The Signed $(|G|-1)$ subdomination number of Product Graphs

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Abstract. In this paper, the singed $(|G|-1)$ - subdomination number of product graphs G, such as $P_2 \times P_n$, $P_2 \times C_n$, $P_3 \times P_n$, are determined by classified discussion and exhaustived method.

1 Introduction

For terminology and notation not defined here we refer to [1].

Let graph $G=(V,E)$ with vertex set V and edge set E, the order of G is denoted by $|G|$. For a vertex $u \in V(G)$, the open neighborhood of u is $N_G(u) = \{v \in (G) | uv \in E(G)\}$, and the closed neighborhood of u is $N_G[u] = N_G(u) \cup \{u\}$. $N_G(u)$ and $N_G[u]$ are abbreviated by $N(u)$ and $N[u]$ when no confusion is possible.

Given graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. The Caresian product $G_1 \times G_2$ of two graphs G_1 and G_2 , is a graph with vertex set $V(G_1 \times G_2)=V_1 \times V_2$, edge set $E(G_1 \times G_2)=\{(u_1,v_1)(u_2,v_2) | u_1=u_2 \text{ and } v_1v_2 \in E(G_2), or\ v_1=v_2 \text{ and } u_1u_2 \in E(G_1)\}$. Let $f: V \to \{-1,1\}$, we define $f[u]=\sum_{v \in N[u]} f(v)$. If for every $u \in V(G)$, where $f[u] \geq 1$, then the function f is called a signed dominating function of G. $f(V)$ is defined as $f(V)=\sum_{u \in V} f(u)$. The signed domination number of graph G is defined as $\gamma_s(G)=\min\{f(V) | f \text{ is a signed dominating function of } G\}$. For an integer $1 \leq k \leq |G|$, a signed k -subdominating function of G is a function $f: V \to \{-1,1\}$ such that $f[u] \geq 1$

For at least k vertices u of $V(G)$. The signed k -subdomination number of graph G is defined as $\gamma_s^k(G)=\min\{f(V) | f \text{ is a signed k-subdominating function of } G$.

Since the signed domination number of graphs has a wide application background, for example, the establishment of transportation posts and material supply points, etc., and the

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calculation of the k - subdomination number is NP complete problem, so the study of domination number of graphs has positive significance.

We number the vertices $u_i(i=1,2,\cdots,m)$ of a path P_m from left to right, number the vertices $v_j(j=1,2,\cdots,n)$ of a cycle C_n counter clockwise from vertex v_1. In this paper, we are interested in signed $(2n-1)$ - subdomination number of product graph $P_2\times P_n$ and $P_2\times C_n$, signed $(3n-1)$ - subdomination number of product graph $P_3\times P_n$.

2 Signed $(2n-1)$ - subdomination number of graph $P_2\times P_n$

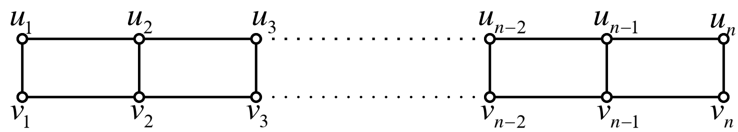

Fig. 1. Graph $P_2\times P_n$.

Let f be a signed $(2n-1)$ - subdominating function of G. We have the following results.

Lemma 2.1 *If vertex* $u\in P_2\times P_n$ *, where* $f[u]\le 0$ *, then:*

(1) u *can't be* u_1 *,* v_1 *,* u_n *or* v_n .

(2) If $u=u_i(2\le i\le n-1)$ *, then* $f(u_i)=1, f(u_{i-1})=f(u_{i+1})=-1$ *, and* $f(v_i)=1$ *.Similarly,* $u=v_i(2\le i\le n-1)$.

Proof. (1) Suppose $u=u_1$, where $f[u_1]\le 0$.

Case 1. If $f(u_1)=-1$, then $f(v_1)=f(u_2)=1$. Otherwise, if $f(v_1)=-1$, then $f[v_1]=f(u_1)+f(v_1)+f(v_2)\le -1$, contradicting the fact that f is a signed $(2n-1)$ - subdominating function of $P_2\times P_n$. So $f(v_1)=1$. Similarly, $f(u_2)=1$. Then, we have $f[u_1]=f(u_1)+f(v_1)+f(u_2)=1$, it is a contradiction.

