Event-triggered $H_{\infty}$ state estimation for time-varying neural networks with variance-constraint and fading measurements

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Abstract. This paper addresses the event-triggered $H_{\infty}$ state estimation problem for a class of discrete recurrent neural networks subject to variance-constraint and fading measurements. The phenomena of fading measurements are described by introducing a set of mutually independent random variables, which reflect that each sensor has individual missing probability. In addition, for the purpose of energy saving, an event-triggered $H_{\infty}$ state estimation scheme is used for time-varying neural networks to determine whether the measurement output is transmitted to the estimator or not. Some sufficient conditions are obtained to guarantee that the estimation error system satisfies both estimation error variance constraint and prescribed $H_{\infty}$ performance requirement. Finally, the feasibility of the proposed event-triggered $H_{\infty}$ state estimation method is verified by a numerical example.

1 Introduction

In the past decades, the analysis and design of recursive neural networks (RNNs) have attracted great attention due to their powerful advantages, such as showing dynamic time behaviour and using internal memory to process arbitrary input sequences. Accordingly, the RNNs have been successfully applied to broad areas such as speech recognition, pattern recognition and associate memory. Hence, more and more researchers have paid attention to the state estimation problem for time varying system. In [1], an event-based recursive input and state estimator has been designed to ensure that the covariance of the estimation error has an upper bound at any time for the linear discrete time-varying systems.

Since the perfect measurements cannot be always available, especially in the unreliable network circumstances, the state estimation problems with fading measurements have aroused the extensive research. To be specific, the stochastic stability of a modified unscented Kalman filter problem has been analyzed in [2] for a class of nonlinear systems with stochastic nonlinearities and multiple fading measurements. In addition, a novel envelope-constrained performance criterion has been proposed in [3] to better quantify the transient dynamics of the filtering error process over the finite horizon. Moreover, it is worth noting that few scholars have studied the state estimation of time-varying neural networks with fading measurements.

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Different from the optimal estimation of minimum error covariance, the variance-constrained estimation method can provide a more loose technique, where the upper bound constraints are introduced to reflect the estimation accuracy. For example, in [4], the finite-horizon state estimator has been designed to provide the estimation scheme for time-varying complex networks, where both the $H_\infty$ performance requirements and prescribed variance constraints on the estimation error can be guaranteed simultaneously. On the other hand, compared with the time-triggered control scheme, the advantage of the event triggered scheme is that it can effectively realize the sharing of communication resources. For example, a new event-triggered distributed state estimation strategy has been presented in [5], which can ensure the existence of the desired estimator gains and the exponentially stability in the mean square of the estimation error dynamics simultaneously.

Motivated by the aforementioned discussion, we handle the event-triggered $H_\infty$ state estimation problem for discrete time-varying RNNs subject to variance-constraint and fading measurements. The main work conducted can be listed: 1) the event-triggered $H_\infty$ state estimation problem is, for the first time, investigated for a class of discrete time-varying stochastic RNNs subject to variance-constraint and fading measurements; 2) a new solvable method is given for addressed variance-constrained state estimation problem based on the recursive linear matrix inequalities (RLMIs); and 3) the usual literature considers the augmented system satisfying both the prescribed $H_\infty$ performance requirement and the estimation error variance constraints, but we analyse the estimation error system directly with same order of original system, which may reduce the computational complexity.

2 Problem formulation and preliminaries

In this paper, the addressed discrete time-varying neural networks are described by

