Some results about g-frames in Hilbert spaces

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Abstract. The concept of g-frame is a natural extension of the frame. This article mainly discusses the relationship between some special bounded linear operators and g-frames, and characterizes the properties of g-frames. In addition, according to the operator spectrum theory, the eigenvalues are introduced into the g-frame theory, and a new expression of the best frame boundary of the g-frame is given.

Keywords: g-frames, Orthogonal projection, Eigenvalues.

1 Introduction

The concept of frame in Hilbert space was proposed by Duffin and Schaeffer[1] when they studied the non-harmonic Fourier series. It is essentially a generalization of bases in Hilbert space, which can express any element of the space linearly. In particular, the framework is redundant. In other words, the way of linear representation using the framework is not necessarily unique. This nature of the framework makes it widely used. Such as image processing[2], sampling theory[3] and neural networks[4].

With the continuous in-depth study of frame theory by scholars, frame theory has been promoted in various ways. In the literature [5], Professor Sun Wenchang extended the extended analysis function of the frame from the inner product operation to the more general bounded linear operator for the first time, and obtained the concept of g-frame. At the same time, some researches are made on the nature of g-frames. This article mainly discusses the properties of g-frames with special bounded linear operators. In addition, the spectrum theory of bounded linear operators is combined with the g-frames, and the eigenvalues are introduced into the g-frame theory.

About notations: Let U , V be two separable Hilbert spaces and $\left\{V_j\right\}_{j\in J}$ be a closed subspace sequence of V . $B\left(U,V\right)$ denote a collection of all bounded linear operators from U to V . $R\left(K\right)$ and $N\left(K\right)$ are called respectively the range and the kernel of K ; K^* be the adjoint operator of K . Let I_U be a unit operator for U . Z is the set of integers, and $J\subset Z$.

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2 Preliminaries

In this part, we'll start with some essential concepts and a series of related learning. For more on how g-frames were discovered, see the article[5]. Besides, a preliminary understanding of the g-frames can be found in Han's book, see[6].

Definition2.1.[5]A sequence of operators $\left\{\Lambda_j \in B\left(U,V_j\right): j\in J\right\}$ is said to be a g-frame for Hilbert space U with respect to sequence of Hilbert spaces $\left\{V_j\right\}_{j\in J}$, if there exists two constants A and B, $0<A\le B<+\infty$,for any vector $f\in U$,

$$A\|f\|^2 \le \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \le B\|f\|^2 \tag{1}$$

The inequality (1) is called a g-frame inequality. The numbers A and B are respectively called the lower frame bound and upper frame bound. A g-frame $\left\{\Lambda_j\right\}_{j\in J}$ is called a tight g-frame if $A=B$ and a Parseval g-frame if $A=B=1$. Moreover, we call $\left\{\Lambda_j\right\}_{j\in J}$ a g-Bessel sequence when only the right inequality of (1) holds. We call $\left\{\Lambda_j\right\}_{j\in J}$ an exact g-frame if it ceases to be a g-frame whenever anyone of its elements is removed. If $\left\{f:\Lambda_j f=0, j\in J\right\}=\{0\}$, then we say that $\left\{\Lambda_j\right\}_{j\in J}$ is g-complete.

Define space $\ell^2\left(\left\{V_j\right\}_{j\in J}\right)$ as follows:

$$\ell^2\left(\left\{V_j\right\}_{j\in J}\right)=\left\{\left\{f_j\right\}_{j\in J}: f_j\in V_j, j\in J, \sum_{j\in J}\left\|f_j\right\|^2<+\infty\right\}.$$

The associated operator of g-Bessel sequence $\left\{\Lambda_j\right\}_{j\in J}$ are defined:

$$T_\Lambda: U\to \ell^2\left(\left\{V_j\right\}_{j\in J}\right),\ T_\Lambda f=\left\{\Lambda_j f\right\}_{j\in J}, \forall f\in U .$$

