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The patterns of traveling wave on shallow water modeled by Green-Naghdi equation

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Abstract. The Green-Naghdi equations are a shallow water waves model which play important roles in nonlinear wave fields. By using the trial equation method and the Complete discrimination system for the polynomial we obtained the classification of travelling wave patterns. Among those patterns, new singular patterns and double periodic patterns are obtained in the first time. And we draw the graphs which help us to understand the dynamics behaviors of the Green-Naghdi model intuitionally.

Keywords: Green-Naghdi equations, Soliton, Shallow water waves, Trial equation method.

1 Introduction

Green-Naghdi equations is a shallow water waves model which can be written as following[1]

$$v_t + vv_x + u_x = \frac{1}{3u}[u^3(vv_{xx} + v_{xt} - v_x^2)]_x , \tag{1}$$

$$u_t + (uv)_x = 0 \quad , \tag{2}$$

where u(x, t) is the free upper surface and v(x, t) is the horizontal velocity of the fluid. Green-Naghdi model describes a kind weekly dispersive nonlinear shallow 1water waves motion which can simulate the propagation of solitons in dispersive media. Compared weekly dispersive nonlinear equations named Boussinesq equations basing small amplitude assumption, Green-Naghdi equations apply to big amplitude question, thus Green-Naghdi equations have more extensive application. In 1953, Serre[2] deduced one-dimensional weekly dispersive shallow water waves model in even bottom. In 1969, Su and Gardner[3] gained same shallow water waves equations by different method. In 1976, Green and Naghdi[4] deduced two-dimensional weekly dispersive shallow water waves equations system (1) and (2) namely Green-Naghdi equations now. In 1987, Santos[5] et al obtained one-dimensional Green-Naghdi model in uneven bottom. In 2010, Dias and Milewski[6] generalized the equations to general form. Due to the im portance of the equations for shallow water waves in physics, the Green-Naghdi equations have been studied by various

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methods. For example, in 2010, Metayer and Gavrilyuk[7] referred a mix-numerical method to deal with the movement of solitons. In 2011, Bonneton et al referred two methods to solve the equations including finite volume method and mix method of finite volume and finite differences. In this paper, we adopt the trial equation method[8–16] to solve the model, and obtain a series of solutions including solitary waves pattern, blow up patterns, periodic wave patterns and periodic blow-up patterns and double periodic wave patterns.[1] Among those, we confirm two new patterns which are singular patterns and double periodic patterns. In addition, we draw the graphs of some patterns to show the dynamical behaviors of the patterns intuitively. The paper is organized as follows:

In section two, we introduce reduction of Green-Naghdi equations. In section three, we give different patterns of travelling wave. In section four, we obtain the expression of the solutions and draw the graphs of some solutions. In last section, we offer the simply conclusion.

2 Reduction of Green-Naghdi equations

We take the traveling wave transformation u = u(ξ), v = v(ξ), ξ = x + ωt and set them into equation(1) and (2) to get

$$3u[\omega v' + vv' + u' = [u^3(vv'' + \omega v'') - (v')^2]_x, \tag{3}$$

$$u = \frac{C_0}{\omega + v}, \tag{4}$$

where C_0 is a constant .Then,setting (4) into (3) and integrating once, we gain

$$C_0^3(v')^2 - C_0^3(\omega + v)v'' + 3C_0 v(\omega + v)^3 + \frac{3}{2}C_0^2(\omega + v) - C_1(\omega + v)^3 = 0, \tag{5}$$

where C_1 is an integral constant. Next, we take the trial equation to solve the equation (5). It is just like

$$v'' = a_m v^m + a_{m-1}v^{m-1} + \ldots + a_2 v^2 + a_1 v + a_0 . \tag{6}$$

We can get m = 3 with the balance principle obviously. Then we can obtain

$$(v')^2 = \frac{a_3}{2}v^4 + \frac{2a_2}{3}v^3 + a_1 v^2 + 2a_0 v + d_0 . \tag{7}$$

where d_0 is an integral constant. Then setting (7) into(5),

we get

$$r_4 v^4 + r_3 v^3 + r_2 v^2 + r_1 v + r_0 = 0 , \tag{8}$$

where

$$\begin{cases} r_4 = 3C_0 - \dfrac{a_3 {C_0}^3}{2} \\ r_3 = 9C_0\omega - a_3C_0^3\omega - \dfrac{a_2C_0^3}{3} - C_1 \\ r_2 = 9C_0\omega^2 - a_2C_0^3\omega - 3C_1\omega \\ r_1 = a_0C_0^3 - a_1C_0^3\omega + 3C_0\omega^3 + \dfrac{3}{2}C_0^2 - 3C_1\omega^2 \\ r_0 = C_0^3 d_0 - a_0C_0^3\omega + \dfrac{3}{2}C_0^2\omega - C_1\omega^3 \end{cases} . \tag{9}$$

