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A transient dynamic process in a structural nonlinear system “beam – two-parameter foundation”

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Abstract. A method for analytical assessment of dynamic added stress in elastic loaded beam resting on elastic two-parameter Pasternak’s foundation due to sudden destruction a part of foundation is proposed. Equations of static bending, natural and forced oscillations are written in a matrix form using state vectors including deflection, rotational angles, bending moments, and shear forces at arbitrary cross section of a beam and also using the matrices of the initial parameters influence on the stress-strain state in arbitrary cross section. The influence of foundation failure on beam’s stress-strain state, taking into account a relation between the stiffness parameters of foundation, is analyzed. The condition of smallness for the shear stiffness parameter (Pasternak’s parameter) in comparison with the stretching-compressing stiffness parameter (Vinkler’s parameter) is accepted. It is shown that the accounting of Pasternak’s parameter reduces the level of dynamic added stress in a beam when sudden destructing of a foundation. The factor of sudden defect occurrence in the system “beam – foundation” increases considerably the internal forces in a beam in comparison with quasistatic formation of the same defect.

1 Introduction

In this work, a problem of construction a mathematical model for the dynamical process in a load-bearing beam resting on Pasternak’s two-parametrical foundation [1] during sudden occurrence a defect in the form of destruction a part of foundation. Before the defect occurs, the stress-strain state of all the construction is determined by a static influence. The sudden defect appearance leads to reduction in the overall construction stiffness. This reduced stiffness does not already provide with static stability of all the system. The occurring inertial forces cause a dynamical response, the beam begins to move and this results in re-distribution and growth of strain and stress. Due to the dynamical added stress, violations in the established performance or loss of bearing capacity along with progressive destruction are possible. At present, investigations on forced oscillations for the non-linear system “beam-foundation” (i.e. a system that changes a proper calculation scheme under loading) are practically absent in the literature. In several works, manifestations of the structural non-linearity and its after-effects are studied during complete or partial

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destruction of foundation [2-4] and during sudden altering of boundary conditions [5, 6]. Note that a foundation was supposed to be Winkler's one-parametrical in all the works.

2 Problem statement

An elastic Bernoulli-Euler beam of bending stiffness *EI* rests on Pasternak's elastic foundation (K_1 and K_2 are proper stiffness parameters) along all beam's length *L*. Beam's endpoints are free. Both uniformly distributed load *q* and a foundation reaction influence on beam's surface. It is supposed that at the moment *t*=0 a part of foundation is destructed. As the result, the beam comes into movement $v = v(x,t)$ and beam's strain and stress obtain dynamical increments. Solution of the problem is carried out in Cartesian coordinates *x*, *y*. All sizes and deflections are related to beam's length *L*. The unknown quantities are deflections and bending moments (both static and dynamic) along with frequencies and modes of free and forced oscillations of the beam partially supported by Pasternak's elastic foundation (Figure 1).

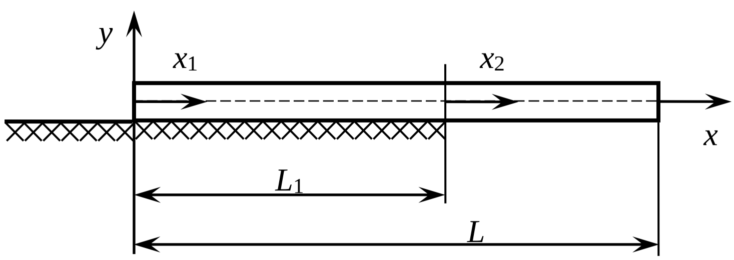


Fig. 1. A beam partially resting on elastic foundation.

3 Results

First, find natural bending oscillations of a free beam partially resting on Pasternak's elastic foundation.

3.1 Natural frequencies of a beam partially supported by an elastic foundation ($0 \le \xi_1 \le \nu$)

The equation of natural bending oscillations is as follow:

$$\frac{\partial^4 w_1}{\partial \xi_i^4} - 4\beta^2 \frac{\partial^2 w_1}{\partial \xi_i^2} + 4\alpha^4 \left(w_1 + \frac{\partial^2 w_1}{\partial \tau^2} \right) = 0 \qquad (1)$$

where $\nu = \frac{L_1}{L}$; $\xi_i = \frac{x_i}{L}$; $w_i = \frac{v_i}{L}$ $(i = 1, 2)$ – deflection; $\alpha^4 = \frac{K_1 L^4}{4EI}$; $\beta^2 = \frac{K_2 L^2}{4EI}$; – bending and shear stiffness of a foundation; *B* – width of beam's cross section. By introduction of two parameters with dimension [frequency] called by "conditional frequencies"