We call T_Λ the analysis operator of $\left\{\Lambda_j\right\}_{j\in J}$. For the adjoint operator of T_Λ,

$$T_\Lambda^*: \ell^2\left(\left\{V_j\right\}_{j\in J}\right)\to U,\ T_\Lambda^*\left\{f_j\right\}_{j\in J}=\sum_{j\in J}\Lambda_j^* f_j .$$

where any $\left\{f_j\right\}_{j\in J}\in \ell^2\left(\left\{V_j\right\}_{j\in J}\right)$. And T_Λ^* is called the synthesis operator of $\left\{\Lambda_j\right\}_{j\in J}$. Therefore, we can obtain the frame operator of $\left\{\Lambda_j\right\}_{j\in J}$, i.e. $S:U\to U$,

$$Sf=T_\Lambda^* T_\Lambda f=\sum_{j\in f}\Lambda_j^*\Lambda_j f,\ \forall f\in U .$$

It is easy to prove that S is a self-adjoint, positive, and reversible bounded linear homeomorphism. For any vector $f\in U$,

$$A\|f\|^2\le \langle Sf,f\rangle=\left\langle \sum_{j\in J}\Lambda_j^*\Lambda_j f, f\right\rangle=\sum_{j\in J}\left\langle \Lambda_j f,\Lambda_j f\right\rangle=\sum_{j\in J}\left\|\Lambda_j f\right\|^2\le B\|f\|^2 .$$

The reconstruction formula is as follows:

$$f = S^{-1}Sf = \sum_{j\in J} S^{-1}\Lambda_j^{*}\Lambda_j f = SS^{-1}f = \sum_{j\in J}\Lambda_j^{*}\Lambda_j S^{-1} f .$$

Note that $\widetilde{\Lambda_j} = \Lambda_j S^{-1}$, then $\left\{\widetilde{\Lambda_j}\right\}_{j\in J}$ is a g-frame of U with respect to $\left\{V_j\right\}_{j\in J}$, and the frame boundaries are B^{-1} and A^{-1} respectively. Call $\left\{\widetilde{\Lambda_j}\right\}_{j\in J}$ the canonical dual g-frame of $\left\{\Lambda_j\right\}_{j\in J}$.

Definition2.2. Let V_1, V_2 be two Hilbert spaces and $\Lambda \in B(V_1, V_2)$. If there exists a linear bounded operator $\Lambda^{\dagger} : R(\Lambda) \to V_1$, such that $\Lambda\Lambda^{\dagger} f = f$, $\forall f \in R(\Lambda)$, then we call $\Lambda^{\dagger}$ a pseudo-inverse of Λ.

Lemma2.3.[8] The operator Λ and its pseudo-inverse $\Lambda^{\dagger}$ satisfy the following relationship:

$$N(\Lambda^{\dagger}) = R^{\perp}(\Lambda) \ , R(\Lambda^{\dagger}) = N^{\perp}(\Lambda) .$$

Lemma2.4.[8] Let $\Lambda \in B(V_1, V_2)$. If Λ is bounded and the $R(\Lambda)$ is closed, then there exists a pseudo-inverse $\Lambda^{\dagger} : R(\Lambda) \to V_1$ such that $\Lambda\Lambda^{\dagger} f = f$, $\forall f \in R(\Lambda)$.

Definition2.5. If A is a Hermite matrix, then $R(A,x) = \dfrac{x^H A x}{x^H x}(x \neq 0)$ is the Rayleigh quotient of matrix A.

Lemma2.6.[9] If $A \in C^{n\times n}$ is a Hermite matrix and $\lambda_1 \le \lambda_2 \le \cdots \lambda_n$ is the eigenvalue of the matrix A, then

$$\min_{x\neq 0} R(A,x) = \lambda_1, \ \max_{x\neq 0} R(A,x) = \lambda_n .$$

3 Main results

This section will be expanded based on the above lemmas, and some theorems about the properties of g-frames will be obtained. A new expression for the best frame boundary.

Lemma3.1.[7] If U be a Hilbert space, and every bounded positive definite operator $S : U \to U$ has a unique bounded positive square root W. If S is self-adjoint, W is self-adjoint. If S is reversible, W is reversible. S, and $SW = WS$.

