Several generalizations on Wolstenholme Theorem

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Abstract. This paper generalizes Wolstenholme theorem on two aspects. The first generalization is a parameterized form: let $p > k + 2, k \geq 1, \forall t \in \mathbb{Z}$, then

$$\frac{(pt + p - 1)!}{(pt)!} \sum_{m=0}^{k-1} (-1)^m \sum_{1 \leq i_1 < \dots < i_{k-m} \leq p-1} \frac{p^{k-(m+1)}}{\prod_{l=1}^{k-m} (pt + i_l)} \equiv 0 \pmod{p^{k+1}}.$$

The second generalization is on composite number module:

Let $\frac{1}{a}$ be the x in congruent equation $ax \equiv 1 \pmod{m} (1 \leq x < m)$,

if $m \geq 5$, then

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^2 \equiv \frac{m}{6} [2m\varphi(m) + \prod_{p|m} (1-p)] \pmod{m};$$

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^3 \equiv \frac{m^2}{4} [m\varphi(m) + \prod_{p|m} (1-p)] \pmod{m};$$

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^4 \equiv \frac{m}{30} [6m^3\varphi(m) + 10m^2 \prod_{p|m} (1-p) - \prod_{p|m} (1-p^3)] \pmod{m};$$

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^r \equiv m^r \sum_{d|m} \mu(d) \left(\frac{m}{d}\right)^{-r} \sum_{k=1}^{\frac{m}{d}} k^r \pmod{m}.$$

Where $\varphi(x)$ is Euler function, $\mu(x)$ is Möbius function.

Keywords: Wolstenholme theorem, Congruent, Composite number module, Möbius transform.

1. Introduction

Congruence modulo theory has wide applications in computer science, coding theory, and information security. It has a famous Wolstenholme theorem: if $p > 3$ is a prime number,

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then the sum of $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1}$ must have a numerator that can be evenly divided

by p^2 , or $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 0 \pmod{p^2}$.

This theorem has important applications and has been closely watched for its generalization[1-8]. Early in 1889, Leudesdorf [5] generalized Wolstenholme theorem by applying Bauer congruence: let $s_m = \sum_{\substack{k=1 \\ (k,m)=1}}^{m-1} \frac{1}{k}$ and $m > 1$, if $2 \nmid m$, $3 \nmid m$, then

$s_m \equiv 0 \pmod{m^2}$; if $2 \nmid m$, $3 \mid m$, then $s_m \equiv 0 \pmod{\frac{m^2}{3}}$. If $2 \mid m$, $3 \mid m$, then

$s_m \equiv 0 \pmod{\frac{m^2}{6}}$. If $m = 2^\alpha$ ($\alpha \geq 2$), then $s_m \equiv 0 \pmod{\frac{m^2}{4}}$. In 1995, Zeng [8] gives

a uniform expression: if $m \geq 5$, then $s_m \equiv -\frac{m^2}{12} (2m\varphi(m) + \prod_{p \mid m} (1-p)) \pmod{m^2}$.

In the following, this theorem will be generalized in two ways: a. in the case of general composite modulo; b. in the case of the same theorem with arguments.

2. Main results

2.1 Generalization in the case of composite modulo

In the following Wolstenholme theorem will be generalized in the case of general composite number m .

Definition For $m > 1$, if $(a, m) = 1$, then let the solution x of congruence equation $ax \equiv 1 \pmod{m}$ ($0 \leq x < m$) be the inverse of a modulus m , namely a^{-1} or $\frac{1}{a}$.

It can be observed that, if $(a, m) = (b, m) = 1$, then

$$(ab)^{-1} \equiv a^{-1}b^{-1} \pmod{m}, \quad a^{-1} + b^{-1} \equiv (ab)^{-1}(a+b) \pmod{m}.$$

Möbius transformation will be used to prove the following:

Theorem 2.1.1 If $m \geq 5$, then

$$(i) \quad \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^2 \equiv \frac{m}{6} [2m\varphi(m) + \prod_{p \mid m} (1-p)] \pmod{m};$$

$$(ii) \quad \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^3 \equiv \frac{m^2}{4} [m\varphi(m) + \prod_{p \mid m} (1-p)] \pmod{m};$$

$$(iii) \quad \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k}\right)^4 \equiv \frac{m}{30} [6m^3\varphi(m) + 10m^2 \prod_{p \mid m} (1-p) - \prod_{p \mid m} (1-p^3)] \pmod{m}.$$

Lemma 2.1.2 Let $s_k(n) = n^{-k} \sum_{j=1}^n j^k$, $s_k^*(n) = n^{-k} \sum_{\substack{(j,n)=1, \\ 1 \leq j \leq n}} j^k$, then

$$\sum_{d|n} s_k^*(d) = s_k(n).$$

$$\begin{aligned} \text{Proof } s_k(n) &= n^{-k} \sum_{j=1}^n j^k = n^{-k} \sum_{d|n} \sum_{\substack{(j,n)=d, \\ 1 \leq j \leq n}} j^k = \sum_{d|n} n^{-k} \sum_{\substack{(j,n)=d, \\ 1 \leq j \leq n}} j^k \\ &= \sum_{d|n} n^{-k} \sum_{\substack{(t, \frac{n}{d})=1, \\ 1 \leq t \leq \frac{n}{d}}} t^k d^k = \sum_{d|n} \left(\frac{n}{d} \right)^{-k} \sum_{\substack{(t, \frac{n}{d})=1, \\ 1 \leq t \leq \frac{n}{d}}} t^k = \sum_{d|n} s_k^* \left(\frac{n}{d} \right) = \sum_{d|n} s_k^*(d) \end{aligned}$$

The lemma is proved.

