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Multiobjective fractional programming problems and the sufficient condition involving $\mathrm{H_b}-(p,r)-\eta$-invex function

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Abstract. On the basis of arcwise connected convex functions and $(p,r)-\eta$- invex functions,we established $\mathrm{H_b}-(p,r)-\eta$-invex functions.Based on the generalized invex assumption of new functions,the solutions of a class of multiobjective fractional programming problems are studied,and the sufficient optimality condition for the feasible solutions of multiobjective fractional programming problems to be efficient solutions are established and proved.

1 Introduction

Multiobjective programming is a branch of mathematical programming. The idea of multiobjective optimization was first put forward by French economist V. pareto in 1896. Multiobjective programming is widely used in many practical problems, such as economy, management, military affairs, science and engineering design. Convex function is one of the most widely used concepts in modern mathematics. In view of its importance in mathematical programming theory and application, many researches are devoted to popularizing these concepts to expand their application scope.M.A.Hanson studied the sufficiency of the Kuhn-Tucker[1];Avriel M and Zang I put forward a new class of generalized convex functions, and on this basis, gives some regularity conditions satisfying the characteristics of local-global minimum[2]; In reference[3], a new class of generalized convex functions is defined by means of symmetric gradient.In reference[4-5], the optimality conditions and Wolfe-type duality of two classes of invariant convex multiobjective nonlinear programming are proposed. In reference[6-12], the concepts of arc, arc connected set and arc connected convex function are established , and several new types of arc connected convex functions on this basis are defined; Tadeusz Antczak extended the generalized concepts of functions and sets and the convexity of functions, and defined invariant convex functions and invariant convex functions [13-14].

In this paper, a new class of generalized convex functions-invariant convex functions is defined on the basis of arc-wise connected convex functions and invariant convex functions. Based on the generalized invariant convex assumption of new functions, the solutions of a class of multiobjective fractional programming problems are studied, and some optimality

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sufficient conditions for the feasible solutions of multiobjective fractional programming problems to be efficient solutions are established and proved.

2 Notation and function definition

Throughout the paper,let R^n be the n- dimensional Euclidean apace and R_+^n be its non-negative orthant and let X be a nonempty open subset of R^n .We use the following conventions for vectors in R^n :

$x^1 > x^2$ *if and only if* $x_i^1 > x_i^2$, *for all* $i = 1,2,\cdots,n$;

$x^1 \square x^2$ *if and only if* $x_i^1 \geq x_i^2$, *for all* $i = 1,2,\cdots,n$;

$x^1 \geq x^2$ *if and only if* $x_i^1 \square x_i^2$, *for all* $i = 1,2,\cdots,n$ *and* $x^1 \neq x^2$;

In this paper,we consider the following multiobjective fractional programming problem:

$$(FP)\begin{cases} \min \dfrac{f(x)}{g(x)} = (\dfrac{f_1(x)}{g_1(x)}, \dfrac{f_2(x)}{g_2(x)}, \ldots, \dfrac{f_k(x)}{g_k(x)})^T \\ s.t. \quad h(x) \square 0 \\ \qquad x \in X \subset R^n, \end{cases}$$

where $f_i, g_i : X \to R, i = 1,2,\ldots k$ and $h : X \to R^m$ are differentiable functions.Without loss of generality,we can assume that $f_i(x) \square 0, g_i(x) > 0, i = 1,2,\ldots k$ for all $x \in X$ .Let $X_0 = \{x \in X : h(x) \square 0\}$ be the set of all feasible solutions of (FP).

Denote $X_0 = \{x \in X \subset R^n \mid g(x) = (g_1(x), g_2(x), \cdots, g_m(x))^T \square 0,\}$.

