained. Numerical simulations verify our results.

When $\tau_1 = \tau_2 = 0$, we prove the local stability of the positive equilibrium applying Routh-Hurwitz criterion. Further, we study the existence of a Hopf bifurcation in three different cases $\tau_1 \neq 0, \tau_2 = 0$, $\tau_1 = 0, \tau_2 \neq 0$ and $\tau_1 = \tau_2 \neq 0$ at the positive equilibrium. In case one the threshold value of τ_1 is 12.85. In case two the threshold of τ_2 is 7.94. In case three the threshold is $\tau_1 = \tau_2 = 7.94$. we only considered two equal delays. It would be more complicated if two delays have different value and deserve more study.

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