Theorem3.2. If $\left\{\Lambda_j\right\}_{j\in J}$ is a g-frame for U with respect to $\left\{V_j\right\}_{j\in J}$. The frame operator is S. $\left\{\Lambda_j S^{-1/2}\right\}_{j\in J}$ is the g-Parseval frame for U with respect to $\left\{V_j\right\}_{j\in J}$, and the frame operator is I_U.

Proof. For any $f \in U$,

$$\begin{aligned}\sum_{j\in J}\left\|\Lambda_j S^{-1/2} f\right\|^2 &= \sum_{j\in J}\left\langle \Lambda_j S^{-1/2} f, \Lambda_j S^{-1/2} f\right\rangle = \sum_{j\in J}\left\langle \Lambda_j^{*}\Lambda_j S^{-1/2} f, S^{-1/2} f\right\rangle \\ &= \left\langle SS^{-1/2} f, S^{-1/2} f\right\rangle = \left\langle \left(S^{-1/2}\right)^{*} S^{1/2} f, f\right\rangle\end{aligned}$$

Since $S^{*} = S$, according to Lemma 3.1, $\left(S^{-1/2}\right)^{*} = S^{-1/2}$.

Therefore, $\sum_{j\in J}\left\|\Lambda_j S^{-1/2} f\right\|^2=\|f\|^2$, $\left\{\Lambda_j S^{-1/2}\right\}_{j\in J}$ is a g-Parseval frame for U with respect to $\left\{V_j\right\}_{j\in J}$.

Let K be its frame operator, $\langle Kf,f\rangle=\langle f,f\rangle\Rightarrow K=I_U$. So the frame operator is I_U.

Theorem3.3. If $\left\{\Lambda_j\right\}_{j\in J}$ is a unitary operator sequence for U with respect to $\left\{V_j\right\}_{j\in J}$. $\left\{\Lambda_j\right\}_{j\in J}$ is a g-frame if and only if $0<|J|<+\infty$, where $|J|$ represents the number of elements in the set J.

Proof. For any $j\in J$, $f\in U$, $\left\|\Lambda_j f\right\|=\|f\|$,

$$\sum_{j\in J}\left\|\Lambda_j f\right\|^2=\sum_{j\in J}\|f\|^2=|J|\|f\|^2.$$

From the definition, $0<|J|<+\infty$ is a sufficient and necessary condition for $\left\{\Lambda_j\right\}_{j\in J}$ to be the g-frame for U with respect to $\left\{V_j\right\}_{j\in J}$.

Corollary3.4. Each unitary operator itself forms a g-Parseval frame.

Corollary3.5. If $\left\{\Lambda_j\right\}_{j\in J}$ is a g-frame for U with respect to $\left\{V_j\right\}_{j\in J}$. A and B are the frame boundaries. If $\left\{\Gamma_j\in B\left(V_j,W_j\right):j\in J\right\}$ is a unitary operator sequence, then $\left\{\Gamma_j\Lambda_j\right\}_{j\in J}$ is a g-frame for U with respect to $\left\{W_j\right\}_{j\in J}$. The frame boundaries are also A, B.

Proof. For any $f\in U$, $j\in J$, $\left\|\Gamma_j\Lambda_j f\right\|=\left\|\Lambda_j f\right\|$, since $\Gamma_j\Lambda_j:U\to W_j, j\in J$, and

$$A\|f\|^2\le\sum_{j\in J}\left\|\Gamma_j\Lambda_j f\right\|^2=\sum_{j\in J}\left\|\Lambda_j f\right\|^2\le B\|f\|^2$$

So $\left\{\Gamma_j\Lambda_j\right\}_{j\in J}$ is a g-frame for U with respect to $\left\{W_j\right\}_{j\in J}$. The frame boundaries are A and B.

Lemma3.6.[5] Let $\left\{V_j\right\}_{j\in J}$ be a sequence of subspaces of U and P_j be the orthogonal projection on V_j .$\left\{V_j\right\}_{j\in J}$ is called a frame of subspaces if there exist positive constants A and B, such that

$$A\|f\|^2\le\sum_{j\in J}\left\|P_j f\right\|^2\le B\|f\|^2,\forall f\in U$$

Obviously, a frame of subspaces is a g-frame for U with respect to $\left\{V_j\right\}_{j\in J}$.

