Almost anti-periodic solution of inertial neural networks model on time scales

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Abstract. In this work, since the importance of investigation of oscillators solutions, an methodology for proving the existence and stability of almost anti-periodic solutions of inertial neural networks model on time scales are discussed. By developing an approach based on differential inequality techniques coupled with Lyapunov function method. A numerical example is given for illustration.

Keywords: Dynamical systems, Time scales, Exponential stability, Almost-anti periodic solution, Inertial Neural Networks.

1 Introduction

In these ten years, neural networks (NNs) have been successfully applied to numerous fields such as synchronization, control, stability and stabilization of periodic and automorphic solutions ([1], [2], [3],[4], [5], [6],[7], [8], [9], [10], [11], [17]). The aim of this work is to introduce the concept of almost anti-periodic functionson time scales. Then we look for a solution of almost anti-periodic type for dynamic systems modeling Inertial Neural Networks (INNs). The (INNs) model represent a class of artificial neural networks, introduced by Wheeler and Schieve [12] in 1997. Mathematically, it's a second order dynamic system,the term derived from the first order describes the inertial term which naturally covers many fields of application: biology, physics, artificial intelligence. On the other hand, the choice of time scales justified, indeed the theory of time scales is developed by Hilger S., which made it possible to treat continuous-timeand discrete-time equations simultaneously. A time scale is an arbitrary non empty closed subset of R denoted by T. Hilger notably defined the Δ-derivative as follows:

The function $f : T \to R$ is said to possess a Δ-derivative f^Δ if there exists $\varepsilon > 0$ such that

$$\|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\| \le \varepsilon\|\sigma(t) - s\|, \text{ for all } \quad s \in T^{\kappa}.$$

For more details, the books [13, 14] summarize much of time scale calculus.

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The innovation of this work is essentially reflected in three parts. First, application to the class of almost anti-periodic functions, this kind of function introduced in literature. Second, we generalize this concept on time scales. Finally, we will give a sufficient condition guarantees the stability which is based on the construction of a Lyapunov function associated with the system (1).

2 Assumptions and main results

We start by defining the notion of almost anti-periodic functions on time scales.

Definition 2.1 T is called periodic time scale if there exists $\theta > 0$ such that if $t \in T$ then $\theta \pm t \in T$.

Definition 2.2 Let $\varepsilon > 0$, we call $\theta > 0$ an ε-antiperiod for φ if $\|\varphi(t+\theta) + \varphi(t)\| \leq \varepsilon$, $t \in T$. We denote the set of all ε-antiperiods for φ by $Vap(\varphi, \varepsilon)$. It is sait that φ is almost anti-periodic if for each $\varepsilon > 0$ the set $Vap(f, \varepsilon)$ is relativety dense in T.

In this work, we consider almost-anti-periodic solutions of the following models for inertial neural networks with varying delays on time scales

$$x_i^{\Delta\Delta}(t) = -a_i(t)x_i^{\Delta}(t) - b_i(t)x_i(t) + \sum_{j=1}^{n} c_{ij}(t)f_j\left(x_j(t)\right) + \sum_{j=1}^{n} d_{ij}(t)g_j\left(x_j(t)\right) + S_i(t) \quad (1)$$
$$x_i(s) = \Psi(s), \quad x_i^{\Delta}(s) = ¥(s)$$

We assume for all $t, u, v \in T$ and $i, j \in [1 \cdots n]$

(P1) For f_j and g_j are all non-decreasing functions with $f_j(0) = g_j(0) = 0$, , and there exists $L_j^f,\ L_j^g > 0$

such that

$$\left|f_j(u) - f_j(v)\right| \leq L_j^f|u - v|, c_{ij}(t+w)f_j(t) = -c_{ij}(t)f_j(-t)$$
$$\left|g_j(u) - g_j(v)\right| \leq L_j^g|u - v|, d_{ij}(t+w)g_j(t) = -c_{ij}(t)g_j(-t)$$

Remark 1 The property (P1) states a sufficient condition to obtain existence and uniqueness of solution of the system (1). This result is a consequence of a Picard-Lindelof theorem of local uniqueness and existence of first-order systems of nonlinear delay dynamic equations on time scales [15], to delve deeper, you can see [16].