Case 2. If $f(u_1)=1$. To make $f[u_1]=f(u_1)+f(v_1)+f(u_2)\le 0$, then $f(v_1)=f(u_2)=-1$, so $f[v_2]=f(v_1)+f(v_2)+f(v_3)+f(u_2)\le 0$, contradicting the fact that f is a signed $(2n-1)$ - subdominating function of $P_2\times P_n$.

Conclusion of Case 1 and 2, u can't be u_1 , v_1 , u_n or v_n .

(2) Let $u=u_i$, where $f[u_i]\le 0$ $(2\le i\le n-1)$. Similarly as Case 1 in (1), we have $f(u_i)=1$. Since $f[u_i]=f(u_{i-1})+f(u_i)+f(u_{i+1})+f(v_i)\le 0$, so there are at least two vertices of $\{u_{i-1}, u_{i+1}, v_i\}$, which are adjacent to u_i , whose values are assigned -1.

Case 1. If $f(u_{i-1})=f(u_{i+1})=f(v_i)=-1$, we have $f[v_{i-1}]=f(v_{i-2})+f(v_{i-1})+f(v_i)+f(u_{i-1})\le 0$ and $f[v_{i+1}]=f(v_i)+f(v_{i+1})+f(v_{i+2})+f(u_{i+1})\le 0$, contradicting the fact that f is a signed $(2n-1)$ - subdominating function of $P_2\times P_n$.

Case 2. If $f(u_{i-1})=1$, $f(u_{i+1})=f(v_i)=-1$, we have $f[v_{i+1}]=f(v_i)+f(v_{i+1})+f(v_{i+2})+f(u_{i+1})\le 0$, contradicting the fact that f is a signed $(2n-1)$ - subdominating function of $P_2\times P_n$. Similarly, if $f(u_{i-1})=f(v_i)=-1$, $f(u_{i+1})=1$, a contradiction too.

In conclusion, the result is true.

Next, we give the result about the singed $(2n-1)$ -subdomintion number of graph $P_2\times P_n$.

Theorem 2.2 *For any integer* n $(n\ge 3)$, $\gamma_s^{2n-1}(P_2\times P_n)=2n-2[\frac{2n+3}{4}]$.

Proof. Note that, if $n=2$, there are only four vertices in graph $P_2\times P_2$. By Lemma 2.1, all of them cannot be u, where $f[u]\le 0$. Therefore, it is only when $n\ge 3$ that we can define the signed $(2n-1)$ -dominating function of graph $P_2\times P_n$.

Let f be the minimal signed $(2n-1)$ - subdominating function, that is, $f(P_2\times P_n)=\gamma_s^{2n-1}(P_2\times P_n)$, denote $P=\{v\in P_2\times P_n | f(v)=1\}$ and $M=\{v\in P_2\times P_n | f(v)=-1\}$, the order of them is denoted by $p=|P|, m=|M|$, where $p+m=2n$. The number of edges between two sets is denoted by $e(P,M)$.

Notice that, f is a signed $(2n-1)$ - subdominating function of $P_2\times P_n$. According to Lemma 2.1, only one vertex of P is adjacent to two vertices of M, the last $p-1$ vertices are adjacent to at most one vertex of M. So $e(P,M)\le p+1$. For a vertex of M, if it is located at a corner, then it must be adjacent to two vertices in P, otherwise it is adjacent to three vertices in P. It implies that $e(P,M)\ge 3m-2$. Since $3m-2\le e(P,M)\le p+1$, so $2n=p+m\ge 4m-3$, then we have $m\le\frac{2n+3}{4}$, namely $m\le[\frac{2n+3}{4}]$. Then, we have $\gamma_s^{2n-1}(P_2\times P_n)=2n-2m\ge 2n-2[\frac{2n+3}{4}]$.