$$\omega_{01} = \sqrt{\frac{K_1}{\rho A}}, \quad \omega_{02} = \frac{1}{2L}\sqrt{\frac{K_2}{\rho A}}$$

where ρ– material density and *A* – cross section area, transform the equation (1) to the form

$$\frac{\partial^4 w_1}{\partial \xi_1^4} - 4\bar{\omega}_{02}^2 \frac{\partial^2 w_1}{\partial \xi_1^2} + \bar{\omega}_{01}^2 \left(w_1 + \frac{\partial^2 w_1}{\partial \tau^2} \right) = 0 \tag{2}$$

where $\bar{\omega}_{0i} = \frac{\omega_{0i}}{\omega_e}$ $(i=1,2)$ – relative conditional frequencies; $\omega_e = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}$ – etalon frequency, and $\tau = \omega_{0i} t$ – dimensionless time. Suppose that oscillations are harmonic and solve the equation (2) by separation of variables:

$$w_1(\xi_1, \tau) = W_1(\xi_1)\sin\bar{\omega}\tau$$

where $\bar{\omega} = \frac{\omega}{\omega_{01}}$ – dimensionless unknown frequency. As the result, obtain the equation for natural modes:

$$W_1^{IV} - 4\bar{\omega}_{02}^2 W_1'' + \left(\bar{\omega}_{01}^2 - \tilde{\omega}^2\right) W_1 = 0 \tag{3}$$

where $\tilde{\omega} = \frac{\omega}{\omega_Э}$ – dimensionless unknown frequency. The structure of equation (3) supposes the following variants of solution.

3.1.1 Variant 1: $\tilde{\omega} = \bar{\omega}$

If $\tilde{\omega} = \bar{\omega}$ then the deflection function is as follow:

$$W_1(\xi_1) = A_1 + A_2\xi_1 + A_3 ch2\beta\xi_1 + A_4 sh2\beta\xi_1 \tag{4}$$

where A_j $(j = 1 \div 4)$ – integration constants. In congenial works [2-6], an efficiency of the initial parameters method in combination with a vector-matrix representation of an arbitrary beam's cross section state is demonstrated in order to analyze both displacements and stresses when interacting with foundation. An analogous approach is used in the present work.

Replacing the integration constants A_j by the internal parameters of the first segment

$$W_{01} = W_1(0), \quad W_{01}' = W_1'(0), \quad W_{01}'' = W_1''(0), \quad W_{01}''' = W_1'''(0)$$

reduce the function (4) to the form

$$W_1 = W_{01} + \xi_1 W_{01}' + \frac{1}{4\beta^2}(ch2\beta\xi_1 - 1)W_{01}'' + \frac{1}{8\beta^3}(sh2\beta\xi_1 - 2\beta\xi_1)W_{01}'''$$

Now, the state of arbitrary section ξ_1 can be represented by the matrix equation

$$\bar{W}_1(\xi_1) = V_{11}(\xi_1)\bar{W}_{01} \tag{5}$$

where $\bar{W}_1(\xi_1) = \{W_1(\xi_1)\ W_1'(\xi_1)\ W_1''(\xi_1)\ W_1'''(\xi_1)\}$ – state vector of an arbitrary section ξ_1; $\bar{W}_{01} = \{W_{01}\ W_{01}'\ W_{01}''\ W_{01}'''\}$ – initial parameters vector;

$$V_{11}(\xi_1)=\begin{pmatrix} 1 & \xi_1 & \frac{1}{4\beta^2}(ch2\beta\xi_1-1) & \frac{1}{8\beta^3}(sh2\beta\xi_1-2\beta\xi_1) \\ 0 & 1 & \frac{1}{2\beta}sh2\beta\xi_1 & \frac{1}{4\beta^2}(ch2\beta\xi_1-1) \\ 0 & 0 & ch2\beta\xi_1 & \frac{1}{2\beta}sh2\beta\xi_1 \\ 0 & 0 & 2\beta sh2\beta\xi_1 & ch2\beta\xi_1 \end{pmatrix}$$

is a matrix-function describing the influence of initial parameters on the state of cross section ξ_1.

