Probabilistic modelling of prestressed concrete roof girders

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Abstract. The paper describes several parts of complex stochastic modelling of precast prestressed concrete girders failing in shear. Experimental studies have been performed on scaled elements as well as on full-scale girders. These tests serve as basis for developing the deterministic nonlinear model and subsequent probabilistic assessment of structural resistance. The combination of nonlinear finite element method and probabilistic analysis is a strong tool for the realistic modelling of structures, but it is extremely time consuming, especially when 3D nonlinear model with many input random parameters is analysed. Therefore, sensitivity analyses have been performed and a surrogate model has been developed. Safety formats are utilized and compared with fully probabilistic approach to determine design value of ultimate capacity of girders.

1 Motivation

Non-linear finite element modelling is used more frequently for design and analysis of structures nowadays. Also, in last decade, it is more frequent to use reliability analysis in practice. The combination of nonlinear finite element method and reliability analysis is a strong tool for realistic modelling of structures. On the other hand, it is still high time consuming to perform large non-linear finite element models with many stochastic input variables as in case of mathematical model of precast prestressed concrete roof girder failing in shear produced by Franz Oberndorfer GmBH presented in this paper.

Sensitivity analysis (e.g. [1]) is a crucial step in computational modelling and assessment. It allows the identification of the parameter or set of parameters that have the greatest influence on the model output. It consequently provides a useful insight into which model input contributes most to the variability of the model output. The application of sensitivity analysis can be summarized as: understanding the input–output relationship, determining to what extent uncertainty in structural model parameters contribute to the overall variability in the model output, identifying the important and influential parameters that drive model outputs and magnitudes and guiding future experimental designs. Sensitivity analysis is very often considered to be a supporting method providing information important for decision making.

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Since the computational time needed to obtain ultimate load-bearing capacity of analysed roof girder is relatively high due to utilization of nonlinear finite element method analysis (one calculation takes approximately 8 hours using standard hardware), the utilization of a surrogate model is an appropriate way of reducing the computational effort to a reasonable level. The general principle of surrogate modelling (also known as metamodeling) is to replace the original model, which requires enormous computational effort, with an approximated (simpler) model whose evaluation is not so time-consuming. In the case of the practical application presented in this paper surrogate model created for approximation of original mathematical model was based on polynomial chaos expansion (PCE, [2]).

2 Mathematical model of concrete girder

The analyzed structure is a full-scale LDE7 roof girder produced by Franz Oberndorfer GmbH & Co KG in Austria. Details of FEM and stochastic model are described in the following paragraphs. Surrogate modelling is a part of the long-term research project including laboratory experiments [3] and mathematical modelling [4]. The girder is made from concrete C50/60 and is prestressed by 2x8 strands in each web (Cables - St 1570/1770 – F93). Strands are located in following distances from bottom: 70 and 7x40 mm. It has a TT-shaped cross-section and the total length is 30.00 m and the height is 0.50 m at ends and 0.90 in the middle. The reinforcement and geometry of the beam is symmetrical according to middle cross-sectional and longitudinal plane. The load was applied 4.125 m from support above both webs and the ultimate limit state is represented by the critical value of the force applied during the simulation (the peak of load-deflection diagram). The geometry of the girder, the cross-section and a place of applied load can be seen in Fig 1.
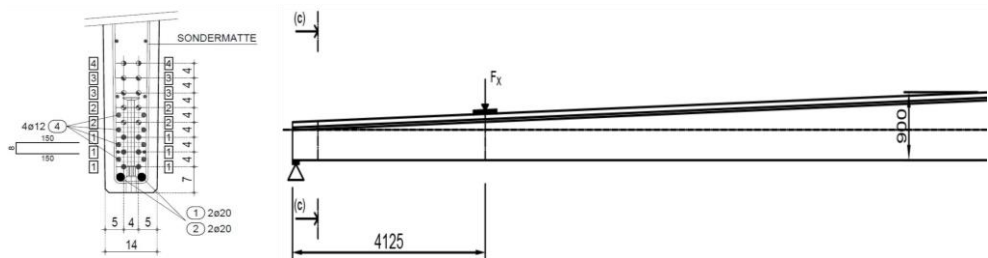

Fig. 1. Scheme of the prestressed concrete roof girder failing in shear.