(P2) There exists $\alpha_i \geq 0, \gamma_i \geq 0$ and $\beta_i > 0$ satisfying

$$E_i(t) < 0, \ 4E_i(t)F_i(t) > G_i(t), \ t \in \mathbb{T} \cap \mathbb{R}^+ \quad (2)$$

where

$$E_i(t) = \alpha_i\gamma_i - \alpha_i^2 a_i(t) + \frac{1}{2}\alpha_i^2 \sum_{j=1}^{n}(|c_{ij}(t)|L_j^f + |d_{ij}(t)|L_j^g)$$

$$F_i(t) = -b_i(t)\alpha_i\gamma_i + \frac{1}{2}\sum_{j=1}^{n}(|c_{ij}(t)|L_j^f + |d_{ij}(t)|L_j^g)|\alpha_i\gamma_i| + \frac{1}{2}\sum_{j=1}^{n}\alpha_j^2(|c_{ij}(t)|L_j^f + |d_{ij}^{+}(t)|L_j^g) + \frac{1}{2}\sum_{j=1}^{n}(|c_{ij}(t)|L_j^f + |d_{ij}^{+}(t)|L_j^g)|\alpha_i\gamma_i|$$

$$G_i(t) = \beta_i + \gamma_i^{\,2} - \alpha_{\,i}\gamma_i a_i(t) - \alpha_{\,i}^{\,2} b_i(t)$$

(P3) (a) a_i, b_i, c_{ij} and d_{ij} periodic functions.
(b) $\Psi(s)$, $¥(s)$ and S_i almost anti-periodic functions.

(P4) There exists $\lambda > 0$ such that

$$E_i^{\lambda}(t) < 0,\ \ 4E_i^{\lambda}(t)F_i^{\lambda}(t)\ >\ (G_i^{\lambda}(t))^2,\ \ t \in \mathbb{T} \cap \mathbb{R}^+ \tag{3}$$

$$E_i^{\ \lambda}(t) = \alpha_{\,i}^{\,2} + \alpha_{\,i}\gamma_i - \alpha_{\,i}^{\,2} a_i(t) + \frac{1}{2}\alpha_{\,i}^{\,2}\sum_{j=1}^{n}(|c_{ij}(t)|L_j^f + |d_{ij}(t)|L_j^g)$$

$$\begin{aligned} F_i^{\ \lambda}(t) = \lambda\beta_i + \lambda\gamma_i^{\,2} - b_i(t)\alpha_{\,i}\gamma_i + \frac{1}{2}\sum_{j=1}^{n}(|c_{ij}(t)|L_j^f \\ + |d_{ij}(t)|L_j^g)|\alpha_{\,i}\gamma_i| \\ + \frac{1}{2}\sum_{j=1}^{n}\alpha_{\,j}^{\,2}(|c_{ij}(t)|L_j^f + |d_{ij}^{+}(t)|L_j^g e^{2\lambda}) + \frac{1}{2}\sum_{j=1}^{n}(|c_{ij}(t)|L_j^f \\ + |d_{ij}^{+}(t)|L_j^g e^{2\lambda})|\alpha_{\,i}\gamma_i| \end{aligned}$$

$$G_i^{\ \lambda}(t) = \ 2\lambda\alpha_{\,i}\gamma_i + \beta_i + \gamma_i^{\,2} - \alpha_{\,i}\gamma_i a_i(t) - \alpha_{\,i}^{\,2} b_i(t)$$