Definition 2.1 A feasible solution $x^* \in X_0$ of (FP) is said to be an efficient solution of (FP) if there exist no other feasible solution $x \in X_0$ such that

$$\frac{f_i(x)}{g_i(x)} \square \frac{f_i(x^*)}{g_i(x^*)} \qquad \textit{for all } i = 1,2,\ldots,\mathrm{k},$$

and

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)} \quad \textit{for some } t \in \{1,2,\ldots,\mathrm{k}\}.$$

Definition2.2 [2] The set $X \subset R^n$ is said to be arcwise connected (AC) if,for every pair of points $x_1, x_2 \in X$,there exists a continuous vector-valued function $H_{x_1,x_2}(\cdot)$,called an arc,defined on the unit interval $[0,1] \subset R$,and with values in X such that

$$H_{x_1,x_2}(0) = x_1 \qquad H_{x_1,x_2}(1) = x_2$$

In the sequel, $H_{x_1,x_2}(\cdot)$ will denote a continuous arc connecting x_1 with x_2 .Note that every convex set in R^n is AC ,since the function

$$H_{x_1,x_2}(\theta) = (1-\theta)x_1 + \theta x_2$$

is an arc in the sense of the above definition.Thus,the concept of arcwise connected sets is a generalization of convex sets.Similarly,we have the following definition.

Definition2.3 [2] A real function f ,defined on the AC set $X \subset R^n$ is called wise connected (CN) if,for every $x_1, x_2 \in X$,there exists an arc H_{x_1,x_2} in X satisfying

$$f(H_{x_1,x_2}(\theta)) \le (1-\theta) f(x_1) + \theta f(x_2)$$

for $0 \le \theta \le 1$.

Definition2.4 [12] Let f: $X \to R$,where $X \subset R^n$ is an AC set.Let x_1, $x_2 \in X$ and H_{x_1,x_2} is an arc connected x_1 and x_2 in X .The function f is said to posses a right derivative,denoted by $f^+(H_{x_1,x_2}(0))$,with respect to the arc H_{x_1,x_2} at $t=0$ if

$$f^+(H_{x_1,x_2}(0)) = \lim_{t \to 0^+} \frac{f(H_{x_1,x_2}(t)) - f(x_1)}{t}.$$

Clearly,if f: $X \to R$ has a right derivative with respect to the arc H_{x_1,x_2} at $t=0$,then

$$f[H_{x_1,x_2}(t)] = f(x_1) + tf^+(H_{x_1,x_2}(0)) + \theta\delta(t)$$

where $t \in [0,1] and \delta : [0,1] \to R satisfied \lim_{t \to 0^+} \delta(t) = 0.$

Definition2.5 [13] The differentiable function $f : M \to R$ is said to be $B-(p,r)$- invex with respect to η and b at $u \in X$ on X if there exist a function $\eta : M \times M \to R^n$ and a function $b : X \times X \to R_+$ such that,for all $x \in M$,the inequalities

$$\frac{1}{r} b(x,u)(e^{r(f(x)-f(u))} - 1) \geqq \frac{1}{p} \nabla f(u)(e^{p\eta(x,u)} - I) \qquad (p \ne 0, r \ne 0)$$

$$\frac{1}{r} b(x,u)(e^{r(f(x)-f(u)} - 1) \geqq \nabla f(u)\eta(x,u) \qquad (p = 0, r \ne 0)$$

$$b(x,u)(f(x) - f(u)) \geqq \frac{1}{p} \nabla f(u)(e^{p\eta(x,u)} - I) \qquad (p \ne 0, r = 0)$$

$$b(x,u)(f(x) - f(u)) \geqq \nabla f(u)\eta(x,u) \qquad (p = 0, r = 0)$$

holds. f is said to be $B-(p,r)$- invex with respect to η and b on X if it is $B-(p,r)$- invex with respect to the same η and b at each $u \in X$ on X .

On the basis of arcwise connected convex functions and $(p,r)-\eta$- invex functions,we established $\mathrm{H_b}-(p,r)-\eta$- invex functions.

Definition2.6 Let $X \in R^n$ is an arc set,the differentiable real function f is said to be $H_b-(p,r)-\eta$- invex with respect to η and b at $u \in X$ on X if there exist a function $\eta : M \times M \to R^n$, $\varphi : M \times M \to R^n$ and a function $b : X \times X \to R_+$ such that,for all $x \in M$,the inequalities

$$\frac{1}{r} b(x,u)(e^{r(f(x)-f(u))} - 1) \geqq \frac{1}{p} f^+(H_{x,u}(0))\varphi^T(x,u)(e^{p\eta(x,u)} - I) \quad p \ne 0, r \ne 0$$

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1) \geqq f^{+}(H_{x,u}(0))\varphi^{T}(x,u)\eta(x,u) \qquad p=0,r\neq 0$$

$$b(x,u)(f(x)-f(u)) \geqq \frac{1}{p}f^{+}(H_{x,u}(0))\varphi^{T}(x,u)(e^{p\eta(x,u)}-I) \qquad p\neq 0,r=0$$

$$b(x,u)(f(x)-f(u)) \geqq f^{+}(H_{x,u}(0))\varphi^{T}(x,u)\eta(x,u) \qquad p=0,r=0$$

holds. f is said to be $H_b-(p,r)-\eta$ - invex with respect to η and b on X if it is $H_b-(p,r)-\eta$ - invex with respect to the same η and b at each $u\in X$ on X .