Finite element model was created in Atena Science software focused on non-linear fracture mechanics of concrete structures [5] and it consists of 61784 finite elements. The geometry of the beam, supports and reinforcement was created exactly according to drawings provided by the manufacturer. The '3D Nonlinear Cementitious 2' material model was used for the concrete [5]. The steel reinforcement and pre-stressing tendons were modelled using 1D elements with a multilinear stress vs. strain diagram with hardening. Prestressing was applied in the form of initial strain in the tendons. Prestress losses (immediate and long-term) were taken into account according to fib Model Code [6]. The evaluation of FEM is very accurate (difference between experiment and numerical model is negligible), but it is also highly time consuming (one calculation takes approximately 8 hours), therefore employment of the surrogate model is necessary to perform stochastic analysis. Herein, we use theory of polynomial chaos expansion for surrogate modelling, which is briefly described in following section.

3 Polynomial chaos expansion

The software developed by authors was employed for the creation of a polynomial chaos expansion (PCE) surrogate model, more details about methodology and the software can be found in [7]. In the following paragraphs, only a brief theoretical background is presented. Assume a probability space $(\Omega, \mathscr{F}, \mathcal{P})$, where Ω is an event space, $\mathscr{F}$ is a σ-algebra on Ω and $\mathcal{P}$ is a probability measure on $\mathscr{F}$. If the input vector of the mathematical model is random vector $X(\omega)$, $\omega \in \Omega$, then random model response $Y(\omega)$ is a random variable. Considering $Y = \mathcal{M}(X)$ has the finite variance σ^2, the polynomial chaos expansion according to Soize & Ghanem is in the following form [8]:

$$Y = \mathcal{M}(X) = \sum_{\alpha \in \mathbb{N}^M} \beta_\alpha \psi_\alpha(X) \tag{1}$$

where M represents the number of input random variables, β_α are unknown deterministic coefficients and Ψ_α are multivariate basis functions orthonormal with respect to the joint probability density function (PDF) of X.

Standard normal input variables are assumed in standard Wiener-Hermite PCE, therefore X must be transformed into standard normal uncorrelated space ξ using Nataf transformation [9] assuming Gaussian copula or generally by Rosenblatt transformation in case of known joint PDF [10]. Then Hermite polynomials can be used as basis functions due to their orthogonality with respect to the PDF of ξ. In practical applications, it is necessary to use PCE truncated to a finite number of terms P. Truncated set of basis functions $\mathcal{A}^{M,p}$ is dependent on given maximal polynomial order p as follows:

$$\mathcal{A}^{M,p} = \{\alpha \in \mathbb{N}^M : |\alpha| = \sum_{i=1}^{M} \alpha_i \leq p\} \tag{2}$$

The cardinality of the truncated set is given by the number of permutations:

$$card\ \mathcal{A}^{M,p} \equiv P = \binom{M+p}{p} = \frac{(M+p)!}{M!p!} \tag{3}$$

Experimental design contains sample points in M-dimensional space and corresponding results of the original model. Size of ED must be higher than P and it is highly dependent on size of the stochastic model and maximal polynomial order p as can be seen in Eq. 3. Moreover, sample points should uniformly cover whole input space, what can be achieved by Latin Hypercube Sampling method [11]. The original mathematical model must be evaluated to obtain results corresponding to generated sample points. Once basis functions are created and experimental design is prepared, PCE coefficients can be estimated by ordinary least square regression method.