3.1.2 Variant 2: $\tilde{\omega}\neq\bar{\omega}_{01}$

Equations similar to equation (5) are built for events $\tilde{\omega}>\bar{\omega}_{01}$:

$$\bar{W}_1(\xi_1)=V_{12}(\xi_1)\bar{W}_{01} \tag{6}$$

and $\tilde{\omega}<\bar{\omega}_{01}$:

$$\bar{W}_1(\xi_1)=V_{13}(\xi_1)\bar{W}_{01} \tag{7}$$

where matrices $V_{12}(\xi_1)$ and $V_{13}(\xi_1)$ take the form

$$V_{12}(\xi_1)=\begin{pmatrix} B_4(\xi_1) & B_3(\xi_1) & B_2(\xi_1) & B_1(\xi_1) \\ (rp)^2B_1(\xi_1) & B_4(\xi_1) & B_2'(\xi_1) & B_2(\xi_1) \\ (rp)^2B_2(\xi_1) & (rp)^2B_1(\xi_1) & B_2''(\xi_1) & B_2'(\xi_1) \\ (rp)^2B_3(\xi_1) & (rp)^2B_2(\xi_1) & B_2'''(\xi_1) & B_2''(\xi_1) \end{pmatrix} \quad p=\sqrt{2\left(\sqrt{\beta^4+\gamma^4}+\beta^2\right)} \quad r=\sqrt{2\left(\sqrt{\beta^4+\gamma^4}-\beta^2\right)} \quad \gamma=\sqrt[4]{\frac{\tilde{\omega}^2-\bar{\omega}_{01}^2}{4}}$$

$$B_1(\xi_1)=\frac{rshp\,\xi_1-p\sin r\,\xi_1}{rp\left(r^2+p^2\right)};\quad B_2(\xi_1)=\frac{chp\,\xi_1-\cos r\,\xi_1}{r^2+p^2};$$

$$B_3(\xi_1)=\frac{r^3shp\,\xi_1+p^3\sin r\,\xi_1}{rp\left(r^2+p^2\right)};\quad B_4(\xi_1)=\frac{r^2chp\xi_1+p^2\cos r\,\xi_1}{r^2+p^2}.$$

$$V_{13}(\xi_1)=\begin{pmatrix} F_4(\xi_1) & F_3(\xi_1) & F_2(\xi_1) & F_1(\xi_1) \\ F_4'(\xi_1) & F_3'(\xi_1) & F_2'(\xi_1) & F_1'(\xi_1) \\ F_4''(\xi_1) & F_3''(\xi_1) & F_2''(\xi_1) & F_1''(\xi_1) \\ F_4'''(\xi_1) & F_3'''(\xi_1) & F_2'''(\xi_1) & F_1'''(\xi_1) \end{pmatrix}$$

$$F_1(\xi_1)=\frac{cha\,\xi_1\sin b\,\xi_1}{2b\left(a^2+b^2\right)}-\frac{sha\,\xi_1\cos b\,\xi_1}{2a\left(a^2+b^2\right)};\quad F_2(\xi_1)=\frac{sha\,\xi_1\sin b\,\xi_1}{2ab};$$

$$F_3(\xi_1)=\frac{3a^2-b^2}{2a\left(a^2+b^2\right)}sha\xi_1\cos b\,\xi_1-\frac{a^2-3b^2}{2b\left(a^2+b^2\right)}cha\xi_1\sin b\,\xi_1;$$

$$F_4(\xi_1)=cha\xi_1\cos b\,\xi_1-\frac{a^2-b^2}{2ab}sha\xi_1\sin b\,\xi_1;$$

$$a=\sqrt{\delta^2+\beta^2}\ ;\ b=\sqrt{\delta^2-\beta^2}\ ;\ \delta=\sqrt[4]{\frac{\bar{\omega}_{01}^2-\tilde{\omega}^2}{4}}.$$

3.1.3 Free segment ($0\le\xi_2\le 1-\nu$)

The equation of natural bending oscillations for this segment is of the form [4]

$$\frac{\partial^4 w_2}{\partial\xi_2^4}+4\alpha^4\frac{\partial^2 w_2}{\partial\tau^2}=0 \tag{8}$$

and the state of arbitrary section $0\le\xi_2\le 1-\nu$ is determined by the matrix equations

$$\bar{W}_2(\xi_2)=V_2(\xi_2)V_{1j}(\nu)\bar{W}_{01}\ (j=1\div 3) \tag{9}$$

where the influence matrix of the first segment on the state of the second segment takes the form

