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Unified theory of beam bending within flexoelectricity with including piezoelectricity

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Abstract. The behaviour of small size dielectric elastic beams is described within higher-grade theory with including electric polarization. The coupling between strain gradients and polarization is incorporated into the constitutive laws in the form of flexoelectricity, while piezoelectricity is involve in the classical form. Both the governing equations and boundary conditions are derived using variational formulation for electro-elastic continuous media and deformation assumptions employed in three various beam bending theories such as the classical theory (Euler-Bernoulli theory), the 1st order shear deformation theory (Timoshenko theory) and 3rd order shear deformation theory. The unified formulation allows switching between theories with various bending assumptions by a proper selection of two key factors.

1 Introduction

In non-centre-symmetric dielectric crystals, the polarization vector is related to the 2nd order strain tensor through the 3rd order piezoelectric tensor which must vanish for all dielectrics with inversion-centre symmetry. Therefore piezoelectricity is not observed in centre-symmetric dielectric crystals [1,2]. However a net electrical dipole moment is generated also upon application of non-uniform strain, i.e. strain gradients, even in originally centre-symmetric dielectric crystals. The existence of non-uniform strain due to relative displacements between the centres of oppositely charged ions is physically possible only provided that the centre-symmetry is broken and the contribution of macroscopic strain gradients to induced polarization is known as flexoelectric effect [3,4]. Thus the flexoelectric effect can be incorporated into macroscopic phenomenological theory by consideration of higher-grade continuum theory involving also the 2nd order derivatives of displacements besides the strains. Having used such a continuum model, we shall deal with behaviour of elastic dielectric beams under electro-mechanical loading [5,6]. The 1D formulation will be derived in a unified form with including the deformation assumptions of three theories for bending of elastic beams. Making use such a unified formulation, one can switch between three various theories by a proper selection of two key factors. The derivation of the governing equations and the boundary conditions is performed in a consistent way with using variational principle.

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2 Stationary electro-elasticity with including piezo- and flexo-electric effects

In contrast to higher-order theories, the number of degrees of freedom is not changed as compared with classical theory, i.e. the independent field variables are the same as in classical theory, but some additional field-gradient measures appear in higher-grade theories. Therefore also the number of governing equations is not changed, while the order of the differential equations as well as and the number of boundary conditions are increased. Assuming small derivatives of field variables, the general linear constitutive laws can be derived from the quadratic energetic functional of the derivatives of field variables. In case of dielectric solids, the electric enthalpy can play the role of the energetic functional. Assuming the higher-grade theory of dielectric solids with including the 2nd order derivatives of field variables, the electric enthalpy density is considered as quadratic functional

$$\begin{aligned} \mathrm{H}(\varepsilon_{ij}, E_i, u_{i,jk}) = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2} g_{ijklmn} u_{i,jk} u_{l,mn} - \frac{1}{2}\chi_{ij} E_i E_j - e_{ijk} E_i \varepsilon_{jk} - \\ - f_{ijkl} E_i u_{j,kl} + d_{ijkl} E_{i,j} u_{k,l} \end{aligned} \tag{1}$$

with elastic displacements $u_i(\mathbf{x})$ and electric field scalar potential $\phi(\mathbf{x})$ playing the role of degrees of freedom, while the field gradients such as elastic strains ε_{ij}, second gradient of displacements $u_{i,jk}$ and intensity of electric field E_i are defined by standard formulae

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}. \tag{2}$$

Note that the index following a comma denotes the partial derivative with respect to the corresponding Cartesian coordinate. In Eq. (1), c_{ijkl} is the tensor of elastic coefficients, g_{ijklmn} is the tensor of material coefficients introduced in strain gradient elasticity, χ_{ij} is the tensor of material dielectric coefficients, and e_{ijk}, f_{ijkl}, d_{ijkl} are the piezoelectric, direct flexoelectric and converse flexoelectric coefficients, respectively. The third-rank piezoelectric tensor vanishes in crystalline centrosymmetric dielectrics. In the above formulation, the contribution to the piezoelectric as well as flexoelectric polarization is considered as a response to an applied macroscopic strains and its gradients. Bearing mind the bulk contribution to the polarization, the direct and converse flexoelectricity terms in (1) can be expressed in only one term [5,6] as $-h_{ijkl} E_i u_{j,kl}$, because of using the integration by parts in variational formulation. The governing equations remain unchanged.