Theorem 3.7. If $\left\{\Lambda_j\right\}_{j\in J}$ be the bounded linear operator sequence of U with respect to $\left\{V_j\right\}_{j\in J}$. P be the orthogonal projection from U to its closed subspace V.

(1) If $\left\{\Lambda_j\right\}_{j\in J}$ be a g-frame for U with respect to $\left\{V_j\right\}_{j\in J}$. S be the frame operator. Then $\left\{\Lambda_j P\right\}_{j\in J}$ is a g-frame for V with respect to $\left\{V_j\right\}_{j\in J}$, and the frame operator is PSP.

(2) If $\{\Lambda_j\}_{j\in J}$ be a g-frame for V with respect to $\{V_j\}_{j\in J}$, and the frame operator is S. The orthogonal projection P from U to V can be given by:

$$Pf=\sum_{j\in J}\Lambda_j^{*}\widetilde{\Lambda_j}f,\forall f\in U.$$

(3) If $\{\Lambda_j\}_{j\in J}$ be a g-frame for U with respect to $\{V_j\}_{j\in J}$, and the analysis operator is T_Λ. The orthogonal projection Q from $\ell^2\left(\{V_j\}_{j\in J}\right)$ to $R\left(T_\Lambda^{*}\right)$ can be given by the following formula:

$$Q\{f_j\}_{j\in J}=\left\{\sum_{i\in J}\widetilde{\Lambda_j}\Lambda_i^{*}f_i\right\}_{j\in J},\ \forall\{f_j\}_{j\in J}\in\ell^2\left(\{V_j\}_{j\in J}\right).$$

Proof.(1) Since P be the orthogonal projection from U to V, when $f\in V$, there is $Pf=f$; when $f\in V^{\perp}$, $Pf=0$. Given $A\|f\|^2\le\sum_{j\in J}\|\Lambda_j f\|^2\le B\|f\|^2$, $\forall f\in U$,

Therefore, $\forall f\in V\subset U$, $A\|f\|^2\le\sum_{j\in J}\|\Lambda_j Pf\|^2=\sum_{j\in J}\|\Lambda_j f\|^2\le B\|f\|^2$,

That is, $\{\Lambda_j P\}_{j\in J}$ is a g-frame for V with respect to $\{V_j\}_{j\in J}$, assuming that its frame operator is K,

$$\langle Kf,f\rangle=\left\langle\sum_{j\in J}\left(\Lambda_j P\right)^{*}\Lambda_j Pf,f\right\rangle=\left\langle\sum_{j\in J}P^{*}\Lambda_j^{*}\Lambda_j Pf,f\right\rangle=\left\langle P^{*}SPf,f\right\rangle\Rightarrow K=P^{*}SP.$$

And because of $P^{*}=P$, so $K=PSP$.

(2) $\forall f\in U=V\oplus V^{\perp}$, on the one hand, when $f\in V$,

$$Pf=\sum_{j\in J}\Lambda_j^{*}\widetilde{\Lambda_j}f=\sum_{j\in J}\Lambda_j^{*}\Lambda_j S^{-1}f=SS^{-1}f=f;$$

On the other hand, since S is a one-to-one mapping, its value range is $R\left(S^{-1}\right)=V$.

Because of $R^{\perp}\left(S^{-1}\right)=N\left(\left(S^{-1}\right)^{*}\right)=N\left(S^{-1}\right)$.

So when $f\in V^{\perp}$, $Pf=\sum_{j\in J}\Lambda_j^{*}\widetilde{\Lambda_j}f=S(0)=0$.

In summary, P is the orthogonal projection from U to V.