Then we let ri = 0(i = 0, ..., 4), and give

$$\begin{cases} a_0 = \dfrac{d}{\omega} + \dfrac{3}{2C_0} - \dfrac{C_1\omega^2}{C_0^3} \\ a_1 = \dfrac{d}{\omega^2} + \dfrac{3}{C_0\omega} + \dfrac{3\omega^2}{C_0^2} - \dfrac{4C_1\omega}{C_0^3} \\ a_2 = \dfrac{9\omega}{C_0^2} - \dfrac{3C_1}{C_0^3} \\ a_3 = \dfrac{6}{C_0^2} \end{cases} . \tag{10}$$

Setting (10) into (7) gives

$$(v')^2 = \frac{3}{C_0^2}v^4 + (\frac{6\omega}{C_0^2} - \frac{2C_1}{C_0^3})v^3 + (\frac{d_0}{\omega^2} + \frac{3}{C_0\omega} + \frac{3\omega^2}{C_0^2} - \frac{4C_1\omega}{C_0^3})v^2 + (\frac{2d_0}{\omega} + \frac{3}{C_0} - \frac{2C_1\omega^2}{C_0^3})v + d_0 . \tag{11}$$

Furthermore, we take the following transformation

$$\begin{cases} \text{v} = T - (\dfrac{C_1}{6C_0} - \dfrac{\omega}{2}) \\ \xi_1 = \dfrac{\sqrt{3}}{C_0}\xi \end{cases} . \tag{12}$$

Setting (12) into (11), we gain the equation

$$(T')^2 = F(T) = T^4 + pT^2 + qT + r , \tag{13}$$

where

$$p = -\frac{\omega}{2} + \frac{C_0}{\omega} + \frac{C_0^2 d_0}{3\omega^2} - \frac{C_1\omega}{3C_0} - \frac{C_1^2}{6C_0^2},$$

$$q = -\omega^3 - 2C_0 - \frac{C_1^3}{27C_0^3} + \frac{5C_1\omega^2}{3C_0} + \frac{C_1^2\omega}{3C_0} + \frac{2C_1C_0d_0}{9\omega^2} + \frac{2C_1}{3\omega} - \frac{8C_1^2\omega}{9C_0^2}, \tag{14}$$

$$r = -\frac{\omega^4}{2} + d_0 + \frac{5C_1\omega^3}{4C_0} - \frac{23C_1\omega^2}{72C_0^2} - \frac{\omega C_1^3}{108C_0^3} - \frac{C_1^4}{432C_0^4} - \frac{13C_0^2d_0}{12} - \frac{7C_0\omega}{4} + \frac{7C_1C_0d_0}{18\omega} + \frac{C_1}{2} + \frac{C_1^2d_0}{108\omega^2} + \frac{C_1^2}{36C_0\omega}. \tag{15}$$

Thus ,we can write (11) as an integral form

$$\int \frac{dT}{\sqrt{T^4 + pT^2 + qT + r}} = \pm(\xi_1 - \xi_0) . \tag{16}$$

Its the complete discrimination system for polynomial F(T) is given as[17-20]

$$\begin{cases} E_2 = 9p^2 - 32pq \\ D_1 = 4 \\ D_2 = -p \\ D_3 = -2p^3 + 8pr - 9q^2 \\ D_4 = -p^3q^2 + 4p^4r + 36pq^2r - 32p^2r^2 - \dfrac{27}{4}q^4 + 64r^3 \end{cases} . \quad (17)$$

In next section, we classify all solutions of equation (16).

3 Patterns of travelling wave

Becase of $u = \dfrac{C_0}{\omega + v}$, we only give the solutions of u.

Solutions I: When $D_2 > 0, D_3 > 0, D_4 = 0$, $F(v) = (v-\lambda)^2(v-\tau)(v-\kappa)$, we get

$$u_1 = \frac{C_0}{\omega + \dfrac{\kappa - \dfrac{\lambda-\tau}{\lambda-\kappa}\coth^2\dfrac{2}{\pm(\xi_1-\xi_0)\sqrt{(\lambda-\tau)(\lambda-\kappa)}}\tau}{1-\dfrac{\lambda-\tau}{\lambda-\kappa}\coth^2\dfrac{2}{\pm(\xi_1-\xi_0)\sqrt{(\lambda-\tau)(\lambda-\kappa)}}}},$$