The symmetry properties of tensors of material coefficients depend on symmetry of elastic dielectric crystals. For crystals of cubic symmetry [7] these tensors are given as

$$\begin{aligned} c_{ijkl} &= (c_{11} - c_{12} - 2c_{44})\delta_{ijkl} + c_{12}\delta_{ij}\delta_{kl} + c_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ h_{ijkl} &= (h_{11} - h_{12} - 2h_{44})\delta_{ijkl} + h_{12}\delta_{ij}\delta_{kl} + h_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ \chi_{ij} &= \chi\delta_{ij} \\ g_{ijklmn} &= c_{11} l'^2 \delta_{ij}\delta_{kn}\delta_{lm} + c_{44} l^2 (\delta_{il}\delta_{jk}\delta_{mn} - \delta_{ij}\delta_{kn}\delta_{lm}) = \\ &= (c_{11} l'^2 - c_{44} l^2)\delta_{ij}\delta_{kn}\delta_{lm} + c_{44} l^2 \delta_{il}\delta_{jk}\delta_{mn} \end{aligned} \tag{3}$$

with $c_{11} - c_{12} - 2c_{44} = 0$ and $h_{11} - h_{12} - 2h_{44} = 0$ for isotropic materials. Note that l and l' are two new material coefficients (micro-length scale parameters) characterizing microstructure of the continuum in higher-grade elasticity theory.

In non-centrosymmetric crystals exhibiting *mm2* class of symmetry with x_3 being the poling axis, the piezoelectric coefficients are given as

$$e_{1jk} = (\delta_{j1}\delta_{k3} + \delta_{j3}\delta_{k1})e_{15}, \quad e_{2jk} = (\delta_{j2}\delta_{k3} + \delta_{j3}\delta_{k2})e_{24},$$
$$e_{3jk} = \delta_{j1}\delta_{k1}e_{31} + \delta_{j2}\delta_{k2}e_{32} + \delta_{j3}\delta_{k3}e_{33} \tag{4}$$

with $e_{32} = e_{31}$, $e_{24} = e_{15}$ for the *mm6* class of symmetry, while $e_{ijk} = 0$ for centrosymmetric crystals.

From (1), we have the constitutive equations

$$\sigma_{ij} := \frac{\partial \mathrm{H}}{\partial \varepsilon_{ij}} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k$$
$$\tau_{ijk} := \frac{\partial \mathrm{H}}{\partial u_{i,jk}} = g_{ijklmn}u_{l,mn} - h_{lijk}E_l \tag{5}$$
$$D_i := -\frac{\partial \mathrm{H}}{\partial E_i} = \chi_{ij}E_j + e_{ijk}\varepsilon_{jk} + h_{ijkl}u_{j,kl}$$

with σ_{ij}, τ_{ijk}, and D_i being the stress tensor, higher order stress tensor, and the electric displacements, respectively.

3 Derivation of the formulations for beam bending

Let us consider a beam of thickness b ($x_2 \in [-b/2, b/2]$), height h ($x_3 \in [-h/2, h/2]$) and the length L ($x_1 \in [0, L]$), with $h = \ L$. Assuming the translational symmetry along x_2, we may write the displacement field distribution as

$$u_i(x_1, x_3) = \delta_{i1}\left\{u(x_1) + \left[c_1\omega(x_3) - x_3\right]w_{,1}(x_1) + c_1\omega(x_3)\varphi(x_1)\right\} + \delta_{i3}w(x_1) \tag{6}$$