Definition 2.3 The system (1) is said to be exponentially stable, if for x(t) and y(t) two solutions of system (1) of initial value $x_i(s) = \Psi^x(s)$, $x_i^{\Delta}(s) = ¥^x(s)$, $y_i(s) = \Psi^y(s)$, $y_i^{\Delta}(s) = ¥^y(s)$, there exists $\lambda > 0$ and $M > 0$ such that for all $t \in \mathbb{T} \cap \mathbb{R}^+$,

$$\begin{aligned} |x_i(t) - y_i(t)| \le M\left|\Psi_i^x - \Psi_i^y\right| e_{-\lambda}(t,s), \left|\Psi_i^x - \Psi_i^y\right| \\ = \max_{1\le i\le n}\{\sup_{-\tau\le s<0}\Psi_i^x(s) - \Psi_i^y(s)\} \end{aligned}$$

$$\begin{aligned} \left|x_i^{\Delta}(t) - y_i^{\Delta}(t)\right| \le M\left|¥_i^{\,x} - ¥_i^{\,y}\right| e_{-\lambda}(t,s), \left|¥_i^{\,x} - ¥_i^{\,y}\right| \\ = \max_{1\le i\le n}\{\sup_{-\tau\le s<0}¥_i^{\,x}(s) - ¥_i^{\,y}(s)\} \end{aligned}$$

Theorem 2.4 Under assumptions (P1)-(P4), the system (1) possesses a globally generalized exponentially stable solution.

Proof.

Step 1: Prove that V is a Lyapunov candidate function associated with system (1) where

$$V(t) = \frac{1}{2}\sum_{j=1}^{n}[\beta_i w_i^2(t) + \left(\alpha_i w_i^{\Delta}(t) + \gamma_i w_i(t)\right)^2]e_{-2\lambda}(t,s) + \frac{1}{2}\sum_{i,j=1}^{n}[\alpha_i^2 + |\alpha_i\gamma_i|]d_{ij}^{+}L_j^{g}e^{2\lambda}\int_0^t w_j^2(s)e_{-2\lambda}(t,s)ds$$

Step 2: Try to write the upper right Dini derivative of V of the following form

$$DV^{+}(t) \leq e_{-2\lambda}(t,s)\left\{\sum_{i=1}^{n} E_i^{\lambda}(t)\left(w_i^{\Delta+}(t) + \frac{G_i^{\lambda}(t)}{2E_i^{\lambda}(t)}w_i^{+}(t)\right)^2 + \sum_{i=1}^{n}\frac{4E_i^{\lambda}(t)F_i^{\lambda}(t) - (G_i^{\lambda}(t))^2}{4E_i^{\lambda}(t)}(w_i^{+}(t))^2\right\}$$

Step 3: Deduce using the property (P2) and (P4) that $DV^{+}(t) \leq 0$.

Remark 2 The advantage of the theory of time scales is to unify continuous and discrete-time and to provide a powerful tool for many applications.

3 Numerical example

Place ourselves in the case where $\mathbb{T} = \frac{1}{4}\mathbb{Z}$, $1 \leq i, j \leq 2$

$$\begin{cases} x_1^{\Delta\Delta}(t) = -\frac{30}{9+|\cos(3t)|}x_1^{\Delta}(t) - \frac{28}{9+|\cos(3t)|}x_1(t) + 0.72|\cos(3t)|f_1(x_1(t) \\ +0.65|\cos(3t)|f_2(x_2(t)) + 0.53|\cos(3t)|g_1(x_1(t)) + 0.32|\cos(3t)|g_2(x_2(t)) + \cos(3t) \end{cases}$$

$$\begin{cases} x_2^{\Delta\Delta}(t) = -\frac{27}{9+|\sin(3t)|}x_2^{\Delta}(t) - \frac{36}{9+|\sin(3t)|}x_2(t) + 0.65|\sin(3t)|f_1(x_1(t) \\ +0.42|\sin(3t)|f_2(x_2(t)) + 0.53|\sin(3t)|g_1(x_1(t)) + 0.66|\sin(3t)|g_2(x_2(t)) + \sin(3t) \end{cases}$$