Definition2.7 Let $X\in R^n$ is an arc set,the differentiable real function f is said to be strict $H_b-(p,r)-\eta$ - invex with respect to η and b at $u\in X$ on X if there exist a function $\eta: M\times M\to R^n$, $\varphi: M\times M\to R^n$ and a function $b: X\times X\to R_+$ such that,for all $x\in M$,the inequalities

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1) > \frac{1}{p}f^{+}(H_{x,u}(0))\varphi^{T}(x,u)(e^{p\eta(x,u)}-I) \qquad p\neq 0,r\neq 0$$

$$\frac{1}{r}b(x,u)(e^{r(f(x)-f(u))}-1) > f^{+}(H_{x,u}(0))\varphi^{T}(x,u)\eta(x,u) \qquad p=0,r\neq 0$$

$$b(x,u)(f(x)-f(u)) > \frac{1}{p}f^{+}(H_{x,u}(0))\varphi^{T}(x,u)(e^{p\eta(x,u)}-I) \qquad p\neq 0,r=0$$

$$b(x,u)(f(x)-f(u)) > f^{+}(H_{x,u}(0))\varphi^{T}(x,u)\eta(x,u) \qquad p=0,r=0$$

holds. f is said to be strict $H_b-(p,r)-\eta$ - invex with respect to η and b on X if it is strict $H_b-(p,r)-\eta$ - invex with respect to the same η and b at each $u\in X$ on X .

Denote $I=(1,1,...,1)\in R^n$.

3 Sufficient optimality condition

Theorem 3.1 Let $x^*\in X_0$ be a feasible solution for (FP) ,if any one of the following conditions are satisfied:

*(i)*there exist $y^*\in R_+^k, z^*\in R^m, v^*\in R^p$ such that

$$\sum_{i=1}^{k} y_i^*[f_i^+(H_{x,x^*}(0))-v_i^* g_i^+(H_{x,x^*}(0))]+\sum_{j=1}^{m} z_j^* h_j^+(H_{x,x^*}(0))=0 \tag{3.1}$$

$$f_i(x^*)-v_i^* g_i(x^*)=0,\ for\ all\ i=1,2,...,\mathrm{k}, \tag{3.2}$$

$$z_j^* h_j(x^*)=0, for\ all\ j=1,2,...,\mathrm{m}, \tag{3.3}$$

$$h_j(x^*)\,\Box\,0, for\ all\ j=1,2,...,m, \tag{3.4}$$

$$y^*\in I, z^*\in R_+^m, \tag{3.5}$$

where $I=\{y\in R^k : y=(y_1,y_2,...,y_k)>0\ and \sum_{i=1}^{k} y_i=1\}$;

(ii) $A(.)=\sum_{i=1}^{k} y_i^*[f_i(.)-v_i^* g_i(.)]+\sum_{j=1}^{m} z_j^* h_j(.)$ is $H_b-(p,r)-\eta$- invex at x^* with respect to η and b ;

(iii) $B(.)=\sum_{i=1}^{k} y_i^*[f_i(.)-v_i^* g_i(.)]$ is $H_b-(p,r)-\eta$- invex at x^* with respect to η and b ;

$C(.)=\sum_{j=1}^{m} z_j^* h_j(.)$ is $H_b-(p,r)-\eta$- invex at x^* with respect to η and b ;

Then x^* is an efficient solution for (FP).