with $u(x_1)$, $\varphi(x_1)$, $w(x_1)$ being axial displacement, rotation of the beam cross-section and deflection, respectively, and $\omega(x_3) := x_3 - c_2\psi(x_3)$, $\psi(x_3) := \frac{4}{3}\left(\frac{x_3}{h}\right)^2 x_3$. The proper selection of two key-factors c_1 and c_2 allows us to switch between three various theories (EBT – Euler-Bernoulli theory, FSDT – 1st order shear deformation theory, i.e. Timoshenko theory, and TSDT – 3rd order shear deformation theory) for bending of the beam

$$c_1 = \begin{cases} 0, \mathrm{EBT} \\ 1, \mathrm{SDT} \end{cases}, \quad c_2 = \begin{cases} 0, \mathrm{FSDT} \\ 1, \mathrm{TSDT} \end{cases}. \tag{7}$$

From (6), one can obtain displacement gradients, strains and $u_{i,jk}$. Owing to the deformation assumptions in beam bending theories, we know explicitly the dependence of all mechanical fields on the x_3-coordinate. In order to get a pure 1D formulation for considered electro-elastic problems in thin beam structures, it is meaningful to adopt the assumption for distribution of electric potential as

$$\phi(x_1, x_3) \approx \phi_0(x_1) + \frac{x_3}{h}\phi_1(x_1) + \left(\frac{x_3}{h}\right)^2 \phi_2(x_1) \tag{8}$$

where $\phi_a(x_1)$, ($a = 0,1,2$) are three new field variables, with two of them being determined by the boundary conditions on the bottom and top surface of the beam as

$$\phi_1(x_1) = A_0(x_1) + A_1 u_{,1}(x_1) + A_2 \varphi_{,1}(x_1) + A_3 w_{,11}(x_1)$$
$$\phi_2(x_1) = B_0(x_1) + B_1 \phi_0(x_1) + B_2 \varphi_{,1}(x_1) + B_3 w_{,11}(x_1) \tag{9}$$

in which $A_{\mathbf{g}}$ and $B_{\mathbf{g}}$ coefficients are specified in Table 1 according to considered either Dirichlet b.c. $\phi(x_1, \pm h/2) = \phi_0(x_1) \pm \frac{1}{2}\phi_1(x_1) + \frac{1}{4}\phi_2(x_1) \equiv \phi^{\pm}(x_1)$, or Neumann b.c. $n_k(x_1, x_3) D_k(x_1, x_3)\big|_{x_3=\pm h/2} = \sigma^{\pm}(x_1)$, $n_k(x_1, x_3)\big|_{x_3=\pm h/2} = \pm\delta_{k3}$.

Table 1. Specification of coefficients used in Eq. (9).

	Dirichlet b.c.	Neumann b.c.
$A_0(x_1)$	$\phi^+(x_1) - \phi^-(x_1)$	$-\frac{h}{2\chi_{33}}\left(\sigma^+(x_1) - \sigma^-(x_1)\right)$
A_1	0	$\frac{he_{31}}{\chi_{33}}$
A_2	0	$\frac{2hh_{44}}{\chi_{33}} c_1(1 - c_2)$
A_3	0	$\frac{h}{\chi_{33}}\left[2h_{44}(c_1 - 1 - c_1 c_2) + h_{12}\right]$
$B_0(x_1)$	$2\left(\phi^+(x_1) + \phi^-(x_1)\right)$	$-\frac{h}{2\chi_{33}}\left(\sigma^+(x_1) + \sigma^-(x_1)\right)$
B_1	-4	0
B_2	0	$\frac{h^2 e_{31}}{2\chi_{33}} c_1(1 - c_2/3)$
B_3	0	$\frac{h^2 e_{31}}{2\chi_{33}}(c_1 - 1 - c_1 c_2/3)$

Thus, $\phi(x_1, x_3)$ is replaced by explicitly known dependence on x_3-coordinate and unknown 1D field variable $\phi_0(x_1)$ as

$$\phi(x_1, x_3) = \left(1 + \left(\frac{x_3}{h}\right)^2 B_1\right)\phi_0(x_1) + \frac{x_3}{h}\left[A_0(x_1) + \frac{x_3}{h} B_0(x_1)\right] + \frac{x_3}{h} A_1 u_{,1}(x_1) +$$
$$+ \frac{x_3}{h}\left(A_2 + \frac{x_3}{h} B_2\right)\varphi_{,1}(x_1) + \frac{x_3}{h}\left(A_3 + \frac{x_3}{h} B_3\right) w_{,11}(x_1) \tag{10}$$