where $f_j(x) = g_j(x) = 0.5\arctan(x)$, $L_j^f = L_j^g = 0.5$ and from $\alpha_i = \gamma_i = 1$, $\beta_i = 5.5$, $\lambda = 0.01$,

the system (4) satisfy all assumptions:

$$4E_1(t)F_1(t) \geq 3 > (G_1(t))^2$$
$$4E_2(t)F_2(t) \geq 3 > (G_2(t))^2$$
$$4E_1^{\lambda}(t)F_1^{\lambda}(t) \geq 7 > (G_1^{\lambda}(t))^2$$
$$4E_2^{\lambda}(t)F_2^{\lambda}(t) \geq 7 > (G_2^{\lambda}(t))^2$$

Theorem 2.4 implies that unique almost anti-periodic solution of system (1) is exponentially stable. We virtue the following transformations in (4)

$$(x_i^{\Delta}(t), x_i^{\Delta\Delta}(t)) = (y_i(t), y_i^{\Delta}(t)). \qquad (5)$$

From (5), the system (4) take the form of first order:

$$\begin{cases} x_1^{\Delta}(t) = y_1(t) \\ y_1^{\Delta}(t) = -\dfrac{30}{9+|\cos(3t)|}y_1(t) - \dfrac{28}{9+|\cos(3t)|}x_1(t) + 0.72|\cos(3t)|f_1(x_1(t) \\ +0.65|\cos(3t)|f_2(x_2(t)) + 0.53|\cos(3t)|g_1(x_1(t)) + 0.32|\cos(3t)|g_2(x_2(t)) + \cos(3t) \end{cases}$$

$$\begin{cases} x_2^{\Delta}(t) = y_2(t) \\ x_2^{\Delta\Delta}(t) = -\dfrac{27}{9+|\sin(3t)|}y_2(t) - \dfrac{36}{9+|\sin(3t)|}x_2(t) + 0.65|\sin(3t)|f_1(x_1(t) \\ +0.42|\sin(3t)|f_2(x_2(t)) + 0.53|\sin(3t)|g_1(x_1(t)) + 0.66|\sin(3t)|g_2(x_2(t)) + \sin(3t) \end{cases}$$

The initial conditions of system (6) are given by folowing almost anti-periodic functions $x_1(s) = -\sin(s)$, $y_1(s) = \sin(s)$, $x_2(s) = \sin(s)$, $y_2(s) = \sin(s) s \in [-0.11, 0]_T$

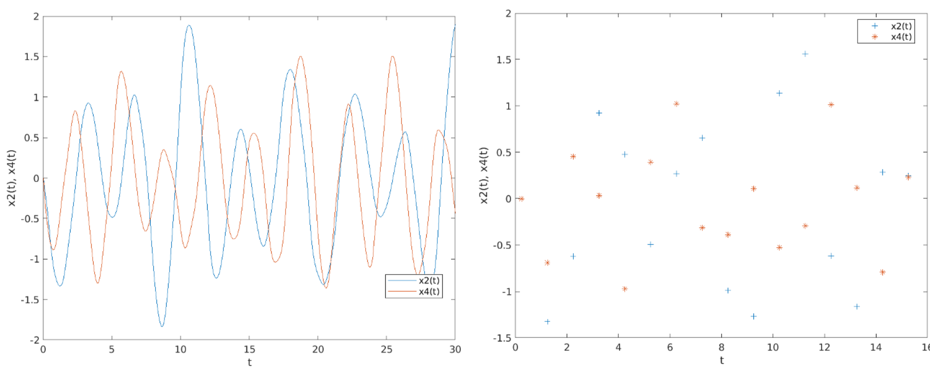

Fig. 3. This analyse proposed in this work, can be extended for a more complex delayed neuronalmodels as example [2] and [0]. These ideas can be the subjects of some future works.

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