Once the PCE approximation is created, it is possible to obtain several important information about mathematical model directly by post-processing of explicit function. An important part of surrogate modelling is an estimation of accuracy of the surrogate model. The advantage of explicit form of PCE is a possibility to obtain Leave-one-out error Q^2 analytically without additional computational demands. Mean value μ and variance σ^2 of approximation Y^{PCE} can be easily estimated from coefficients due to the orthogonality of PCE terms. Moreover, it is also possible to get Sobol indices directly from estimated deterministic coefficients without additional computational demands. Created surrogate model is used for estimation of probability distribution of ultimate shear strength of girder as well as for sensitivity analysis described in following section.

4 Sensitivity analysis

The relative effect of each basic random variable on structural response can be measured using the partial correlation coefficient between each basic input variable and the response variable. With respect to the small-sample simulation techniques of the Monte Carlo type utilized for the reliability assessment of time-consuming nonlinear problems, the most straightforward and simplest approach uses non-parametric rank-order statistical correlation [12]. This method is based on the assumption that the random variable which influences the response variable most considerably (either in a positive or negative sense) will have a higher correlation coefficient than the other variables. Non-parametric correlation is more robust than linear correlation and more resistant to defects in data. It is also independent of probability distribution. Because the model of the structural response is generally nonlinear, a non-parametric rank-order correlation is used by means of the Spearman correlation coefficient:

$$r_{\mathrm{s},i} = 1 - \frac{6 \sum_{j=1}^{N}(q_{ji}-p_j)^2}{N^3-N}, r_{\mathrm{s},i} \in \langle -1; 1 \rangle \qquad (4)$$

where q_{ji} is the rank of a representative value of the random variable X_i in an ordered sample of N simulated values used in the j-th simulation and p_j is the rank of the response variable obtained in the same simulation.

The crude Monte Carlo simulation method can be used for the preparation of random samples. However, it is recommended that a proper sampling scheme should be used, e.g. stratified Latin hypercube sampling. This method utilizes random permutations of the number of layers of the distribution function of the basic random variables to obtain representative values for the simulation. When using this method, the ranks q_{ji} in Equation 4 are directly equivalent to the permutations used in sampling.

One of the most complex type of sensitivity analysis is so called analysis of variance (ANOVA), specifically Sobol indices [13]. This type of global sensitivity analysis aims at quantifying the importance of input variables on the variance of model response. First order Sobol indices can be computed as follows:

$$S_i = \frac{\sigma^2[\mu(Y|X_i)]}{\sigma^2[Y]} \qquad (5)$$

where the variance of conditional expectation is normalized by the variance $\sigma^2[\mu(Y|X_i)]$. If the random variables Y and X_i are independent, then $S_i = 0$ and if Y does depend only on X_i, then $S_i=1$. Note that first order Sobol indices do not take into account the interactions with other variables. For this purpose, total Sobol indices can be computed. It is shown by Sudret [14] that Sobol indices of any order or total Sobol indices can be computed from PCE without additional computational effort due to orthogonality of the PCE terms.

5 Semi-probabilistic approach

It is necessary to reduce desired number of numerical simulations as much as possible in practical applications. Therefore, semi-probabilistic approach assuming several simplifications was developed. These methods are focused on estimation of statistical moments of resistance and estimation of design value of resistance under assumption of lognormal distribution. Coefficient of variation of resistance according to Estimation of Coefficient of Variation (ECov) by Červenka [15] can be estimated as follows:

$$v_f = \frac{1}{1.65} ln\left(\frac{R_m}{R_k}\right) \tag{6}$$

Note that, just 2 simulations of NLFEA are needed in this approach $R_m = R\left(f_{cm}, f_{ym}, a_{nom}, ...\right)$ with mean values of input random variables and R_k using characteristic values (5% quantile). The global resistance safety factor is then calculated as:

$$\gamma_R = exp(\alpha_R \beta v_f) \tag{7}$$

Described concept was adopted in the fib Model Code and design value R_d was later decreased by $\gamma_{Rd} = 1.06$ to take model uncertainties into account:

$$R_d = \frac{R(f_{cm}, f_{ym}, a_{nom}, ...)}{\gamma_R \gamma_{Rd}} \tag{8}$$

Improved ECoV method, where v_R is extended by variability of model and geometrical uncertainties, is proposed by Schlune et al. [16]:

$$v_R = \sqrt{v_g^2 + v_m^2 + v_f^2} \tag{9}$$

where v_g and v_m represents coefficient of variation of geometrical and model uncertainties. The coefficient of variation of material v_f, if material parameters are not correlated, can be calculated as:

$$v_f \approx \frac{1}{R_m} \sqrt{\sum_i^N \left(\frac{R_m - R_{\Delta Xi}}{\Delta X_i} \sigma_{Xi}\right)^2} \tag{10}$$

where the response of construction $R_{\Delta Xi}$ is determined by NLFEA using reduced mean values of material variables by Δ_{Xi} and σ_{Xi} is standard deviation of variable. If the lognormal distribution of material variables is assumed, the reduced values of X_i can be calculated as:

$$X_{\Delta i} = X_{mi} exp(-c \cdot v_{Xi}) \tag{11}$$

where X_{mi} is mean value of *i*-th material characteristic and step size parameter $c = (\alpha_R \beta) / \sqrt{2}$.

A simple advanced method to estimate moments of resistance function, was proposed by Rosenblueth [17]. The expected value of *m*-th moment of function *Y* can be estimated:

$$E\left(Y^m\right) \approx \sum_{i=1}^{2^N} P_i \cdot y_i^m \tag{12}$$

where P_i are weighting factors and y_i^m is a realization of Y^m. The realizations are located at 2^N points where each is a combination of one standard deviation above or below the mean of all the variables.

6 Selected results

The original stochastic model contained 12 random variables in four groups: reinforcement (modulus of elasticity, tensile strength), tendons (modulus of elasticity, tensile strength), prestressing (force, long-term losses, immediate losses) and concrete (compressive strength, tensile strength, modulus of elasticity, fracture energy, density). However, for surrogate modelling a reduced stochastic model, obtained during previous research [18] by sensitivity analysis (Sobol indices and rank-order correlation), was utilized. The reduced stochastic model contains concrete material characteristics and immediate prestressing losses. It is based on laboratory experiments [3] and it contains 5 lognormal random variables as can be seen in Tab. 1: f_c - compressive strength of concrete, E_c - Young's modulus of concrete, f_t - tensile strength of concrete, G_f - fracture energy of concrete and *I.L.U*- uncertainty of calculated immediate prestressing losses.

Table 1. Stochastic model.

Parameter	Mean	Coefficient of Variation [%]	Probability distribution
f_c [MPa]	77	6.4	Lognormal
E_c [GPa]	34.8	10.6	Lognormal
f_t [MPa]	3.9	10.6	Lognormal
G_f [Jm^2]	219.8	12.8	Lognormal
I.L.U [-]	1	10	Lognormal

Statistical correlation among random parameters of concrete should be considered to obtain realistic behaviour of a mathematical model for all realizations of a random vector. The correlation matrix is composed of Spearman correlation coefficients between each pair of random variables and it was considered according to [4].

Table 2. Correlation matrix.

	f_c	E_c	f_t	G_f
f_c	1	0.8	0.7	0.6
E_c	0.8	1	0.5	0.5
f_t	0.7	0.5	1	0.8
G_f	0.6	0.5	0.8	1

Whole process of surrogate modelling using PCE can be divided into 3 parts: pre-processing, processing and post-processing. Pre-processing contains settings of PCE, a sampling of realizations of a random vector, a transformation of realizations to correlated space and an evaluation of the original mathematical model. In this application, author's PCE software was used in cooperation with FReET (Feasible Reliability Engineering Tool), which is multipurpose probabilistic software for the statistical, sensitivity and reliability analysis of engineering problems [19]. Latin Hypercube Sampling method was performed using FReET to generate 100 realizations in uncorrelated space. Because of the assumption of correlation among random variables, they were transformed to correlated space assuming Gaussian Copula using Nataf transformation. Evaluation of the original mathematical model using NLFEA was performed once samples in correlated space were prepared.