If we consider a beam without free bulk electric charge and external body forces, the 1D formulation (governing equations and boundary conditions) can be derived from the variational principle

$$\delta\int_V \mathrm{H}\,dV-\delta W_e=0\ ,\quad \delta W_e=b\int_0^L \bar{t}_3\delta w dx_1 \tag{11}$$

where δW_e is the work of external transversal loading with the axial density $\bar{t}_3(x_1)$. The electric boundary conditions on the bottom and top surfaces of the beam have been yet incorporated into the formulation. It can be seen that

$$\delta\mathrm{H}(\varepsilon_{ij},E_i,u_{i,jk})=\left(c_{ijkl}\varepsilon_{kl}-e_{ijk}E_i\right)\delta\varepsilon_{ij}+\left(g_{ijklmn}u_{ij,k}u_{l,mn}-h_{lijk}E_l\right)\delta u_{ij,k}-$$
$$-\left(\chi_{ij}E_j+e_{ijk}\varepsilon_{jk}+h_{ijkl}u_{j,kl}\right)\delta E_i=\sigma_{ij}\delta\varepsilon_{ij}+\tau_{ijk}\delta u_{ij,k}-D_i\delta E_i$$

Since the dependence of the integrand on transversal coordinates is known, the integrations within the cross-section can be performed explicitly and the volume integration in (11) is reduced to axial integration

$$\begin{aligned}&\int_0^L\Big\{\left(T_{11}^{(u)}+Q_3^{(u)}\right)\delta u_{,1}+\left(T_{111}^{(u)}+Q_1^{(u)}\right)\delta u_{,11}+\left(M_{11}^{(\varphi)}+T^{(\varphi)}+Q_3^{(\varphi)}\right)\delta\varphi_{,1}+\\&+\left(M_{111}^{(\varphi)}+Q_1^{(\varphi)}\right)\delta\varphi_{,11}+\left(T_{13}^{(w\varphi)}-T_{133}^{(w\varphi)}\right)\delta\varphi+\left(T_{13}^{(w\varphi)}-T_{133}^{(w\varphi)}\right)\delta w_{,1}-\\&-\left(M_{11}^{(w)}-T^{(\varphi)}-T^{(w)}-Q_3^{(w)}\right)\delta w_{,11}-\left(M_{111}^{(w)}-Q_1^{(w)}\right)\delta w_{,111}-b\bar{t}_3\delta w+\\&+Q_1^{(\phi_0)}\delta\phi_{0,1}+Q_3^{(\phi_0)}\delta\phi_0\Big\}dx_1=0\end{aligned} \tag{12}$$

where the semi-integral fields are defined as

$$T_{11}^{(u)}(x_1):=\int_{-h/2}^{h/2}\int_{-b/2}^{b/2}\sigma_{11}(x_1,x_3)dx_2dx_3=b\int_{-h/2}^{h/2}\sigma_{11}(x_1,x_3)dx_3 \tag{13}$$

$$T_{13}^{(w\varphi)}(x_1):=bc_1\int_{-h/2}^{h/2}\left[(1-c_2)\kappa+c_2-c_2\left(\frac{2x_3}{h}\right)^2\right]\sigma_{13}(x_1,x_3)dx_3$$

$$M_{11}^{(w)}(x_1):=b\int_{-h/2}^{h/2}\left[x_3-c_1\omega(x_3)\right]\sigma_{11}(x_1,x_3)dx_3$$

$$M_{11}^{(\varphi)}(x_1):=b\int_{-h/2}^{h/2}c_1\omega(x_3)\sigma_{11}(x_1,x_3)dx_3$$

$$T_{111}^{(u)}(x_1):=b\int_{-h/2}^{h/2}\tau_{111}(x_1,x_3)dx_3$$

$$M_{111}^{(w)}(x_1):=b\int_{-h/2}^{h/2}\left[x_3-c_1\omega(x_3)\right]\tau_{111}(x_1,x_3)dx_3$$