$$V_2(\xi_2)=\begin{pmatrix} R_4(\xi_2) & R_3(\xi_2) & R_2(\xi_2) & R_1(\xi_2)\\ \beta_3^4R_1(\xi_2) & R_4(\xi_2) & R_3(\xi_2) & R_2(\xi_2)\\ \beta_3^4R_2(\xi_2) & \beta_3^4R_1(\xi_2) & R_4(\xi_2) & R_3(\xi_2)\\ \beta_3^4R_3(\xi_2) & \beta_3^4R_2(\xi_2) & \beta_3^4R_1(\xi_2) & R_4(\xi_2)\end{pmatrix}\qquad \beta_3=\sqrt{\tilde{\omega}}$$

$$R_1(\xi_2)=\frac{sh\,\beta_3\xi_2-\sin\beta_3\xi_2}{2\beta_3^3};\quad R_2(\xi_2)=\frac{ch\,\beta_3\xi_2-\cos\beta_3\xi_2}{2\beta_3^2};$$

$$R_3(\xi_2)=\frac{sh\beta_3\xi_2+\sin\beta_3\xi_2}{2\beta_3};\quad R_4(\xi_2)=\frac{ch\,\beta_3\xi_2+\cos\beta_3\xi_2}{2}.$$

When finding the equation (9), the conjugation condition $\bar{W}_2(0)=\bar{W}_1(\nu)$ was used. Further, boundary conditions are to be fixed: free endpoints with proper conditions of the form

$$\begin{gathered} W_{01}''=W_{01}'''=0\\ W_2''(1-\nu)=W_2'''(1-\nu)=0 \end{gathered} \tag{10}$$

are considered below.

3.2 Natural frequencies and modes of bending oscillations for a beam with free endpoints partially supported by two-parametrical Pasternak's foundation

In this Subsection, natural frequencies $\tilde{\omega}_n$ and proper modes are calculated for a beam having free endpoints and interacting with Pasternak's elastic foundation by various combinations of unknown $\tilde{\omega}_n$ and known conditional ω_{01} frequencies.

3.2.1 $\tilde{\omega}=\bar{\omega}_{01}$

First, accept a condition that the unknown frequency $\tilde{\omega}$ is equal to the conditional frequency $\bar{\omega}_{01}$ It means that the following problem is considered: for a given beam which oscillates with unknown frequency $\tilde{\omega}$ along with a foundation, we bring into correspondence some conditional free (i.e., without a foundation) beam with a natural frequency the same as for the given beam, i.e. $\tilde{\omega} = \bar{\omega}_{01}$ taking into account the equation

$$\bar{\omega}_{01} = \frac{\omega_0}{\omega_e} = \sqrt{\frac{K_1}{\rho A}} : \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}} = \sqrt{\frac{K_1 L^4}{EI}}$$

A combination of parameters (K_1, L, E, I) for the system “beam-foundation” can be obtained if the supporting length ν and foundation characteristics K_2 are known and yield the frequency $\tilde{\omega}$ along with proper mode of oscillations.

In given case $\tilde{\omega} = \bar{\omega}_{01}$, the states of both the segments are described by the equations (5) and (9). Taking into account the boundary conditions (9), obtain the equation

$$ch\left(\beta_3(1-\nu)\right)\cos\left(\beta_3(1-\nu)\right) = 1 \tag{11}$$

the roots of which are of the form

$$\beta_{31} = 0, \quad \beta_{32}(1-\nu) = 4.73, \quad \beta_{33}(1-\nu) = 7.853, \quad \beta_{3n} = \frac{2n+1}{2}\pi \quad \text{if } n > 3$$

therefore $\lim\limits_{\nu\to 1}\beta_{32} = \lim\limits_{\nu\to 1}\sqrt{\tilde{\omega}} = \lim\limits_{\nu\to 1}\frac{4.73}{1-\nu} = \infty$ if $\nu = 1$ what means the unreality of accepted condition $\tilde{\omega} = \bar{\omega}_{01}$.

3.2.2 Free beam with unfixed endpoints

As is known [7], a free (i.e. without foundation) beam with unfixed endpoints, besides natural frequencies coinciding with frequencies of a beam with fixed endpoints, has also two zero frequencies corresponding to translational and rotational motion of a beam as a rigid body. That is displacements caused by beam oscillations can be supplemented by movement of a rigid body. This combined motion can be described by the function

$$W = C_1 + C_2\xi$$

In the framework of accepted model for the system “beam-foundation”, the presence of an indefinitely small beam’s segment of length $\nu \neq 0$ interacting with foundation excludes the possibility of motion (as a rigid body) for a beam having free endpoints. And calculation of the first natural frequency should be carried out according to the variant (7) and (9) that is under conditions $\tilde{\omega} < \bar{\omega}_{01}$ starting from $\tilde{\omega}_1 = 0$ at $\nu = 0$ and $\bar{\omega}_{01} \neq 0$. In this case, the frequency equation takes the form