$$u_2 = \frac{C_0}{\omega + \dfrac{\kappa - \dfrac{\lambda-\tau}{\lambda-\kappa}\tanh^2\dfrac{\pm(\xi_1-\xi_0)\sqrt{(\lambda-\tau)(\lambda-\kappa)}}{2}\tau}{1-\dfrac{\lambda-\tau}{\lambda-\kappa}\tanh^2\dfrac{\pm(\xi_1-\xi_0)\sqrt{(\lambda-\tau)(\lambda-\kappa)}}{2}}}, \quad (18)$$

$$u_3 = \frac{C_0}{\omega + \dfrac{\dfrac{\tau-\lambda}{\lambda-\kappa}\tan^2[(\xi_1-\xi_0)\dfrac{\sqrt{(\lambda-\kappa)(\tau-\lambda)}}{2}]\tau-\kappa}{\dfrac{\tau-\lambda}{\lambda-\kappa}\tan^2[(\xi_1-\xi_0)\dfrac{\sqrt{(\lambda-\kappa)(\tau-\lambda)}}{2}]-1}}.$$

Solution II: When $D_2 > 0, D_3 = 0, D_4 = 0, E_2 = 0$, $F(v) = (v-\mu)^3(v-\rho)$, we obtain

$$u_4 = \frac{C_0}{\omega + \mu + \dfrac{4(\mu-\rho)}{(\rho-\mu)^2(\xi_1-\xi_0)^2 - 4}}. \quad (19)$$

Solutions III: When $D_4 > 0, D_3 > 0, D_1 > 0$, $F(v) = (v-\beta_1)(v-\beta_2)(v-\beta_3)(v-\beta_4)$, we get

$$u_5 = \frac{C_0}{\omega + \frac{\beta_2(\beta_1-\beta_4)sn^2(\frac{\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}}{2}(\xi_1-\xi_0),m)-\beta_1(\beta_2-\beta_4)}{(\beta_1-\beta_4)sn^2(\frac{\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}}{2}(\xi_1-\xi_0),m)-(\beta_2-\beta_4)}} \tag{20}$$

$$u_6 = \frac{C_0}{\omega + \frac{\beta_4(\beta_2-\beta_3)sn^2(\frac{\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}}{2}(\xi_1-\xi_0),m)-\beta_3(\beta_2-\beta_4)}{(\beta_2-\beta_3)sn^2(\frac{\sqrt{(\beta_1-\beta_3)(\beta_2-\beta_4)}}{2}(\xi_1-\xi_0),m)-(\beta_2-\beta_4)}} \tag{21}$$

where

$$m^2 = \frac{(\beta_1-\beta_4)(\beta_2-\beta_3)}{(\beta_1-\beta_3)(\beta_2-\beta_4)}. \tag{22}$$

Solution IV: When $D_4 > 0, D_2D_3 \le 0$, $F(v) = [(v-h_1)^2 + {s_1}^2][(v-h_2)^2 + {s_2}^2]$

$$u_7 = \frac{C_0}{\omega + \frac{a_1 sn(\chi(\xi_1-\xi_0),\upsilon) + a_2 cn(\chi(\xi_1-\xi_0),\upsilon))}{a_3 sn(\chi(\xi_1-\xi_0),\upsilon) + a_4 cn(\chi(\xi_1-\xi_0),\upsilon)}}, \tag{23}$$

where

$a_1 = h_1a_3 + s_1a_4, a_2 = h_1a_4 - s_1a_3, a_3 = -s_1 - \frac{s_2}{\upsilon_1}, a_4 = h_1 - h_2$,

$E = \frac{(h_1-h_2)^2 + {s_1}^2 + {s_2}^2}{2s_1s_2}$,

$\upsilon_1 = E + \sqrt{E^2+1}$, $\chi = \frac{s_2\sqrt{(a_3^2+a_4^2)(\upsilon_1^2a_3^2+a_4^2)}}{a_3^2+a_4^2}$, $\upsilon^2 = \frac{{\upsilon_1}^2 - 1}{{\upsilon_1}^2}$.