$$M_{111}^{(\varphi)}(x_1):=b\int_{-h/2}^{h/2}c_1\omega(x_3)\tau_{111}(x_1,x_3)dx_3$$

$$T^{(\varphi)}(x_1):=b\int_{-h/2}^{h/2}c_1\left(1-c_2\left(\frac{2x_3}{h}\right)^2\right)\left(\tau_{113}(x_1,x_3)+\tau_{131}(x_1,x_3)\right)dx_3$$

$$T^{(w)}(x_1):=b\int_{-h/2}^{h/2}\left(\tau_{311}(x_1,x_3)-\tau_{113}(x_1,x_3)-\tau_{131}(x_1,x_3)\right)dx_3$$

$$T_{133}^{(w\varphi)}(x_1):=b\int_{-h/2}^{h/2}c_1c_2\frac{8x_3}{h^2}\tau_{133}(x_1,x_3)dx_3$$

$$Q_1^{(\phi_0)}(x_1):=b\int_{-h/2}^{h/2}\left[1+\left(\frac{x_3}{h}\right)^2B_1\right]D_1(x_1,x_3)dx_3$$

$$Q_1^{(u)}(x_1):=bc_1\int_{-h/2}^{h/2}\frac{x_3}{h}A_1D_1(x_1,x_3)dx_3$$

$$Q_1^{(\varphi)}(x_1):=b\int_{-h/2}^{h/2}\frac{x_3}{h}\left(A_2+\frac{x_3}{h}B_2\right)D_1(x_1,x_3)dx_3$$

$$Q_1^{(w)}(x_1):=b\int_{-h/2}^{h/2}\frac{x_3}{h}\left(A_3+\frac{x_3}{h}B_3\right)D_1(x_1,x_3)dx_3$$

$$Q_3^{(\phi_0)}(x_1):=b\int_{-h/2}^{h/2}\frac{2x_3}{h}B_1D_3(x_1,x_3)dx_3$$

$$Q_3^{(u)}(x_1):=b\int_{-h/2}^{h/2}\frac{A_1}{h}D_3(x_1,x_3)dx_3$$

$$Q_3^{(\varphi)}(x_1):=b\int_{-h/2}^{h/2}\frac{1}{h}\left(A_2+\frac{2x_3}{h}B_2\right)D_3(x_1,x_3)dx_3$$

$$Q_3^{(w)}(x_1):=b\int_{-h/2}^{h/2}\frac{1}{h}\left(A_3+\frac{2x_3}{h}B_3\right)D_3(x_1,x_3)dx_3$$

where the shear correction factor $\left[(1-c_2)\kappa+c_2\right]$ is introduced in the FSDT according to Reissner modification of shear stresses. Note that all the integrations in (13) can be performed in closed form. Furthermore, making use of the integration by parts, one can eliminate the derivatives of variations of field variables in the integrand of (12) as

$$\Big\{\left(T_{11}^{(u)}+Q_3^{(u)}-T_{111,1}^{(u)}-Q_{1,1}^{(u)}\right)\delta u+\left(T_{111}^{(u)}+Q_1^{(u)}\right)\delta u_{,1}+ \tag{14}$$

$$+\left(M_{11}^{(\varphi)}+T^{(\varphi)}+Q_3^{(\varphi)}-M_{111,1}^{(\varphi)}-Q_{1,1}^{(\varphi)}\right)\delta\varphi+\left(M_{111}^{(\varphi)}+Q_1^{(\varphi)}\right)\delta\varphi_{,1}+$$

$$+\left(M_{11,1}^{(w)}-T_{,1}^{(\varphi)}-T_{,1}^{(w)}-Q_{3,1}^{(w)}+T_{13}^{(w\varphi)}-T_{133}^{(w\varphi)}-M_{111,11}^{(w)}+Q_{1,11}^{(w)}\right)\delta w-$$