Otherwise, the signed $(2n-1)$ -subdominating function g of $P_2\times P_n$ is defined as follows:

(i) If $n=4k-1$, let $A=\{u_1,u_3,v_5,\cdots,u_{4i-1},v_{4i+1},\cdots,u_n\}$, it implies that $|A|=\frac{n-1}{2}+1=2k=[\frac{2n+3}{4}]$;

(ii) If $n=4k$, let $A=\{u_1,u_3,v_5,\cdots,u_{4i-1},v_{4i+1},\cdots,u_{n-1}\}$, it implies that $|A|=\frac{n}{2}=2k=[\frac{2n+3}{4}]$;

(iii) If $n=4k+1$, let $A=\{u_1,u_3,v_5,\cdots,u_{4i-1},v_{4i+1},\cdots,v_n\}$, it implies that $|A|=\frac{n-1}{2}+1=2k+1=[\frac{2n+3}{4}]$;

(iv) If $n=4k+2$, let $A=\{u_1,u_3,v_5,\cdots,u_{4i-1},v_{4i+1},\cdots,v_{n-1}\}$, it implies that $|A|=\frac{n}{2}=2k+1=[\frac{2n+3}{4}]$.

Let $g(v)=\begin{cases}-1, & \text{if } v\in A;\\ 1, & \text{otherwise,}\end{cases}$ then $\gamma_s^{2n-1}(P_2\times P_n)\le 2n-2[\frac{2n+3}{4}]$.

Consequently, the equality holds.

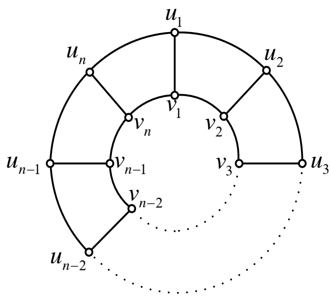

Fig. 2. Graph $P_2 \times C_n$.

3 Signed $(2n-1)$ - subdomination number of graph $P_2 \times C_n$

Theorem 3.1 *For any integer* n $(n \geq 5)$,

$$\gamma_s^{2n-1}(P_2 \times C_n) = \begin{cases} n+2, & \text{if } n \equiv 0(\text{mod}4); \\ 2n-2[\frac{n}{2}], & \text{otherwise.} \end{cases}$$

Proof. In graph $P_2 \times C_n$, every vertex is adjacent to three vertices. Let f be the minimal signed $(2n-1)$ - subdominating function, that is, $f(P_2 \times C_n) = \gamma_s^{2n-1}(P_2 \times C_n)$. Suppose that $u_i \in P_2 \times C_n$, where $f[u_i] \leq 0$. It follows from a same argument as that for Lemma 2.1 that $f(u_i)=1, f(u_{i-1})=f(u_{i+1})=-1, f(v_{i-1})=f(v_i)=f(v_{i+1})=1$, where, the index i is understood to be taken modulo n . Moreover, to ensure that there exsists a unique $u_i \in P_n \times C_n$ such that $f\left[u_i\right] \leq 0$, there must be at least two vertices u', u'' , lying between u_{i-1} and u_{i+1} but not along the direction of u_i , and two corresponding vertices v', v'' such that $f(u')=f(u'')=f(v')=f(v'')=1$. Therefor, we define the signed $(2n-1)$ -subdominating function of graph $P_2 \times C_n$ with $n \geq 5$.

Denote $P=\{v \in P_2 \times C_n | \ f(v)=1\}$ and $M=\{v \in P_2 \times C_n | \ f(v)=-1\}$, the order of them is denoted by $p=|P|, m=|M|$, where $p+m=2n$. The number of edges between two sets is denoted by $e(P,M)$. In M , every vertex is adjacent to three vertices of P , so $e(P,M) \geq 3m$. In P , only one vertex is adjacent to two vertices of M , the last $p-1$ vertices are adjacent to one vertex of M , so $e(P,M) \leq p+1$, then $3m \leq e(P,M) \leq p+1$, so $2n=m+p \geq 4m-1$, $m \leq \frac{n}{2}+\frac{1}{4}$, therefore $m \leq [\frac{n}{2}]$.

Next, we will prove that $m \leq [\frac{n}{2}]-1$ with $n=4k$ $(k \geq 2)$.

Proof by contradiction. Suppose $m > [\frac{n}{2}] - 1$, but the integer m, where $m \le [\frac{n}{2}]$. So $m = [\frac{n}{2}] = 2k (k \ge 2)$. Let $f[u_i] \le 0$, that $f(u_i) = 1$, $f(u_{i-1}) = f(u_{i+1}) = -1$, $f(u_j) = f(v_k) = 1$ $(j = i \pm 2, i \pm 3; k = i, i \pm 1, i \pm 2)$, where, the index is understood to be taken modulo n.