$$Z_1 Z_2 - Z_3 Z_4 = 0 \tag{12}$$

where

$$Z_1 = \beta_3^4 \left(R_2 (1-\nu) F_4 (\nu) + R_1 (1-\nu) F_4'(\nu) \right) + R_1 (1-\nu) F_4''(\nu) + R_3 (1-\nu) F_4'''(\nu);$$
$$Z_2 = \beta_3^4 \left(R_2 (1-\nu) F_3 (\nu) + R_1 (1-\nu) F_3'(\nu) \right) + R_1 (1-\nu) F_3''(\nu) + R_3 (1-\nu) F_3'''(\nu);$$
$$Z_3 = \beta_3^4 \left(R_3 (1-\nu) F_4 (\nu) + R_2 (1-\nu) F_4'(\nu) + R_1 (1-\nu) F_4''(\nu) \right) + R_4 (1-\nu) F_4'''(\nu);$$
$$Z_4 = \beta_3^4 \left(R_3 (1-\nu) F_3 (\nu) + R_2 (1-\nu) F_3'(\nu) + R_1 (1-\nu) F_3''(\nu) \right) + R_4 (1-\nu) F_3'''(\nu).$$
$$W_{1n}(\xi_1) = F_{4n}(\xi_1) - UF_{3n}(\xi_1);$$
$$W_{2n}(\xi_2) = R_{4n}(\xi_2) F_4(\nu) + R_{3n}(\xi_2) F_4'(\nu) + R_{2n}(\xi_2) F_4''(\nu) + R_{1n}(\xi_2) F_4'''(\nu) -$$
$$-U\left(R_{4n}(\xi_2) F_3(\nu) + R_{3n}(\xi_2) F_3'(\nu) + R_{2n}(\xi_2) F_3''(\nu) + R_{1n}(\xi_2) F_3'''(\nu) \right);$$
$$U = \frac{Z_3}{Z_4}.$$

Bending moments in segments (both in free and in resting ones) are determined by the functions

$$M_{1n}(\xi_1) = F_{4n}''(\xi_1) - UF_3''(\xi_1);$$
$$M_{2n}(\xi_2) = \beta_3^4 \left(R_{2n}(\xi_2) F_4(\nu) + R_{1n}(\xi_2) F_4'(\nu) + R_{4n}(\xi_2) F_4''(\nu) + R_{3n}(\xi_2) F_4'''(\nu) \right) -$$
$$-U\left(\beta_3^4 \left(R_{2n}(\xi_2) F_3(\nu) + R_{1n}(\xi_2) F_3'(\nu) \right) + R_{4n}(\xi_2) F_3''(\nu) + R_{3n}(\xi_2) F_3'''(\nu) \right).$$

3.2.3 $\tilde{\omega} > \bar{\omega}_{01}$

Further, accept the condition $\tilde{\omega} > \bar{\omega}_{01}$. The states of segments can be expressed by the matrix equations (6) and (9)

$$\bar{W}_1(\xi_1) = V_{12}(\xi_1)\bar{W}_{01};$$
$$\bar{W}_2(\xi_2) = V_2(\xi_2)V_{12}(\nu)\bar{W}_{01}$$

The frequency equation takes the form

$$T_1T_2 - T_3T_4 = 0 \tag{13}$$

where

$$T_1 = \beta_3^4 R_2 (1-\nu) B_4(\nu) + (rp)^2 \left(\beta_3^4 R_1 (1-\nu) B_1(\nu) + R_4 (1-\nu) B_2(\nu) + R_3 (1-\nu) B_2'(\nu) \right);$$
$$T_2 = \beta_3^4 \left(R_2 (1-\nu) B_3(\nu) + R_1 (1-\nu) B_4(\nu) \right) + (rp)^2 \left(R_4 (1-\nu) B_1(\nu) + R_3 (1-\nu) B_2(\nu) \right);$$
$$T_3 = \beta_3^4 \left(R_3 (1-\nu) B_4(\nu) + (rp)^2 \left(R_2 (1-\nu) B_1(\nu) + R_1 (1-\nu) B_2(\nu) \right) \right) + (rp)^2 R_4 (1-\nu) B_2'(\nu);$$
$$T_4 = \beta_3^4 \left(R_3 (1-\nu) B_3(\nu) + R_2 (1-\nu) B_4(\nu) + (rp)^2 R_1 (1-\nu) B_1(\nu) \right) + (rp)^2 R_4 (1-\nu) B_2(\nu).$$