$$-\left(M_{11}^{(w)}-T^{(\varphi)}-T^{(w)}-Q_3^{(w)}-M_{111,1}^{(w)}+Q_{1,1}^{(w)}\right)\delta w_{,1}-$$

$$-\left(M_{111}^{(w)}-Q_1^{(w)}\right)\delta w_{,11}+Q_1^{(\phi_0)}\delta\phi_0\Big\}\Big|_0^L-$$

$$-\int_0^L\Big\{\left(T_{11,1}^{(u)}+Q_{3,1}^{(u)}-T_{111,11}^{(u)}-Q_{1,11}^{(u)}\right)\delta u+$$

$$+\left(M_{11,1}^{(\varphi)}+T_{,1}^{(\varphi)}+Q_{3,1}^{(\varphi)}+T_{133}^{(w\varphi)}-T_{13}^{(w\varphi)}-M_{111,11}^{(\varphi)}-Q_{1,11}^{(\varphi)}\right)\delta\varphi+$$

$$+\left(M_{11,11}^{(w)}-T_{,11}^{(\varphi)}-T_{,11}^{(w)}-Q_{3,11}^{(w)}+T_{13,1}^{(w\varphi)}-T_{133,1}^{(w\varphi)}-M_{111,111}^{(w)}+Q_{1,111}^{(w)}+b\overline{t}_3\right)\delta w+$$

$$+\left(Q_{1,1}^{(\phi_0)}-Q_3^{(\phi_0)}\right)\delta\phi_0\Big\}\,dx_1=0$$

Introducing the notations

$$\begin{aligned}
&M^{(w)}(x_1):=M_{11}^{(w)}(x_1)-T^{(\varphi)}(x_1)-T^{(w)}(x_1)-Q_3^{(w)}(x_1)-V_{,1}^{(w)}(x_1)\\
&V^{(w)}(x_1):=M_{111}^{(w)}(x_1)-Q_1^{(w)}(x_1)\\
&T^{(w\varphi)}(x_1):=T_{13}^{(w\varphi)}(x_1)-T_{133}^{(w\varphi)}(x_1)\\
&M^{(\varphi)}(x_1):=M_{11}^{(\varphi)}(x_1)+T^{(\varphi)}(x_1)+Q_3^{(\varphi)}(x_1)-m_{,1}^{(\varphi)}(x_1)\\
&m^{(\varphi)}(x_1):=M_{111}^{(\varphi)}(x_1)+Q_1^{(\varphi)}(x_1)\\
&T^{(u)}(x_1):=T_{11}^{(u)}(x_1)+Q_3^{(u)}(x_1)-t_{,1}^{(u)}(x_1)\\
&t^{(u)}(x_1):=T_{111}^{(u)}(x_1)+Q_1^{(u)}(x_1)
\end{aligned}\tag{15}$$

one can simplify Eq. (14) as

$$\begin{aligned}
&\Big\{T^{(u)}\delta u+t^{(u)}\delta u_{,1}+M^{(\varphi)}\delta\varphi+m^{(\varphi)}\delta\varphi_{,1}+\\
&+\left(M_{,1}^{(w)}+T^{w\varphi)}\right)\delta w-M^{(w)}\delta w_{,1}-V^{(w)}\delta w_{,11}+Q_1^{(\phi_0)}\delta\phi_0\Big\}\Big|_0^L-\\
&-\int_0^L\Big\{T_{,1}^{(u)}\delta u+\left(M_{,1}^{(\varphi)}-T^{(w\varphi)}\right)\delta\varphi+\left(M_{,11}^{(w)}+T_{,1}^{(w\varphi)}+b\overline{t}_3\right)\delta w+\\
&+\left(Q_{1,1}^{(\phi_0)}-Q_3^{(\phi_0)}\right)\delta\phi_0\Big\}\,dx_1=0
\end{aligned}\tag{16}$$