Let set $B = \{u_{i+4}, v_{i+3}, v_{i+4}, v_{i+5}\}$, that there is only one vetex v, where $f(v) = -1$. If there is $f(v) = 1$ for any vertex $v \in B$, then, for v_{i+3} and the adjacent three vertices u_{i+3}, v_{i+2} and v_{i+4}, that $f[v_{i+3}] = f[u_{i+3}] = f[v_{i+2}] = f[v_{i+4}] = 4$, contrading the fact that f is the minimal signed $(2n-1)$- subdominating function. Furthermore, if there is $f(v') = f(v'') = -1$ for $v', v'' \in B$, it implies that $f[u_{i+4}] \le 0$, contrading the fact that f is the minimal signed $(2n-1)$- subdominating function. Therefor, there is only one vertex $v \in B$, where $f(v) = -1$.

Let set $A_i = \{u_i, v_i\}$ $(1 \le i \le n)$. In [3], there is a result:

$$|(A_i \cup A_{i+1} \cup \cdots \cup A_{i+t}) \cap M| \le [\frac{t+1}{2}] \tag{3.1}$$

Discuss the following:

Case 1. If $f(v_{i+5}) = -1$, then $f(u_{i+5}) = f(u_{i+6}) = f(v_{i+6}) = 1$. Denote $V_1 = (P_2 \times C_n) / \bigcup_{j=i-2}^{i+6} A_j$. By formula (3.1), there are at most $[\frac{n-9+1}{2}] = 2k-4$ vertices in V_1, whose value are assigned -1. With $u_{i\pm1}$ and v_{i+5}, there are at most $2k-1$ vertices in $P_2 \times C_n$ whose value are assigned -1, a contradiction.

Case 2. If $f(u_{i+4}) = -1$ or $f(v_{i+4}) = -1$, then $f(u_{i+5}) = f(v_{i+5}) = 1$. Denote $V_2 = (P_2 \times C_n) / \bigcup_{j=i-2}^{i+5} A_j$. By formula (3.1), there are at most $[\frac{n-8+1}{2}] = 2k-4$ vertices in V_2, whose value are assigned -1. With $u_{i\pm1}$ and u_{i+4} (or v_{i+4}), there are at most $2k-1$ vertices in $P_2 \times C_n$ that sign -1, a contradiction.

Case 3. If $f(v_{i+3}) = -1$. Denote $V_3 = (P_2 \times C_n) / \bigcup_{j=i-2}^{i+2} A_j$. By formula (3.1), there are at most $[\frac{n-5+1}{2}] = 2k-2$ vertices in V_3, whose value are assigned -1. In [4], to make $2k-2$ vertices in V_3 whose value are assigned -1, if and only if value -1 are assigned at the vertices $u_{i+5}, v_{i+7}, u_{i+9}, v_{i+11}, \cdots$ The remaining vertices are assigned value 1, followed by the same rules. Because there are $n-5 \equiv 3 (\mathrm{mod} 4)$ vertices in V_3, it implies $f(u_{i-3}) = -1$, so $f[u_{i-2}] = f(u_{i-1}) + f(u_{i-2}) + f(u_{i-3}) + f(v_{i-2}) = 0$. So there are at most $2k-3$ vertices in V_3 , whose value are assigned -1. Thus, there are at most $2k-1$ vertices in $P_2 \times C_n$ t whose value are assigned -1, a contradiction.

Based on the above claims, if $n = 4k \ \ (k \geq 2)$, where $m \leq [\frac{n}{2}] - 1 = \frac{n}{2} - 1$. So we have $\gamma_s^{2n-1}(P_2 \times C_n) = 2n - 2m \geq \begin{cases} n+2, & \text{if } n \equiv 0(\text{mod}4); \\ 2n - 2[\frac{n}{2}], & \text{otherwise.} \end{cases}$

Furthermore, the $(2n-1)$ -subdominating function g of $P_2 \times C_n$ is defined as follows:

(i) If $n = 4k+1$, let $A = \{u_1, u_3, v_5, \cdots, u_{4i-1}, v_{4i+1}, \cdots, u_{n-2}\}$, then $|A| = 2k = [\frac{n}{2}]$;

(ii) If $n = 4k+2$, let $A = \{u_1, u_3, v_5, \cdots, u_{4i-1}, v_{4i+1}, \cdots, v_{n-1}\}$, then $|A| = 2k+1 = [\frac{n}{2}]$;

(iii) If $n = 4k+3$, let $A = \{u_1, u_3, v_5, \cdots, u_{4i-1}, v_{4i+1}, \cdots, v_{n-2}\}$, then $|A| = 2k+1 = [\frac{n}{2}]$;

(iv) If $n = 4k$, let $A = \{u_1, u_3, v_5, \cdots, u_{4i-1}, v_{4i+1}, \cdots, v_{n-3}\}$, then $|A| = 2k-1 = \frac{n}{2} - 1$.