$$\begin{align*}
    x_{k+1} &= (A_k + \delta_k A_k) x_k + B_k f(x_k) + C_k v_{1k} \\
    y_k &= \Lambda_k D_k x_k + E_k v_{2k} \\
    z_k &= M_k x_k
\end{align*}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^n$ depict the state vector and the measurement output, $z_k \in \mathbb{R}^m$ is the controlled output, $A_k = \text{diag}\{a_{1,k}, a_{2,k}, \ldots, a_{n,k}\}$ is the state coefficient matrix, $A_k$, $C_k$, $D_k$, $E_k$ and $M_k$ are known real matrices with appropriate dimensions, $v_{1k} \in \mathbb{R}^{n_1}$ and $v_{2k} \in \mathbb{R}^{n_2}$ are Gaussian white noises with zero mean values and covariances $V_1 > 0$ and $V_2 > 0$, respectively. $\delta_k$ is zero mean Gaussian white noise with unity covariance. $B_k = [b_{\xi k}]_{n \times n}$ is the connection weight matrix. $f(x_k)$ is the neuron activation function. The matrix $\Lambda_k = \text{diag}\{\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k\}$ describes the fading measurements phenomenon with $\lambda_i^k$ being mutually independent random variables. Here, $\Lambda_k = \text{diag}\{\lambda_1^k, \lambda_2^k, \ldots, \lambda_m^k\}$ has the probability density function $P_\xi^{(i)}(s)$ on the interval $[\alpha_i, \beta_i]$ ($0 \leq \alpha_i \leq \beta_i \leq 1$) with mathematical expectation $\mu$ and variance $\psi_i (i = 1, 2, \ldots, m)$, where $\mu_i$ and $\psi_i$ are known scalars.

Subsequently, we can define $\Lambda_k = \sum_{i=1}^{\infty} \mu_i G_i$ and $\Lambda_k = \sum_{i=1}^{\infty} \mu_i G_i$, where $G_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 being on the $i$-th term.

To save energy, we define the event generator function $f(x_k, \delta) = \xi_k^T \xi_k - \delta y_k^T y_k > 0$, where $\xi_k = y_k - y_{ij}$ and $\delta$ is the triggering threshold. The execution is triggered as long as...
the condition \( f(\xi_k, \delta) > 0 \) is satisfied. Therefore, the sequence of event-triggered instants \( 0 \leq k_0 \leq k_1 \leq \cdots \leq k_\ell \leq \cdots \) is determined iteratively by \( r_{i+1} = \inf \{ k \in \mathbb{N} \mid k > r_i, f(\xi_k, \delta) > 0 \} \).

In order to estimate the states of neurons, the following state estimator is constructed:

\[
\begin{cases}
\hat{x}_{k+1} = A_k \hat{x}_k + B_k f(\hat{x}_k) + K_k (y_{r_k} - \tilde{y}_k D_k \hat{x}_k) \\
\tilde{z}_k = M_k \hat{x}_k
\end{cases}
\]

where \( \hat{x}_k \) is the estimation of neural state \( x_k \), \( K_k \) is the estimator gain matrix to be determined. Let the estimation error be \( e_k = x_k - \hat{x}_k \) and the controlled output estimation error be \( \tilde{z}_k = z_k - \tilde{z}_k \). The estimation error dynamics can be obtained as follows:

\[
\begin{cases}
e_{k+1} = (A_k - K_k \tilde{z}_k D_k) e_k + \delta_k A_k (\hat{x}_k + e_k) + B_k f(e_k) - K_k [\tilde{z}_k D_k (\hat{x}_k + e_k) + E_k v_{2k} - \tilde{z}_k] + C_k v_{1k} \\
\tilde{z}_k = M_k e_k
\end{cases}
\]

where \( f(e_k) = f(x_k) - f(\hat{x}_k) \) and \( \tilde{z}_k = \Lambda_k - \tilde{z}_k \). Define the covariance matrix of the estimation error \( X_k = \mathbb{E}(e_k e_k^T) \).

The main purpose of this paper is to design a time-varying state estimator (2), the following two requirements are satisfied simultaneously:

(i) Let the scalar \( \gamma > 0 \), the positive define matrices \( W_\rho \) and \( W_\phi \) be given. For the initial state \( e_0 \), the estimation error \( \tilde{z}_k \) satisfies:

\[
J_1 := \mathbb{E} \left\{ \sum_{i=0}^{N-1} (\| \tilde{z}_i \|^2 - \gamma^2 \| v_{i+1} \|_{W_\rho}^2) \right\} - \gamma^2 \mathbb{E} \left\{ e_0^T W_\phi e_0 \right\} < 0
\]

where \( \| v_{i+1} \|_{W_\rho}^2 = v_{i+1}^T W_\rho v_{i+1} \).