Hence, one can deduce the governing equations

$$\begin{aligned}
&T_{,1}^{(u)}(x_1)=0\\
&M_{,1}^{(\varphi)}(x_1)-T^{(w\varphi)}(x_1)=0\\
&M_{,11}^{(w)}(x_1)+T_{,1}^{(w\varphi)}(x_1)=-b\overline{t}_3(x_1)\\
&Q_{1,1}^{(\phi_0)}(x_1)-Q_3^{(\phi_0)}(x_1)=0
\end{aligned}\tag{17}$$

at $x_1\in(0,\,L)$, and the set of boundary restrictions

$$T^{(u)}\delta u\Big|_0^L=0\;,\qquad t^{(u)}\delta u_{,1}\Big|_0^L=0$$

$$M^{(\varphi)}\delta\varphi\Big|_0^L=0\;,\qquad m^{(\varphi)}\delta\varphi_{,1}\Big|_0^L=0$$

$$\left(M_{,1}^{(w)}+T^{w\varphi)}\right)\delta w\Big|_0^L=0\ ,\quad M^{(w)}\delta w_{,1}\Big|_0^L=0\ ,\quad V^{(w)}\delta w_{,11}\Big|_0^L=0 \tag{18}$$

$$Q_1^{(\phi_0)}\delta\phi_0\Big|_0^L=0$$

For the sake of brevity, we concise to expressions of the semi-integral fields (15) in terms of the primary field variables $u(x_1)$, $w(x_1)$, $\varphi(x_1)$ and their derivatives by

$$\begin{aligned}
M^{(w)} &= A^{(w4)}w_{,1111}+A^{(w2)}w_{,11}+A^{(\varphi3)}\varphi_{,111}+A^{(\varphi1)}\varphi_{,1}+\\
&\quad +A^{(u3)}u_{,111}+A^{(u1)}u_{,1}+A^{(\phi2)}\phi_{0,11}+A^{(\phi)}\phi_0+A\\
T^{(w\varphi)} &= B^{(w3)}w_{,111}+B^{(w2)}w_{,11}+B^{(w1)}w_{,1}+B^{(\varphi2)}\varphi_{,11}+B^{(\varphi)}\varphi+\\
&\quad +B^{(u2)}u_{,11}+B^{(\phi1)}\phi_{0,1}+B\\
M^{(\varphi)} &= C^{(w4)}w_{,1111}+C^{(w2)}w_{,11}+C^{(\varphi3)}\varphi_{,111}+C^{(\varphi1)}\varphi_{,1}+\\
&\quad +C^{(u3)}u_{,111}+C^{(u1)}u_{,1}+C^{(\phi2)}\phi_{0,11}+C^{(\phi)}\phi_0+C\\
T^{(u)} &= D^{(w4)}w_{,1111}+D^{(w2)}w_{,11}+D^{(\varphi3)}\varphi_{,111}+D^{(\varphi1)}\varphi_{,1}+\\
&\quad +D^{(u3)}u_{,111}+D^{(u1)}u_{,1}+D^{(\phi2)}\phi_{0,11}+D\\
V^{(w)} &= H^{(w3)}w_{,111}+H^{(w1)}\left(w_{,1}+\varphi\right)+H^{(\varphi2)}\varphi_{,11}+\\
&\quad +H^{(u2)}u_{,11}+H^{(\phi1)}\phi_{0,1}+H\\
t^{(u)} &= -D^{(w4)}w_{,111}-D^{(w1)}\left(w_{,1}+\varphi\right)-D^{(\varphi3)}\varphi_{,11}-\\
&\quad -D^{(u3)}u_{,11}-D^{(\phi2)}\phi_{0,1}+G\\
m^{(\varphi)} &= -C^{(w4)}w_{,111}-C^{(\varphi3)}\varphi_{,11}-C^{(\varphi)}\left(w_{,1}+\varphi\right)-\\
&\quad -C^{(u3)}u_{,11}-C^{(\phi2)}\phi_{0,1}+P\\
Q_1^{(\phi_0)} &= E^{(w3)}w_{,111}+E^{(w1)}w_{,1}+E^{(\varphi2)}\varphi_{,11}+E^{(\varphi)}\varphi+\\
&\quad +E^{(u2)}u_{,11}+E^{(\phi1)}\phi_{0,1}+E\\
Q_3^{(\phi_0)} &= F^{(w2)}w_{,11}+F^{(\varphi1)}\varphi_{,1}+F^{(\phi)}\phi_0+F
\end{aligned} \tag{19}$$