Proof of theorem 2.1.1 For any $1 \leq i < j \leq m-1$ we have $\frac{1}{i} \not\equiv \frac{1}{j}$, otherwise it leads to contradiction $0 \equiv \frac{1}{i} - \frac{1}{j} \equiv \frac{i-j}{ij} \pmod{m}$. Thus when $\frac{1}{i}$ iterates from 1 to $m-1$, $(i, m)=1$, i also iterates from 1 to $m-1$, $(i, m)=1$, thus

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k} \right)^2 \equiv \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^2 \pmod{m}.$$

Then according to lemma 2.1.2 and Möbius transformation,

$$\begin{aligned} \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^2 &= m^2 \left(m^{-2} \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^2 \right) = m^2 s_2^*(m) = m^2 \sum_{d|m} \mu(d) s_2 \left(\frac{n}{d} \right) \\ &= m^2 \sum_{d|m} \mu(d) s_2 \left(\frac{m}{d} \right) = m^2 \sum_{d|m} \mu(d) \left(\frac{m}{d} \right)^{-2} \frac{1}{6} \cdot \frac{m}{d} \left(\frac{m}{d} + 1 \right) \left(\frac{2m}{d} + 1 \right) \\ &= m^2 \sum_{d|m} \mu(d) \left(\frac{m}{3d} + \frac{1}{2} + \frac{d}{6m} \right) = m^2 \left(\frac{1}{3} \varphi(m) + 0 + \frac{1}{6m} \prod_{p|m} (1-p) \right) \\ &= m \left(\frac{m\varphi(m)}{3} + \frac{1}{6} \prod_{p|m} (1-p) \right) = \frac{m}{6} \left(2m\varphi(m) + \prod_{p|m} (1-p) \right) \end{aligned}$$

thus

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k} \right)^2 \equiv \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^2 \equiv \frac{m}{6} \left(2m\varphi(m) + \prod_{p|m} (1-p) \right) \pmod{m}.$$

Hence (i) is proved.

For the same reason

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k} \right)^3 \equiv \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^3 \pmod{m}.$$

Also according to lemma 2.1.2 and Möbius transformation,

$$\begin{aligned}
\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^3 &= m^3 \left(m^{-3} \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^3 \right) = m^3 s_3^*(m) = m^3 \sum_{d|m} \mu(d) s_3 \left(\frac{n}{d} \right) \\
&= m^3 \sum_{d|m} \mu(d) s_3 \left(\frac{m}{d} \right) = m^3 \sum_{d|m} \mu(d) \left(\frac{m}{d} \right)^{-3} \frac{1}{4} \left(\frac{m}{d} \right)^2 \left(\frac{m}{d} + 1 \right)^2 \\
&= m^3 \sum_{d|m} \mu(d) \left(\frac{m}{4d} + \frac{1}{2} + \frac{d}{4m} \right) = m^2 \left(\frac{1}{4} \varphi(m) + 0 + \frac{1}{4m} \prod_{p|m} (1-p) \right) \\
&= m^2 \left(\frac{m\varphi(m)}{4} + \frac{1}{4} \prod_{p|m} (1-p) \right) = \frac{m^2}{4} \left(m\varphi(m) + \prod_{p|m} (1-p) \right).
\end{aligned}$$

Thus

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k} \right)^3 \equiv \sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} k^3 \equiv \frac{m^2}{4} \left(m\varphi(m) + \prod_{p|m} (1-p) \right) \pmod{m}.$$

Therefore, (ii) is proved. (iii) can be proved in a similar way.

From the proof of theorem 2.1.1, for any natural number r , we can conclude:

Theorem 2.1.3 If $m \geq 5$, $r(\geq 2)$ being a natural number, then

$$\sum_{\substack{(k,m)=1, \\ 1 \leq j \leq m}} \left(\frac{1}{k} \right)^r \equiv m^r \sum_{d|m} \mu(d) s_r \left(\frac{n}{d} \right) \equiv m^r \sum_{d|m} \mu(d) \left(\frac{m}{d} \right)^{-r} \sum_{k=1}^{\frac{m}{d}} k^r \pmod{m}.$$