(3) When $\{f_j\}_{j\in J}\in R\left(T_\Lambda\right)$, there is $f\in U$ such that $T_\Lambda f=\{\Lambda_j f\}_{j\in J}=\{f_j\}_{j\in J}$.

$$Q\{f_j\}_{j\in J}=Q\{\Lambda_j f\}_{j\in J}=\left\{\sum_{i\in J}\widetilde{\Lambda_j}\Lambda_i^{*}\Lambda_i f\right\}_{j\in J}=\left\{\widetilde{\Lambda_j}\sum_{i\in J}\Lambda_i^{*}\Lambda_i f\right\}_{j\in J}$$

$$=\left\{\widetilde{\Lambda_j}Sf\right\}_{j\in J}=\{\Lambda_j f\}_{j\in J}=\{f_j\}_{j\in J}$$

When $\{f_j\}_{j\in J}\in R^{\perp}\left(T_\Lambda\right)$, because of $R^{\perp}\left(T_\Lambda\right)=N\left(T_\Lambda^{*}\right)$, so $T_\Lambda^{*}\{f_j\}_{j\in J}=\sum_{j\in J}\Lambda_j^{*}f_j=0$,

Thus $Q\{f_j\}_{j\in J}=\left\{\sum_{i\in J}\widetilde{\Lambda_j}\Lambda_i^{*}f_i\right\}_{j\in J}=\left\{\widetilde{\Lambda_j}(0)\right\}_{j\in J}=0$.

Q is the orthogonal projection from $\ell^2\left(\{V_j\}_{j\in J}\right)$ to $R\left(T_\Lambda^*\right)$.

Theorem 3.8. If $\{\Lambda_j\}_{j\in J}$ be a g-frame for U with respect to $\{V_j\}_{j\in J}$, and the analysis operator is T_Λ.

(1) There is a unique bounded operator $T_\Lambda^+ : \ell^2\left(\{V_j\}_{j\in J}\right) \to U$ that satisfies $T_\Lambda T_\Lambda^+ \{f_j\}_{j\in J} = \{f_j\}_{j\in J}$;

(2) $\{\Lambda_j T_\Lambda^+ T_\Lambda\}_{j\in J}$ is a g-frame for $R\left(T_\Lambda^+\right)$ with respect to $\{V_j\}_{j\in J}$;

(3) $\left(T_\Lambda^*\right)^+ f = \{\widetilde{\Lambda_j} f\}_{j\in J}$, $\forall f \in U$.

Proof. (1) It is easy to prove that T_Λ is a bounded operator and its range $R\left(T_\Lambda\right)$ is closed. According to lemma2.4, there is a unique bounded operator T_Λ^+. According to lemma2.4, there is a unique bounded operator T that satisfies T_Λ^+. Similarly, we can also know that the pseudo-inverse operator of T_Λ^* exists.

(2) When $f \in R\left(T_\Lambda^+\right)$, because of $T_\Lambda\left(T_\Lambda^+ T_\Lambda f\right) = T_\Lambda T_\Lambda^+\left(T_\Lambda f\right) = T_\Lambda f$, so $T_\Lambda^+ T_\Lambda f = f$.

According to lemma2.3, $R^\perp\left(T_\Lambda^+\right) = N\left(T_\Lambda\right)$. So when $f \in R^\perp\left(T_\Lambda^+\right)$, $T_\Lambda^+ T_\Lambda f = T_\Lambda^+(0) = 0$.

Thus, $T_\Lambda^+ T_\Lambda$ is the orthogonal projection from U to $R\left(T_\Lambda^+\right)$. According to theorem 3.7, it can be concluded that $\{\Lambda_j T_\Lambda^+ T_\Lambda\}_{j\in J}$ is a g-frame for $R\left(T_\Lambda^+\right)$ with respect to $\{V_j\}_{j\in J}$.

(3) $\forall f \in U$, on the one hand, $T_\Lambda^*\left(T_\Lambda^*\right)^+ f = f$;

On the other hand, $T_\Lambda^*\{\widetilde{\Lambda_j} f\}_{j\in J} = \sum_{j\in J} \Lambda_j^* \widetilde{\Lambda_j} f = SS^{-1} f = f$. So, $\left(T_\Lambda^*\right)^+ f = \{\widetilde{\Lambda_j} f\}_{j\in J}$, $\forall f \in U$.