without presenting the explicit expressions for the coefficients $A^{(\mathbf{g})}$, $B^{(\mathbf{g})}$, $C^{(\mathbf{g})}$, $D^{(\mathbf{g})}$, $E^{(\mathbf{g})}$, $F^{(\mathbf{g})}$, $H^{(\mathbf{g})}$, A, B, C, D, E, F, H, G, P.
Now the governing equations (17) can be rewritten as

$$\begin{aligned}
&D^{(w4)}w_{,11111}+D^{(w2)}w_{,111}+D^{(\varphi3)}\varphi_{,1111}+D^{(\varphi1)}\varphi_{,11}+\\
&\qquad +D^{(u3)}u_{,1111}+D^{(u1)}u_{,11}+D^{(\phi2)}\phi_{0,111}=-D_{,1}
\end{aligned} \tag{20}$$

$$
\begin{aligned}
&A^{(w4)}w_{,111111}+\left(A^{(w2)}+B^{(w3)}\right)w_{,1111}+B^{(w2)}w_{,111}+B^{(w1)}w_{,11}+\\
&\quad +A^{(\varphi 3)}\varphi_{,11111}+\left(A^{(\varphi 1)}+B^{(\varphi 2)}\right)\varphi_{,111}+B^{(\varphi)}\varphi_{,1}+\\
&\quad +A^{(u3)}u_{,11111}+\left(A^{(u1)}+B^{(u2)}\right)u_{,111}+\\
&\quad +A^{(\phi 2)}\phi_{0,1111}+\left(A^{(\phi)}+B^{(\phi 1)}\right)\phi_{0,11}=-b\bar{t}_3-A_{,11}-B_{,1}
\end{aligned}
\qquad (21)
$$

$$
\begin{aligned}
&C^{(w4)}w_{,11111}+\left(C^{(w2)}-B^{(w3)}\right)w_{,111}-B^{(w2)}w_{,11}-B^{(w1)}w_{,1}+\\
&\quad +C^{(\varphi 3)}\varphi_{,1111}+\left(C^{(\varphi 1)}-B^{(\varphi 2)}\right)\varphi_{,11}-B^{(\varphi)}\varphi+\\
&\quad +C^{(u3)}u_{,1111}+\left(C^{(u1)}-B^{(u2)}\right)u_{,11}+\\
&\quad +C^{(\phi 2)}\phi_{0,111}+\left(C^{(\phi)}-B^{(\phi 1)}\right)\phi_{0,1}=-C_{,1}-B
\end{aligned}
\qquad (22)
$$

$$
\begin{aligned}
&E^{(w3)}w_{,1111}+\left(E^{(w1)}-F^{(w2)}\right)w_{,11}+E^{(\varphi 2)}\varphi_{,111}+\left(E^{(\varphi)}-F^{(\varphi 1)}\right)\varphi_{,1}+\\
&\quad +E^{(u2)}u_{,111}+E^{(\phi 1)}\phi_{0,11}-F^{(\phi)}\phi_0=F-E_{,1}
\end{aligned}
\qquad (23)
$$

Thus, the governing equations are given by the system of the 6th order ordinary differential equations. Similarly, one can rewrite also the Neumann boundary conditions resulting from the boundary restrictions (18) with using the expressions given by (19). From this general formulation, one can obtain the formulations corresponding to deformation assumptions of particular beam bending theories by proper selection of two key-factors c_1 and c_2 .

4 Conclusions

In this paper, we presented the consistent derivation of 1D formulation for behaviour of dielectric elastic beams subject to stationary electro-mechanical loading. The derivation starts from the higher-grade continuum theory for elastic dielectrics with including flexoelectric and piezoelectric effects. The deformation assumptions of three beam bending theories are incorporated in the derived unified formulation and switching among these three theories is allowed by proper selection of two key-factors.

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