Solution V: When $D_2 < 0, D_3 = 0, D_4 = 0$, $F(v) = [(v-l)^2 + h^2]^2$, we obtain

$$u_8 = \frac{C_0}{\omega + h\tan[h(\xi_1-\xi_0)] + l} \tag{24}$$

Solutions VI When $D_2 > 0, D_3 = 0, D_4 = 0$, $E_2 > 0$, $F(v) = (v-\mu)^2(v-\rho)^2$,

$$u_9 = \frac{C_0}{\omega + \frac{\rho-\mu}{2}[\tanh\frac{(\mu-\rho)(\xi_1-\xi_0)}{2} - 1] + \rho},$$

$$u_{10} = \frac{C_0}{\omega + \frac{\rho-\mu}{2}[\coth\frac{(\mu-\rho)(\xi_1-\xi_0)}{2} - 1] + \rho}. \tag{25}$$

Solution VII: When $D_4 = 0, D_2D_3 < 0, F(v) = (v-\beta)^2[(v-h)^2 + s^2]$

$$u_{11} = \frac{C_0}{\omega + \dfrac{[e^{\pm\sqrt{(\beta-h)^2+s^2}(\xi-\xi_0)} - \gamma] + \sqrt{(\beta-h)^2+s^2}(2-\gamma)}{[e^{\pm\sqrt{(\beta-h)^2+s^2}(\xi-\xi_0)} - \gamma]^2 - 1}} \quad . \tag{26}$$

where

$$\gamma = \frac{\beta - 2h}{\sqrt{(\beta-h)^2+s^2}}, \quad \delta = \sqrt{(\beta-h)^2+s^2} - \frac{\beta(\beta-2h)}{\sqrt{(\beta-h)^2+s^2}} . \tag{27}$$

Solution VIII:When $D_4 < 0, D_2D_3 \geq 0$, $F(v) = (v-\mu)(v-\rho)[(v-h)^2 + s^2]$,

$$u_{12} = \frac{C_0}{\omega + \dfrac{f_1 cn\left(\dfrac{\sqrt{-2s\upsilon_1(\mu-\rho)}}{2\upsilon\upsilon_1}(\xi_1-\xi_0),\upsilon\right) + f_2}{f_3 cn\left(\dfrac{\sqrt{-2s\upsilon_1(\mu-\rho)}}{2\upsilon\upsilon_1}(\xi_1-\xi_0),\upsilon\right) + f_4}} \tag{28}$$

where

$f_1 = \frac{1}{2}(\mu-\rho)f_3 - \frac{1}{2}(\mu-\rho)f_4, f_2 = \frac{1}{2}(\mu+\rho)f_4 - \frac{1}{2}(\mu-\rho)f_3$, $f_3 = \mu - h - \frac{s}{\upsilon_1}$,

$f_4 = \mu - h - s\upsilon_1$, $G = \frac{s^2 + (\mu-h)(\rho-h)}{s(\mu-\rho)}$, $\upsilon_1 = G \pm \sqrt{G^2+1}$. $\upsilon^2 = \frac{1}{1+\upsilon_1^2}$.

4 Expression of solutions

In this section, we give concert parameters to obtain the some expression of the solutions. And we draw the graphs of some typical solutions.

Situation 1:

We take $\mu = 1, \rho = -3, C_1 = 0, \omega = 1, C_0 = -\frac{9}{2}, d_0 = \frac{48}{379}$ to get the solution

$$u_1(x,t) = \frac{5 - 10(\frac{-3\sqrt{3}}{2} - 1)^2(x+t)^2}{-2 + 8(\frac{-3\sqrt{3}}{2} - 1)^2(x+t)^2} . \tag{29}$$

We take $h = 1, l = 0, C_1 = 0, \omega = 1, C_0 = -\frac{3}{2}, d_0 = \frac{123}{108}$ to get the solution

$$u_2(x,t) = \frac{3}{-2 + 2\tan[(-\frac{\sqrt{3}}{2} - 1)(x+t)]} . \tag{30}$$

The graphs of the solutions u1 and u2 can be seen in Figure 1 and 2.

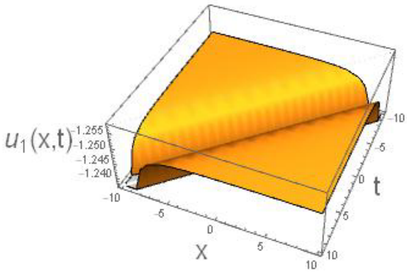

Fig. 1. Solution of . $u_1(x,t)$

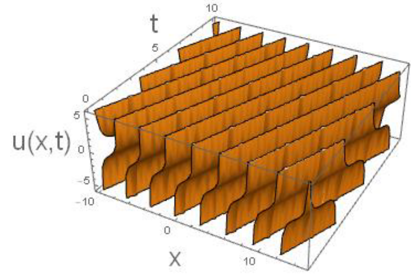

Fig. 2. Solution of $u_2(x,t)$.