(ii) The estimation error covariance satisfies the bounded constraint \( J_2 := \mathbb{E}(e_k e_k^T) \leq \Theta_k \), where \( \Theta_k (0 \leq k < N) \) is a set of pre-defined known matrix.

### 3 Main results

#### Theorem 1

Consider the estimation error dynamics (3), assume that the scalar \( \gamma > 0 \) and \( \lambda > 0 \), matrices \( W_\rho > 0 \) and \( W_\phi > 0 \), state estimator gain matrix \( K_k \) in (2) are given. If \( Q_0 < \gamma^2 W_\phi \) and there exists a series of positive-definite real-value matrices \( \{Q_k\}_{1 \leq k \leq N+1} \) satisfying the following recursive matrix inequality:

\[
\Psi = \begin{bmatrix} \Gamma_{11} & 0 \\ * & \Gamma_{22} \end{bmatrix} < 0
\]

where

\[
\Gamma_{11} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & -R_{22}^T \hat{x}_k \\ * & * & \Psi_{33} \end{bmatrix}, \quad \Gamma_{22} = \text{diag}(\Psi_{44}, \Psi_{55}, \Psi_{66}), \quad \Psi_{12} = A_k^T Q_{k+1} B_k - R_{2k}
\]
\[
\Psi_{11} = 4A_i^T Q_{k_i} A_i + 4D_i^T \bar{\Lambda}_i K_i Q_{k_i} \bar{\Lambda}_i D_i + 2A_i^T Q_{k_i} A_i + 2\lambda\delta D_i^T \bar{\Lambda}_i K_i D_i + 2\lambda\delta \sum_{i=1}^{m} (\psi_i^T)^2 D_i^T (G_i^T)^2 G_i D_i \\
+ 2\sum_{i=1}^{m} (\psi_i^T)^2 D_i^T (G_i^T)^2 K_i^T Q_{k_i} K_i G_i D_i + M_i^T M_i - Q_k - R_k
\]

\[
\Psi_{22} = 4B_i^T Q_{k_i} B_i - I , \quad \Psi_{44} = 4K_i^T Q_{k_i} K_i - \lambda I , \quad \Psi_{66} = E_i^T K_i^T Q_{k_i} E_i + \lambda\delta E_i^T E_i - \gamma^2 W_o
\]

\[
\Psi_{33} = 2\hat{\chi}_i^T A_i^T Q_{k_i} A_i \hat{x}_i + 2\sum_{i=1}^{m} (\psi_i^T)^2 \hat{\chi}_i^T D_i^T (G_i^T)^2 (K_i^T Q_{k_i} K_i G_i D_i \hat{x}_i + 2\lambda\delta \hat{\chi}_i^T \times D_i^T \bar{\Lambda}_i K_i D_i \hat{x}_i + 2\lambda\delta \sum_{i=1}^{m} (\psi_i^T)^2 \hat{\chi}_i^T D_i^T (G_i^T)^2 G_i D_i \hat{x}_i - \hat{\chi}_i^T R_k \hat{x}_i
\]

then, (4) can be achieved for all nonzero \( v_k \) under the initial condition.