The oscillation modes and bending moments, after the frequencies have been evaluated from (30), are determined by the functions

$$W_{1n}\left(\xi_1\right)=B_{4n}\left(\xi_1\right)-U_1B_{3n}\left(\xi_1\right);$$

$$W_{2n}\left(\xi_2\right)=R_{4n}\left(\xi_2\right)B_4\left(\nu\right)+\left(rp\right)^2\left(R_{3n}\left(\xi_2\right)B_1\left(\nu\right)+R_{2n}\left(\xi_2\right)B_2\left(\nu\right)+R_{1n}\left(\xi_2\right)B_2'\left(\nu\right)\right)-$$
$$-U_1\left(R_{4n}\left(\xi_2\right)B_3\left(\nu\right)+R_{3n}\left(\xi_2\right)B_4\left(\nu\right)+\left(rp\right)^2\left(R_{2n}\left(\xi_2\right)B_1\left(\nu\right)+R_{1n}\left(\xi_2\right)B_2\left(\nu\right)\right)\right);$$

$$M_{1n}\left(\xi_1\right)=\left(rp\right)^2\left(B_{2n}\left(\xi_1\right)-U_1B_{1n}\left(\xi_1\right)\right);$$

$$M_{2n}\left(\xi_2\right)=\beta_3^4\left(R_{2n}\left(\xi_2\right)B_4\left(\nu\right)+\left(rp\right)^2R_{1n}\left(\xi_2\right)B_1\left(\nu\right)+\left(rp\right)^2\left(R_{4n}\left(\xi_2\right)B_2\left(\nu\right)+R_{3n}\left(\xi_2\right)B_2'\left(\nu\right)\right)\right)-$$
$$-U_1\left(\beta_3^4\left(R_{2n}\left(\xi_2\right)B_3\left(\nu\right)+R_{1n}\left(\xi_2\right)B_4\left(\nu\right)\right)+\left(rp\right)^2\left(R_{4n}\left(\xi_2\right)B_1\left(\nu\right)+R_{3n}\left(\xi_2\right)B_2\left(\nu\right)\right)\right);$$

$$U_1=\frac{T_3}{T_4}.$$

4 Numerical examples

Deflections and bending moments in the system "beam-foundation" are calculated in this Section. Input data: beam length L=6.7 m; cross section – rectangular of width B=0.25 m and of height h=0.18 m; cross section area A=0.045 m^2; axial inertia moment I=1.215·10-4 m^4; Young's modular for beam's material E=3.05·1010 n/m^2. The material of foundation is varied. Gravel is chosen as a basic material with stiffness

$$\bar{K}_1=7.5\ [\text{mPa/m}^3]=7.5\cdot10^6\ [\text{n/m}^3]$$

The bed coefficient for this material, taking into account the foundation width B=0.25 m, is equal to

$$\bar{K}_1=\bar{K}_1B=1.875\cdot10^6\ [\text{n/m}^2]$$

One of the generalized parameters for the system "beam-foundation" takes the value

$$\alpha=\sqrt[4]{\frac{K_1L^4}{4EI}}=3.976$$

Experimental data concerning the second parameter K_2 are practically absent. According to recommendation [8-14], accept the value