Situation 2:

We take $\mu=1, \rho=-1, C_1=0, \omega=1, C_0=\frac{3}{2}, d_0=\frac{174}{11}. t_0=\frac{3^{\frac{1}{8}}}{2^{\frac{4}{3}}}, C_1=0$ to get the solutions

$$u_3(x,t)=\frac{3}{2[\coth[(\frac{\sqrt{3}}{2}-1)(x+t)-1]} . \quad (31)$$

$$u_4(x,t)=\frac{3}{2[\tanh[(\frac{\sqrt{3}}{2}-1)(x+t)-1]}$$

The graphs of the solutions u3 and u4 can be seen in Figure 3 and 4.

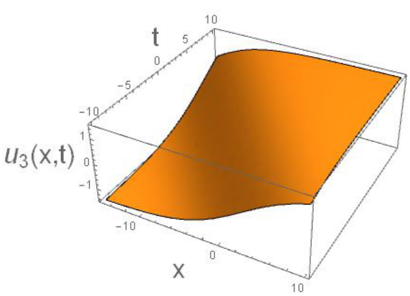

Fig. 3. solution of . $u_4(x,t)$.

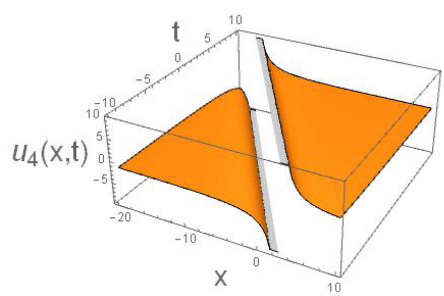

Fig. 4. solution of $u_5(x,t)$.

Situation 3 :

We take $\beta_1=-1, \beta_2=-1, \beta_3=1, \beta_4=1, \omega=1, C_1=0, C_0=-\frac{1}{2}, d_0=-\frac{24}{73}$ to get the solution

$$u_5(x,t)=-\frac{sn^2[(-\frac{\sqrt{3}}{6})(x+t),1]+1}{4} \quad (32)$$

Taking $C_1=0, d_0=\frac{96}{19}, a_1=4-\sqrt{10}$, $a_4=2, \chi=\frac{\sqrt{220+24\sqrt{10}}}{18-4\sqrt{10}}$, $a_2=\sqrt{10}, a_3=2-\sqrt{10}$,

$h_1=1, s_1=1, s_2=1, \upsilon_1=3+\sqrt{10}, \upsilon=\frac{18+6\sqrt{10}}{19+6\sqrt{10}}$ $h_2=-1, C_0=-\frac{1}{2}, \omega=1.$

The equation can be wrote

$$u_6(x,t)=\frac{1}{2-\dfrac{(8+2\sqrt{10})sn(\chi(-\frac{\sqrt{3}}{6}-1)(x+t),\upsilon)+2\sqrt{10}cn(\chi(-\frac{\sqrt{3}}{6}-1)(x+t),\upsilon)}{(2-\sqrt{10})sn(\chi(-\frac{\sqrt{3}}{6}-1)(x+t),\upsilon)+2cn(\chi(-\frac{\sqrt{3}}{6}-1)(x+t),\upsilon)}} \tag{33}$$

The graphs of the solutions u5 and u6 can be seen in Figure 5 and 6.

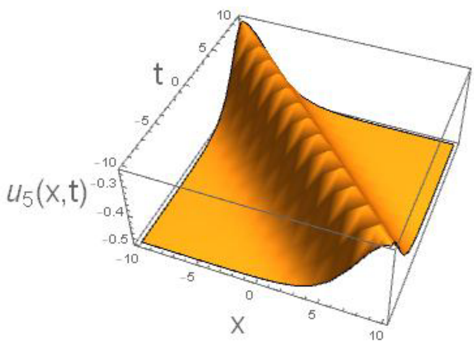

Fig. 5. solution of $u_5(x,t)$.

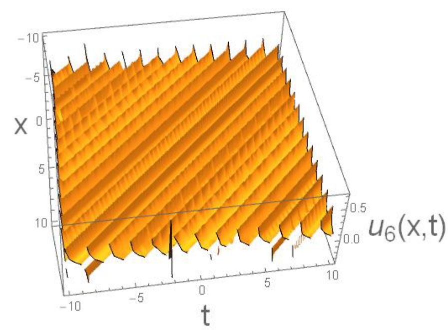

Fig. 6. solution of $u_6(x,t)$.

5 Conclusion

In this paper, we gained a series of solutions of the Green-Naghdi equa tions using the trial equation method. Besides solitary waves pattern, blow-up patterns, periodic wave patterns and periodic blow-up patterns and double pe riodic waves patterns there are two new patterns which are singular patterns and double periodic patterns. We realize them in figures with concrete param eters which help us to understand the dynamics behaviors of the Green-Naghdi model.

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