The experimental design contains 100 realizations of a random vector and corresponding results of a mathematical model. Adaptive polynomial order algorithm and model selection algorithm was employed using the author's PCE software. Processing of PCE is fully automatic without any user's action and it takes a few seconds to be finished.

Obtained accuracy of PCE measured by Leave-one-out Error $Q^2 = 0.955$. The explicit form of PCE was then analysed during post-processing and statistical information of mathematical model was obtained without any additional computation demands. The mean value of distribution $\mu = 284$ kN and standard deviation $\sigma = 28.8$ kN were obtained directly from PCE coefficients, and hypothesis of a Gaussian distribution was confirmed by Kolmogorov-Smirnoff test.

The surrogate model was utilized for the estimation of the probability distribution of resistance and two types of sensitivity analysis. Distribution of resistance is crucial for estimation of design value of resistance R_d or failure probability p_f during reliability analysis of structure. Nevertheless, sensitivity analysis is important for detection of failure mode and the influence of input variables to model output.

Although there are several types of sensitivity analysis, interpretation of results is often problematic. Moreover, in practical applications dependence among random variables should be considered, therefore results might be highly affected by statistical correlation.

Herein, two methods of sensitivity analysis were performed: nonparametric rank-order statistical correlation and Analysis of Covariance (ANCOVA) [20], generalization of ANOVA for correlated variables. Note that, both methods have a different meaning. On the one hand, nonparametric correlation measures the strength and direction of association among variables and on the other hand, ANCOVA measures the contribution of input variable to the output variance. From this point of view, nonparametric sensitivity represents the relative effect of variables to structural response and ANCOVA represents an absolute effect on the variability of response.

Sensitivity analysis based on non-parametric rank-order correlation ρ_c in correlated space is affected by correlation, thus another SA ρ_u was performed for an uncorrelated sample containing 1 million simulations. For correct comparison, Sobol' indices S_{tot} were decomposed to uncorrelated index S_u and correlative index S_c. Obtained results can be seen in Tab. 3.

Table 3. Influence of correlation in sensitivity analysis.

	f_c	E_c	f_t	G_f	$I.L.U$
ρ_c	0.991	0.766	0.849	0.774	0.264
S_{tot}	0.48	0.08	0.25	0.11	0.08
ρ_u	0.743	0.147	0.418	0.202	0.264
S_u	0.28	0.01	0.09	0.02	0.08
S_c	0.20	0.07	0.16	0.09	0

The design of prestressed girder was performed by nonlinear finite element modelling and safe design value of ultimate shear capacity was obtained by application of safety formats according to Eurocode (EN 1990 and EN 1992-2), semi-probabilistic approach and LHS method. Target reliability of structural member is given by target reliability index $\beta = 3.8$. Commonly used Partial Safety Factor (PSF) method has lowest computational requirements, only 1 simulation is needed for the estimation of R_d. In this case, laboratory experiments were used to estimate statistical parameters of concrete material characteristics and design values of input variables according to EN 1990 (PSF). For the comparison, another NLFEA was performed with design values of material characteristics obtained from tables in EN documents (PSF EN tab.) with usage of normative safety factors γ_M – for reinforcement and tendons $\gamma_s = 1.15$ and $\gamma_c = 1.5$ for concrete. Another normative approach according to EN 1992-2 is using global safety factor and 1 numerical simulation. For the determination of input values of material characteristics, statistical parameters obtained

from laboratory experiments were used. The reference value of ultimate shear capacity was obtained by Latin Hypercube Sampling method with 30 simulations. In this case, model uncertainty was included by coefficients of uncertainty applied on losses of prestressing calculated according to fib Model Code. Full stochastic model and correlation matrix were assumed in full probabilistic approach. In this work, the full probabilistic method was performed using software FReET and simulations were generated by LHS-mean.