Proof If conditions (ii) holds,then

$$\frac{1}{r}b(x,u)(e^{r(A(x)-A(x^*))}-1) \square \frac{1}{p}A^+(H_{x,x^*}(0))\varphi^T(x,x^*)(e^{p\eta(x,x^*)}-I). \tag{3.6}$$

The above inequality together with (3.2),imply

$$A(x) \square A(x^*). \tag{3.7}$$

Suppose contrary to the result that x^* is not an efficient solution of (FP).Then there exists a feasible solution x of (FP) such that

$$\frac{f_i(x)}{g_i(x)} \square \frac{f_i(x^*)}{g_i(x^*)}=v_i^* \quad for\ i=1,2,...,k,$$

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(x^*)}{g_t(x^*)}=v_t^* \quad for\ some\ t\in\{1,2,...,k\},$$

That is,

$$f_i(x)-v_i^* g_i(x) \square f_i(x^*)-v_i^* g_i(x^*) \ \ for\ i=1,2,...k,$$

$$f_t(x)-v_t^* g_t(x) < f_i(x^*)-v_t^* g_t(x^*) \ \ for\ some\ t\in\{1,2,...,k\}.$$

By (3.5) and the above inequalities,we have

$$\sum_{i=1}^{k} y_i^*[f_i(x)-v_i^* g_i(x)] < \sum_{i=1}^{k} y_i^*[f_i(x^*)-v_i^* g_i(x^*)]. \tag{3.8}$$

By the feasibility of x and from (3.3) and (3.5),we have

$$\sum_{j=1}^{m} z_j^* h_j(x) \square \sum_{j=1}^{m} z_j^* h_j(x^*). \tag{3.9}$$

On adding (3.8) and (3.9),we obtain

$$\sum_{i=1}^{k} y_i^*[f_i(x)-v_i^* g_i(x)]+\sum_{j=1}^{m} z_j^* h_j(x) < \sum_{i=1}^{k} y_i^*[f_i(x^*)-v_i^* g_i(x^*)]+\sum_{j=1}^{m} z_j^* h_j(x^*),$$

i.e.,

$$A(x) < A(x^*),$$

which contradicts (3.7).

If condition (iii) holds.From the $H_b-(p,r)-\eta$ - invexity of $C(.)$,we get

$$\frac{1}{r}(e^{r(C(x)-C(x^*))}-1)\square\ \frac{1}{p}C^+(H_{x,x^*}(0))\varphi^T(x,x^*)(e^{p\eta(x,x^*)}-I). \qquad (3.10)$$

From (3.9) and (3.10),we obtain

$$\frac{1}{p}C^+(H_{x,x^*}(0))(e^{p\eta(x,x^*)}-I)\square\ 0,$$

i.e.,

$$\frac{1}{p}\sum_{j=1}^{m}z_j^*h_j^+(H_{x,x^*}(0))(e^{p\eta(x,x^*)}-I)\square\ 0. \qquad (3.11)$$

By (3.1),(3.11),we obtain

$$\frac{1}{p}[\sum_{i=1}^{k}y_i^*[f_i^+(H_{x,x^*}(0))-v_i^*g_i^+(H_{x,x^*}(0))]](e^{p\eta(x,x^*)}-I)\square\ 0,$$

i.e.,

$$\frac{1}{p}B^+(H_{x,x^*}(0))(e^{p\eta(x,x^*)}-I)\square\ 0. \qquad (3.12)$$

From the $H_b-(p,r)-\eta$ - invexity of $B(.)$,we have

$$\frac{1}{p}(e^{r(B(x)-B(x^*))}-1)\square\ \frac{1}{p}B^+(H_{x,x^*}(0))(e^{p\eta(x,x^*)}-I). \qquad (3.13)$$

From (3.12) and (3.13),we obtain

$$B(x)\square\ B(x^*),$$

i.e.,

$$\sum_{i=1}^{k}y_i^*[f_i(x)-v_i^*g_i(x)]\square\ \sum_{i=1}^{k}y_i^*[f_i(x^*)-v_i^*g_i(x^*)]. \qquad (3.14)$$

If x^* is not an efficient solution of (FP),then we get (3.8) in the same way.But (3.8) contradicts (3.14).Therefore, x^* is an efficient solution of (FP).This completes the proof.

4 Conclusion

In this paper,we established $\mathrm{H_b}-(p,r)-\eta$ - invex function on the basis of arcwise connected convex functions and $(p,r)-\eta$ - invex functions.Based on the generalized invariant convex assumption of new functions, the solutions of a class of multiobjective fractional programming problems are studied, and the optimality sufficient conditions for the feasible solutions of multiobjective fractional programming problems to be efficient solutions are established and proved.

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