2.2 Generalization in a parameter form

Theorem 2.2.1 Let prime number $p > k + 2$, $k \geq 1$, $\forall t \in \mathbb{Z}$, then

$$\frac{(pt + p - 1)!}{(pt)!} \sum_{m=0}^{k-1} (-1)^m \sum_{1 \leq i_1 < \dots < i_{k-m} \leq p-1} \frac{p^{k-(m+1)}}{\prod_{l=1}^{k-m} (pt + i_l)} \equiv 0 \pmod{p^{k+1}}.$$

Lemma 2.2.2 ([7]) Let the number of solutions for congruence equation:

$$f(x) = a_n x^n + \dots + a_1 x + a_0 \equiv 0 \pmod{p}$$

Be greater than n , where p is a prime, and $a_i (i = 0, 1, 2, \dots, n)$ is an integer. Then

$$p | a_i \quad (i = 0, 1, \dots, n).$$

Lemma 2.2.3 For any prime number p , $\forall t \in \mathbb{Z}$, the coefficients of polynomial

$$f(x) = [x - (tp + 1)][x - (tp + 2)] \cdots [x - (tp + p - 1)] - x^{p-1} + 1$$

Can be evenly divided by p .

Proof Let $g(x) = [x - (tp + 1)][x - (tp + 2)] \cdots [x - (tp + p - 1)]$, then $1, 2, \dots, p - 1$ are the solutions of the congruence equation

$$g(x) \equiv 0 \pmod{p}.$$

According to Fermat's little theorem, $1, 2, \dots, p - 1$ are also the solutions of congruence equation

$$h(x) = x^{p-1} - 1 \equiv 0 \pmod{p}.$$

Thus there are $p-1$ solutions of the congruence equation

$$f(x) = g(x) - h(x) \pmod{p},$$

where $f(x)$ is a $p-2$ th order polynomial. According to lemma 2.2.2, all its coefficients can be evenly divided by p . The theorem is proved.

Proof of theorem 2.2.1 Let

$$\begin{aligned} g(x) &= [x - (tp + 1)][x - (tp + 2)] \cdots [x - (tp + p - 1)] \\ &= x^{p-1} - s_1 x^{p-2} + s_2 x^{p-3} + \cdots - s_{p-2} x + \frac{(pt + p - 1)!}{(pt)!}. \end{aligned} \quad (1)$$

where $s_j (j = 1, 2, \dots, p-2)$ is an integer, and

$$s_{p-m} = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq p-1} \frac{1}{(pt + i_1)(pt + i_2) \cdots (pt + i_{m-1})} \cdot \frac{(pt + p - 1)!}{(pt)!} \quad (2)$$

where $m = 2, 3, \dots, p-1$. According to lemma 2.2.3, $p \mid s_j (j = 1, 2, \dots, p-2)$.

Let $x = (2t + 1)p$ in (1), since $g[(2t + 1)p] = \frac{(pt + p - 1)!}{(pt)!}$, according to (1),

$$\frac{(pt + p - 1)!}{(pt)!} = [(2t + 1)p]^{p-1} - s_1 [(2t + 1)p]^{p-2} + \cdots - s_{p-2} [(2t + 1)p] + \frac{(pt + p - 1)!}{(pt)!}.$$

Hence

$$[(2t + 1)p]^{p-1} - s_1 [(2t + 1)p]^{p-2} + \cdots - s_{p-2} [(2t + 1)p] = 0.$$

Thus

$$\begin{aligned} &[(2t + 1)p]^k \{ [(2t + 1)p]^{p-k-1} - s_1 [(2t + 1)p]^{p-k-2} + \cdots + (-1)^{k-1} s_{p-k-1} \} \\ &+ (-1)^k [(2t + 1)p]^{k-1} s_{p-k} + \cdots + [(2t + 1)p]^2 s_{p-3} - [(2t + 1)p] s_{p-2} = 0. \end{aligned} \quad (3)$$

Since $p > k + 2, k \geq 1, p \mid s_j (j = 1, 2, \dots, p-2)$,

$$[(2t + 1)p]^k \{ [(2t + 1)p]^{p-k-1} - s_1 [(2t + 1)p]^{p-k-2} + \cdots + (-1)^{k-1} s_{p-k-1} \} \equiv 0 \pmod{p^{k+1}}. \quad (4)$$

According to (3) and (4),

$$(-1)^k [(2t + 1)p]^{k-1} s_{p-k} + \cdots + [(2t + 1)p]^2 s_{p-3} - [(2t + 1)p] s_{p-2} \equiv 0 \pmod{p^{k+1}}. \quad (5)$$

Theorem 2.2.1 is proved by (5) and (2).

Let $t = 0$ in theorem 2.2.1, then

Inference 2.2.4 Let prime number $p > k + 2$, then

$$(p-1)! \sum_{m=0}^{k-1} (-1)^m \sum_{1 \leq i_1 < \cdots < i_{k-m} \leq p-1} \frac{p^{k-(m+1)}}{\prod_{s=1}^{k-m} i_s} \equiv 0 \pmod{p^{k+1}}.$$

In theorem 2.2.1, $t = 0, k = 1$ is the case of Wolstenholme theorem. An extra parameter t makes the theorem more applicable and more convenient to use.

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