Design values of ultimate shear capacity determined by normative safety formats, semi-probabilistic methods and PDF of resistance obtained by PCE are depicted in Fig. 2. As can be seen, the highest R_d is determined by fully probabilistic approach, this value is assumed to be the reference. On the other hand, the most conservative design values were obtained by normative approaches.
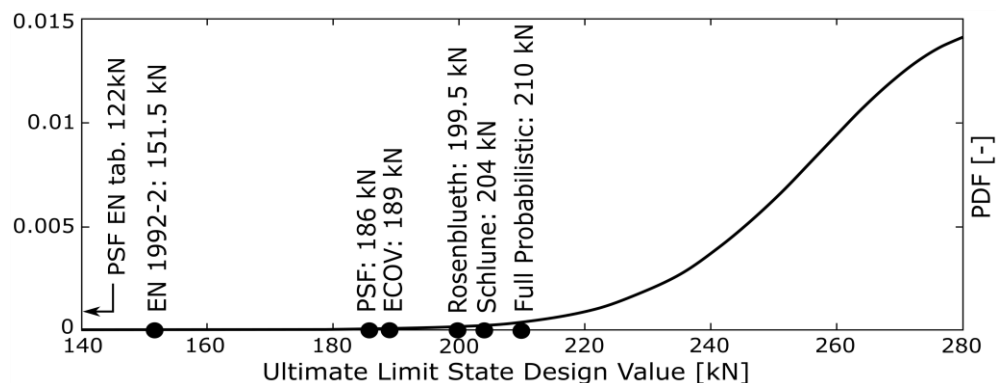

Fig. 2. Comparison of determined design value of ultimate shear strength of analysed girders by various methods: normative approach, semi-probabilistic approach and LHS simulation.

7 Discussion and conclusion

The results of the uncorrelated sample represented by ρ_u and S_u are in compliance with the predicted shear failure of prestressed concrete roof girders. It is clear from Tab. 3, that main influence on the ultimate limit state of girders (assuming uncorrelated variables) have compressive strength and tensile strength of concrete. Negligible influence of fracture energy can be explained by strut inclination method for shear capacity implemented in Eurocode EN 1992, where shear is resisted by concrete struts acting in compression when cracks already occurred, thus fracture energy has no influence at that time. Second important information is the pure influence of correlation S_c, which can play significant role as in case of E_c and G_f.

It is clear that sensitivity analysis plays important role in the stochastic analysis of structures, however, it is important to understand obtained information using different types of sensitivity analysis. Moreover, there can be a high influence of correlation among material characteristics. Therefore, in case of concrete structures, it is important to understand the differences between results for correlated and uncorrelated random variables. As can be seen in this section, it is more suitable to perform sensitivity analysis for uncorrelated variables to identify important variables for failure mode of structures. Although sensitivity analysis is highly computationally demanding, it can be efficiently circumvented by surrogate modelling, especially in case of PCE for estimation of Sobol' indices.

Herein, several approaches to determine design value – so called safety formats are described. They are assessed for shear capacity of prestressed concrete girders–

computationally time-consuming problem in the light of advanced reliability methods. The range of design values obtained is really large (from 122 kN to 210 kN). Normative approaches are naturally conservative - lowest 3 values. The best of normative methods is PSF based on laboratory experiments as expected. Advanced reliability approaches resulted in high design values (highest 3 values). Fully probabilistic method represented by LHS determined the highest R_d. The most efficient method in this application was ECoV modified by Schlune, but when full stochastic model was assumed the standard ECoV by Červenka would be much more suitable regarding computational demand of needed simulations and obtained accuracy of estimated Rd. Semi-probabilistic approaches represent generally useful tools to determine design value of resistance without time-consuming statistical simulation. They provide usually very good results and can be recommended as superior technique in comparison with normative approaches.

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