**Theorem 2:** Consider the estimation error dynamics (3). Let the estimator gain matrix \( K_i \) in (2) be given. Under the initial condition \( P_0 = X_0 \), if there exists a set of positive-definite matrices \( \{P_i\}_{1 \leq k \leq N+1} \) satisfying:

\[
P_{k+1} \geq \Theta(P_k) \tag{6}
\]

where

\[
\Theta(P_k) = 4A_i P_k A_i^T + 4K_i \bar{\Lambda}_i D_i P_k D_i^T \bar{\Lambda}_i K_i^T + 2A_i^T \hat{x}_i \hat{\chi}_i^T A_i^T + 2A_i \hat{x}_i \hat{\chi}_i^T A_i^T + 4\text{tr}(P_k) B_k Y_k^T
\]

\[
+ 2\sum_{i=1}^{m} (\psi_i^T)^2 K_i G_i D_i P_i D_i^T (G_i^T)^2 K_i^T + 2\sum_{i=1}^{m} (\psi_i^T)^2 K_i G_i D_i P_i D_i^T (G_i^T)^2 K_i^T + K_i E_i V_i E_i^T K_i^T
\]

\[
+ 4\delta \text{tr}(\bar{\Lambda}_i D_i \hat{x}_i \hat{\chi}_i^T D_i^T (\bar{\Lambda}_i^T)) K_i K_i^T + 4\delta \text{tr}(\bar{\Lambda}_i D_i \hat{x}_i \hat{\chi}_i^T D_i^T (\bar{\Lambda}_i^T)) K_i K_i^T + 4\delta \text{tr}(\bar{\Lambda}_i D_i \hat{x}_i \hat{\chi}_i^T D_i^T (\bar{\Lambda}_i^T)) K_i K_i^T
\]

\[
+ 4\delta \text{tr}(\sum_{i=1}^{m} (\psi_i^T)^2 G_i^T D_i \hat{x}_i \hat{\chi}_i^T D_i^T (G_i^T)^2) K_i K_i^T + 4\delta \text{tr}(\sum_{i=1}^{m} (\psi_i^T)^2 G_i^T D_i \hat{x}_i \hat{\chi}_i^T D_i^T (G_i^T)^2) K_i K_i^T
\]

\[
+ 4\delta \text{tr}(\sum_{i=1}^{m} (\psi_i^T)^2 G_i^T D_i \hat{x}_i \hat{\chi}_i^T D_i^T (G_i^T)^2) K_i K_i^T + 4\delta \text{tr}(E_i V_i E_i^T) + C_i V_i C_i^T
\]

\[
Y = (\rho + \rho^{-1})U_j^2/2(1 - \rho) + U_j^2/\rho(1 - \rho)
\]

and other elements are defined in Theorem 1, then we have \( P_k \geq X_k \) (\( \forall k \in \{1,2,\ldots,N+1\} \)).

**Theorem 3:** Given the scalar \( \gamma \), matrices \( W > 0 \), \( W > 0 \), and a set of pre-defined variance upper bound matrices \( \{\Theta_k\}_{0 \leq k \leq N+1} \), under the initial conditions \( \{Q_0 \leq \gamma^2 W_o \}, \{E[e_i e_i^T] = P_0 \leq \Theta_0 \} \), if there exist the following series of positive definite matrices \( \{Q_k\}_{0 \leq k \leq N+1}, \{P_k\}_{0 \leq k \leq N+1} \) and matrix \( \{K_k\}_{0 \leq k \leq N} \) with appropriate dimensions satisfying the following RLMIs:

\[
\begin{bmatrix}
\Xi_{11} & \Xi_{12} \\
* & \Pi_{22}
\end{bmatrix} < 0
\tag{7}
\]

\[
\begin{bmatrix}
-P_{k+1} & Y_{12} \\
* & Y_{22}
\end{bmatrix} < 0
\tag{8}
\]

\[
G_{2,k+1} - \Theta_{k+1} \leq 0
\tag{9}
\]
with the parameter updated by $\Qbar_k = Q_k^{-1}$, where

$$\Xi_{11} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \ast & \Pi_{22} & \Pi_{23} \end{bmatrix}, \quad \Xi_{12} = \begin{bmatrix} \Pi_{14} & \Pi_{15} & 0 & 0 & 0 \\ 0 & 0 & \Pi_{26} & \Pi_{27} & 0 \end{bmatrix}, \quad \Xi_{22} = \text{diag}(\Pi_{44}, \Pi_{55}, \Pi_{66}, \Pi_{77}, \Pi_{88})$$