Let $g(v) = \begin{cases} -1, & \text{if v} \in \text{A}; \\ 1, & \text{otherwise,} \end{cases}$ then we have

$$\gamma_s^{2n-1}(P_2 \times C_n) \leq \begin{cases} n+2, & \text{if } n \equiv 0(\text{mod}4); \\ 2n - 2[\frac{n}{2}], & \text{otherwise.} \end{cases}$$

Consequently, the equality holds.

4 Signed $(3n-1)$ - subdomination number of graph $P_3 \times P_n$

Graph $P_3 \times P_n$ is a grid diagram with three rows and n columns.(see Figure 3) The vertex sets on the first, second, and third rows are denoted as $U = \{u_1, u_2, \cdots, u_n\}$, $V = \{v_1, v_2, \cdots, v_n\}$ and $W = \{w_1, w_2, \cdots, w_n\}$. The column i is denoted as $A_i = \{u_i, v_i, w_i\} (1 \leq i \leq n)$.

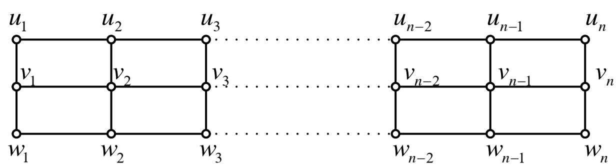

Figure 3. Graph $P_3 \times P_n$.

Suppose that $v \in P_3 \times P_n$, where $f[v] \le 0$. Because this graph is symmetric, for vertex v, we have the following results:

(1) If $v = u_1$, then $f(u_1) = 1$, $f(u_2) = f(v_1) = -1$;

(2) If $v = v_1$, then $f(v_1) = 1$, $f(u_1) = f(w_1) = -1$;

(3) If $v = u_i$ $(2 \le i \le n-1)$, then $f(u_i) = 1, f(u_{i-1}) = f(u_{i+1}) = -1, f(v_i) = 1$;

Or $v = u_i (2 \le i \le n-2)$, then $f(u_i) = 1, f(u_{i-1}) = 1, f(v_i) = f(u_{i+1}) = -1$;

Or $v = u_i (3 \le i \le n-1)$, then $f(u_i) = 1, f(u_{i-1}) = f(v_i) = -1, f(u_{i+1}) = 1$;

Or $v = u_i (2 \le i \le n-1)$, then $f(u_i) = f(v_i) = -1, f(u_{i-1}) = f(u_{i+1}) = 1$;

(4) v cannot be v_2;

(5) If $v = u_i (3 \le i \le n-2)$, then $f(u_i) = 1, f(u_{i-1}) = f(u_{i+1}) = f(v_i) = -1$;

(6) If $v = v_3 (3 \le i \le n-2)$, then $f(v_{i-1}) = f(v_i) = f(v_{i+1}) = -1$, $f(u_i) = f(w_i) = 1$.

Consequently, we have the following lemma.

Lemma 4.1 (1) $\gamma_s^{3n-1}(P_3 \times P_3) = 3$;

(2) $\gamma_s^{3n-1}(P_3 \times P_5) = 5$;

(3) $\gamma_s^{3n-1}(P_3 \times P_8) = 10$.

Theorem 4.2 *For any integer* $n \ge 2$ *and* $n \ne 3,5,8$, $\gamma_s^{3n-1}(P_3 \times P_n) = n + 2\left\lceil \frac{n-2}{5} \right\rceil$.

Proof. Let vertex set $P = \{v \in P_3 \times P_n | f(v) = 1\}$ and $M = \{v \in P_3 \times P_n | f(v) = -1\}$, the order of them is denoted by $p = |P|, m = |M|$, where $p + m = 3n$. We use mathematical induction for n.