Theorem 3.9. If $\{\Lambda_j\}_{j\in J}$ is a tight g- frame for U with respect to $\{V_j\}_{j\in J}$, and the frame bound is A. If there exists $J_1 \subset J$ such that $\sum_{j\in J} \|\Lambda_j\|^2 < A$, then $\{\Lambda_j\}_{j\in J\setminus J_1}$ is a g-frame for U with respect to $\{V_j\}_{j\in J\setminus J_1}$, and the frame bounds are $A - \sum_{j\in J} \|\Lambda_j\|^2$ and A respectively.

Proof. For any $f \in U$,

$$A\|f\|^2 = \sum_{j\in J} \|\Lambda_j f\|^2 \le \sum_{j\in J\setminus J_1} \|\Lambda_j f\|^2 + \sum_{j\in J_1} \|\Lambda_j f\|^2 \le \sum_{j\in J\setminus J_1} \|\Lambda_j f\|^2 + \|f\|^2 \sum_{j\in J_1} \|\Lambda_j\|^2 .$$

So,

$$\left(A - \sum_{j\in J_1} \|\Lambda_j\|^2\right)\|f\|^2 \le \sum_{j\in J\setminus J_1} \|\Lambda_j f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 = A\|f\|^2 .$$

Obviously, $\{\Lambda_j\}_{j\in J\setminus J_1}$ is a g-frame for U with respect to $\{V_j\}_{j\in J\setminus J_1}$, and the frame bounds are $A-\sum_{j\in J}\|\Lambda_j\|^2$ and A respectively.

If U is an n-dimensional separable Hilbert space. From linear algebra, the linear operator on it can be completely determined by its eigenvalues and the smallest polynomial. Therefore, the operator eigenvalues can be introduced into the g-frame. If $\{\Lambda_j\}_{j\in J}$ is a bounded linear operator sequence for U with respect to $\{V_j\}_{j\in J}$, then $\Lambda_j^*\Lambda_j \in C^{n\times n}\left(j\in J\right)$ is a Hermite matrix, so $\Lambda_j^*\Lambda_j\left(j\in J\right)$ has n non-negative eigenvalues.

Theorem 3.10. If U is an n-dimensional separable Hilbert space. $\{\Lambda_j\}_{j\in J}$ is a g-frame for U with respect to $\{V_j\}_{j\in J}$, and the frame bounds are A and B respectively. Let $0\le \lambda_{j,1}\le\cdots\le\lambda_{j,n}\left(j\in J\right)$ be the n non-negative eigenvalues of $\Lambda_j^*\Lambda_j\left(j\in J\right)$. If there exists $i\in J$ such that $0<\lambda_{i,1}\left(i\in J\right)$. The best frame boundaries of $\{\Lambda_j\}_{j\in J}$ can be represented by $A=\sum_{j\in J}\lambda_{j,1}, B=\sum_{j\in J}\lambda_{j,n}$.

Proof. According to Lemma 2.6, $\min_{f\neq 0} R\left(\Lambda_j^*\Lambda_j, f\right)=\lambda_{j,1}$, $\max_{f\neq 0} R\left(\Lambda_j^*\Lambda_j, f\right)=\lambda_{j,n}\left(j\in J\right)$.

And because of $R\left(\Lambda_j^*\Lambda_j, f\right)=\dfrac{f^H\Lambda_j^*\Lambda_j f}{f^H f}=\dfrac{\|\Lambda_j f\|^2}{\|f\|^2}\left(f\neq 0\right)$, $A\|f\|^2\le\sum_{j\in J}\|\Lambda_j f\|^2\le B\|f\|^2$, so

$$\sum_{j\in J} R\left(\Lambda_j^*\Lambda_j, f\right)=\frac{\sum_{j\in J}\|\Lambda_j f\|^2}{\|f\|^2}\left(f\neq 0\right).$$

Therefore, $A\le\sum_{j\in J} R\left(\Lambda_j^*\Lambda_j, f\right)\le B\left(f\neq 0\right)$.

And because of $A>0$, when there exists $0<\lambda_{i,1}\left(i\in J\right)$,

$$A=\min_{f\neq 0} R\left(\Lambda_j^*\Lambda_j, f\right)=\sum_{j\in J}\lambda_{j,1}, B=\max_{f\neq 0} R\left(\Lambda_j^*\Lambda_j, f\right)=\sum_{j\in J}\lambda_{j,n}.$$

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