$$Y_{12} = [\Phi_{12}, \Phi_{13}, \Phi_{14}, \Phi_{15}, \Phi_{16}, \Phi_{17}, \Phi_{18}, \Phi_{19}],$$

$$Y_{22} = \text{diag}(\Phi_{22}, \Phi_{23}, \Phi_{44}, \Phi_{55}, \Phi_{66}, \Phi_{77}, \Phi_{88}, \Phi_{99})$$

$$\Pi_{11} = -R_{k-1} - Q_k, \quad \Pi_{12} = [\ast -R_{k-1} - \lambda \hat{d}_k], \quad \Pi_{13} = [0 \ 0 \ 0 \ A_k^T \ A_k^T], \quad \Pi_{35} = \text{diag}(-I, -\Qbar_{k+1}, -I)$$

$$\Pi_{14} = \sqrt{\lambda \delta} \sum_{j=i}^m \psi_j D_k^T (G_k^j)^T K_k + \sqrt{2 \lambda \delta} \sum_{i=1}^m \psi_i \xi_i D_k^T (G_k^i)^T M_k^T$$

$$\Pi_{22} = \begin{bmatrix} -I & -2R_{k-1}^T \hat{d}_k \\ \ast & -\lambda \hat{d}_k R_{k-1} \hat{d}_k \end{bmatrix}, \quad \Pi_{23} = \begin{bmatrix} 0 & 0 & 0 & B_k^T \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_{38} = \begin{bmatrix} 2K_k^T \quad 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_{39} = \begin{bmatrix} 2A_k \quad 0 \\ 0 & 0 \end{bmatrix}$$

$$\Phi_{12} = \begin{bmatrix} 2A_k & 2K_k & \hat{d}_k & P_k \end{bmatrix} \sqrt{2A_k \hat{d}_k}$$

$$\Phi_{14} = \begin{bmatrix} \sqrt{2 \lambda \delta} \sum_{j=i}^m \psi_j K_k G_k^j D_k P_k \\ 2 \sqrt{\lambda \delta} \sum_{j=i}^m \psi_j K_k G_k^j D_k \hat{d}_k \\ 2 \sqrt{\lambda \delta} \sum_{j=i}^m \psi_j K_k G_k^j D_k \hat{d}_k \hat{d}_k \end{bmatrix}$$

$$\Phi_{16} = \begin{bmatrix} \sqrt{2 \lambda \delta} \sum_{j=i}^m \psi_j K_k G_k^j D_k \hat{d}_k \hat{d}_k \\ 0 \end{bmatrix}, \quad \Pi_{18} = \text{diag}(-\Qbar_{k+1}, -\Qbar_{k+1}, -\Qbar_{k+1}, -I)$$

$$\Phi_{18} = \begin{bmatrix} \sqrt{2 \lambda \delta} \sum_{j=i}^m \psi_j K_k G_k^j D_k \hat{d}_k \hat{d}_k \\ 0 \end{bmatrix}, \quad \Pi_{38} = \begin{bmatrix} 2K_k^T \quad 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_{39} = \begin{bmatrix} 2A_k \quad \text{tr}(P_k) \end{bmatrix} \sqrt{2A_k \text{tr}(P_k)}$$

$$\Phi_{44} = \text{diag}(-P_k, -\text{tr}(P_k), -P_k, -\text{tr}(P_k), -I)$$

$$\Phi_{48} = \text{diag}(-\text{tr}(\hat{d}_k D_k P_k D_k^T \hat{d}_k), -\text{tr}(\hat{d}_k D_k P_k D_k^T \hat{d}_k), -\text{tr}(\hat{d}_k D_k P_k D_k^T \hat{d}_k), -\text{tr}(\hat{d}_k D_k P_k D_k^T \hat{d}_k), -I)$$

$$\Phi_{66} = \text{diag}(-\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k), -\text{tr}(\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k)))$$

$$\Phi_{77} = \text{diag}(-\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k), -\text{tr}(\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k)))$$

$$\Phi_{88} = \text{diag}(-\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k), -\text{tr}(\text{tr}(\hat{d}_k D_k e_k \xi_k D_k^T \hat{d}_k)))$$
then, the $H_\infty$ performance requirements and the upper constraints of estimation error covariance can be satisfied simultaneously.