$$K_2=0.35K_1=0.65\cdot10^6\ [\text{n}].$$

Then

$$\beta=\sqrt[4]{\frac{K_2L^4}{4EI}}=1.41.$$

Taking into consideration the relations

$$4\alpha^4=\bar{\omega}_{01}\text{ and }\beta=\bar{\omega}_{02}$$

accept the values

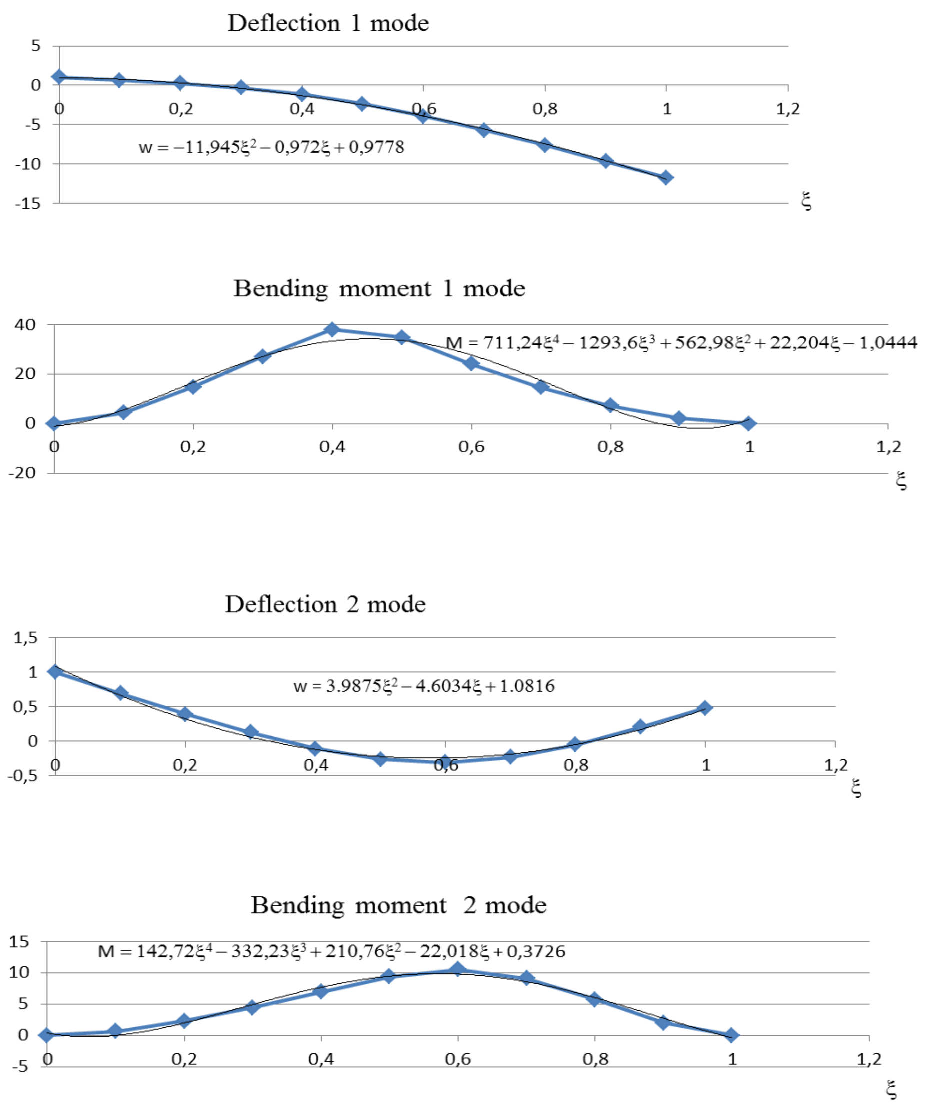
$$\bar{\omega}_{01}=32\text{ and }\bar{\omega}_{02}=1.41$$

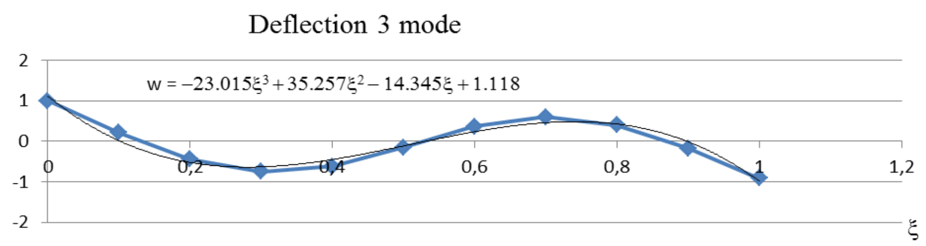
as basic relative conditional frequencies. In Table 1, the first three dimensionless frequencies $\tilde{\omega}_1-\tilde{\omega}_3$ obtained from (12) and (13) for the two combinations of conditional frequencies $\bar{\omega}_{01}$ and $\bar{\omega}_{02}$ characterizing the general stiffness of the system "beam – foundation" are adduced along with length ν of the indestructed part of foundation after its partial destruction.

Table 1. The first three dimensionless frequencies.

$\bar{\omega}_{01}$ / $\bar{\omega}_{02}$	$\tilde{\omega}_1\left(\tilde{\omega}<\bar{\omega}_0\right)$			$\tilde{\omega}_2\left(\tilde{\omega}>\bar{\omega}_0\right)$			$\tilde{\omega}_3\left(\tilde{\omega}>\bar{\omega}_0\right)$		
	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$	$\nu=0.25$	$\nu=0.5$	$\nu=0.75$
11/0	0.822	2.742	5.899	23.24	23.71	24.29	61.9	62.2	62.4
11/0.23	0.812	2.746	5.857	23.24	23.73	24.35	61.9	62.2	62.4
32/0	2.187	5.83	12.96	33.56	34.14	37.61	63.7	66.0	67.8
32/1.41	2.15	5.64	12.46	33.48	34.06	38.4	63.9	67.5	70.1