If n is $2,4,6,7,9,10,11,12,13$, we have the $(3n-1)$ - subdomination numbers of $P_3 \times P_n$ are $2,3,5,6,7,8,9,10,10$, all of them satisfy $m \le n - \left\lceil \frac{n-2}{5} \right\rceil$.

Suppose $n = l \ge 14$, where $m \le l - \left\lceil \frac{l-2}{5} \right\rceil$.

If $n = l + 5$. In [4], we know that $|(A_{l+1} \cup A_{l+2} \cup \cdots \cup A_{l+5}) \cap M| \le 4$, note that $|(A_1 \cup A_2 \cup \cdots \cup A_l) \cap M| \le l - \left\lceil \frac{l-2}{5} \right\rceil$, we have

$$m = |M| \le 4 + l - \left\lceil \frac{l-2}{5} \right\rceil = l + 5 - \left\lceil \frac{l+5-2}{5} \right\rceil = n - \left\lceil \frac{n-2}{5} \right\rceil.$$

So, $\gamma_s^{3n-1}(P_3 \times P_n) = 3n - 2m \ge n + 2\left\lceil \frac{n-2}{5} \right\rceil$.

Furthermore. If $n \ge 2$, and $n \ne 3,5,8$. The $(3n-1)$ - subdominating function g of $P_3 \times P_n$ is defined as follows:

(i) If $n = 5k + 2$, let $A = \{u_1, w_1, v_3, u_5, w_5, v_7, \cdots, v_{5i-2}, u_{5i}, w_{5i}, v_{5i+2}, \cdots, v_{5k-2},$

$u_{5k}, w_{5k}, v_{5k+2}\}$, it implies that $|A| = 4k+2 = n - \left\lceil \frac{n-2}{5} \right\rceil$;

(ii) If $n = 5k+3 (k \neq 0,1)$, let $A = \{u_1, w_1, v_3, u_5, w_5, v_7, \cdots, v_{5i-2}, u_{5i}, w_{5i}, v_{5i+2}, \cdots, v_{5k-2}, u_{5k}, w_{5k}, v_{5k+2}\}$, it implies that $|A| = 4k+2 = n - \left\lceil \frac{n-2}{5} \right\rceil$;

(iii) If $n = 5k+4$, let $A = \{u_1, w_1, v_3, u_5, w_5, v_7, \cdots, v_{5i-2}, u_{5i}, w_{5i}, v_{5i+2}, \cdots, v_{5k+3}\}$, it implies that $|A| = 4k+3 = n - \left\lceil \frac{n-2}{5} \right\rceil$;

(iv) If $n = 5k (k \geq 2)$, let $A = \{u_1, w_1, v_3, u_5, w_5, v_7, \cdots, v_{5i-2}, u_{5i}, w_{5i}, v_{5i+2}, \cdots, v_{5k-2}, u_{5k}\}$, it implies that $|A| = 4k = n - \left\lceil \frac{n-2}{5} \right\rceil$;

(v) If $n = 5k+1 (k \geq 1)$, let $A = \{u_1, w_1, v_3, u_5, w_5, v_7, \cdots, v_{5i-2}, u_{5i}, w_{5i}, v_{5i+2}, \cdots, v_{5k-2}, u_{5k}, w_{5k}\}$, it implies that $|A| = 4k+1 = n - \left\lceil \frac{n-2}{5} \right\rceil$.

Let $g(v) = \begin{cases} -1, & \text{if } v \in A; \\ 1, & \text{otherwise,} \end{cases}$ then we have $\gamma_s^{3n-1}(P_3 \times P_n) \leq n + 2\left\lceil \frac{n-2}{5} \right\rceil$.

Consequently, the equality holds.

References

1. J.A. Bondy, U.S.R.Murty, Graph Theory with Applications [M]. London: Macmillan, 1977.
2. E.J. Cockayne, C.M. Mynhardt, On a Generalization of Singed Dominating Functions of Graphs [J], Ars. Combin., **43** (1996).
3. Xuedong Zhang, Signed Domination Number in a Ladder $P_2 \times P_n$ and a Circular Ladder $P_2 \times C_n$ [J], Journal of Henan Institute of Education(Natural Science). **13(4)** (2004).
4. Baogen Xu, Junxia Zhang, Guang Li, Signed Domination Numbers for Two Classes of Product Graphs [J], Journal of Henan University of Science and Technology(Natural Science), **41(2)** (2020)