4 A numerical example

In this section, the parameters of time-varying RNNs (1) are given as follows:

$$A_k = \begin{bmatrix} -0.1\sin(2k) & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_k^r = \begin{bmatrix} -0.2 & 0.2 \\ -0.1 & -0.1\sin(2k) \end{bmatrix}, \quad B_k = \begin{bmatrix} -0.1 & -0.14 \end{bmatrix}$$

$$C_k = \begin{bmatrix} -0.1 & -0.3\sin(2k) \end{bmatrix}^T, \quad D_k = \begin{bmatrix} -0.5\sin(k) & 1.5 \\ -0.1 & -0.14 \end{bmatrix}, \quad E_k = \begin{bmatrix} -0.27\sin(2k) & 0.15 \end{bmatrix}^T$$

$$M_k = \begin{bmatrix} -0.02 & -0.08\sin(3k) \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda = 0.01, \quad \delta = 0.6, \quad \rho = 0.45$$

In addition, the activation functions can be taken as $f(x_k) = \begin{bmatrix} \tanh(x_{1,k}) \tanh(0.8x_{2,k}) \end{bmatrix}^T$ with $x_k = \begin{bmatrix} x_{1,k} \ x_{2,k} \end{bmatrix}^T$. It is easy to obtain that $\Sigma_o = \text{diag}(0.2,0.2), \Sigma_1 = \text{diag}(0.3,0.3)$ and $\Sigma_2 = \text{diag}(0.5,0.5)$. Let the disturbance attenuation level $\gamma = 0.9$ and $N = 100$, the weighted matrices $W_o(i) = W_o(2) = 1$, upper bound matrices $\{\Theta_k\}_{i \in \mathbb{N}} = \text{diag}(0.3,0.3)$ and covariances as $V_1 = V_2 = 1$. Then, by solving RLMIs (7)-(9), $K_k$ is designed as follows:

$$K_1 = \begin{bmatrix} -0.0013 & 0.0954 \\ 0.0970 & 0.0984 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0266 & 0.0878 \\ 0.0811 & 0.0827 \end{bmatrix}, \quad K_3 = \begin{bmatrix} -0.0145 & 0.0898 \\ 0.1077 & 0.0786 \end{bmatrix}$$

![Fig. 1.](image-url) (a) the controlled output $z_{1,k}, z_{2,k}$ and their estimations; (b) $X_k$ and its upper bound.

Set the initial states $x_0 = \begin{bmatrix} -1.2 & 0.3 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} -0.2 & 0.6 \end{bmatrix}^T$. Let $\mu_k^1 = 0.55, \mu_k^2 = 0.65, \psi_k^1 = 0.3905$ and $\psi_k^2 = 0.4153$. In addition, the simulation results are shown in Fig. 1, which illustrate that the proposed event-triggered $H_\infty$ state estimation method is effective.

5 Conclusion

In this paper, we have discussed the event-triggered $H_\infty$ state estimation problem for a class of discrete RNNs subject to variance-constraint and fading measurements. The phenomena of fading measurements have been described by introducing a set of mutually independent random variables. Applying the RLM technique, some criteria have been established to
guarantee the prescribed $H_\infty$ performance and the estimation error covariance constraints of the estimation error system. Finally, the feasibility of the proposed method has been verified by a numerical example.

This work was supported in part by the National Natural Science Foundation of China under Grants 12071102.

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