Proper modes along with bending moments are adduced in Figure 2





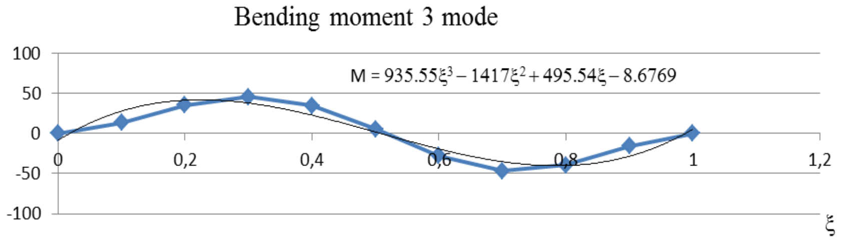


Fig. 2. The first three modes of beam's vibration.

Forced beam's oscillations cased by sudden partial destruction of the foundation, on which the beam rests, are described by the equation

$$\frac{\partial^4 w_{dyn}}{\partial \xi^4} - 4\beta^2 \frac{\partial^2 w_{dyn}}{\partial \xi^2} + 4\alpha^4 \left(w_{dyn} + \frac{\partial^2 w_{dyn}}{\partial \tau^2} \right) = \bar{q} \tag{14}$$

where $\bar{q} = \frac{ql^3}{EI}$ – dimensionless intensity of an evenly distributed load; $w_{dyn} = w_{dyn}(\xi, \tau)$ – deflection function for an arbitrary beam's cross section ξ $(0 \le \xi \le 1)$ depending on time τ. Let us separate variables in equation (14) using the series

$$w_{dyn} = \sum_{n=1}^{\infty} Q_n(\tau) W_n(\xi), \tag{15}$$

where $W_n = W_n(\xi)$ – eigenfunctions obtained by conjunction of the eigenfunctions for cross sections $W_{1n}(\xi_1)$ and $W_{2n}(\xi)$; $Q_n = Q_n(\tau)$ – unknown time functions. Substituting (15) into (14), we obtain the following equations to determine the functions $Q_n(\tau)$:

$$\frac{d^2 Q_n}{d\tau^2} + \bar{\omega}_n^2 Q_n = R_n \tag{16}$$

where

$$R_n = \frac{1}{\bar{\omega}_0^2} \frac{\int_0^1 \bar{q} W_n(\xi) d\xi}{\int_0^1 W_n^2(\xi) d\xi}.$$

The common solution of equation (14) takes the form

$$w_{dyn} = \sum_{n=1}^{\infty}\left(D_{1n}\cos\bar{\omega}_n\tau + D_{2n}\sin\bar{\omega}_n\tau + \frac{R_n}{\bar{\omega}_n^2}\right)W_n(\xi). \tag{17}$$

The integration constants D_{1n} and D_{2n} can be determined from the initial conditions

$$w_{dyn}(\xi,0) = w_{st}(\xi), \quad \left.\frac{\partial w_{dyn}}{\partial\tau}\right|_{\xi,0} = 0, \tag{18}$$

where $w_{st}(\xi)$ – static deflection of the beam entirely resting on elastic foundation. A beam resting freely on Pasternak's elastic foundation and loaded with evenly distributed loading $\bar{q} \equiv const$ is embedded without flexure into foundation by depth

$$w_{st}(\xi) = \frac{\bar{q}}{4\alpha^4}. \tag{19}$$

From 2-nd condition (18), it follows

$$D_{2n} = 0. \tag{20}$$

From 1-st condition (18), we obtain

$$\sum_{n=1}^{\infty}\left(D_{1n} + \frac{R_n}{\bar{\omega}_n^2}\right)W_n(\xi) = w_{cm}. \tag{21}$$

Multiplying both parts of (21) by $W_n(\xi)$ and integrating by ξ from 0 to 1, we obtain

$$D_{1n} = B_n - \frac{R_n}{\bar{\omega}_0^2}, \quad B_n = \frac{\int_0^1 w_{cm}W_n(\xi)d\xi}{\int_0^1 W_n^2(\xi)d\xi}. \tag{22}$$

Substituting (20) and (22) into series (17) and taking into account the equality

$$1 - \cos\bar{\omega}_n\tau = 2\sin^2\frac{\bar{\omega}_n}{2}\tau,$$

we obtain

$$w_{dyn}(\xi,\tau) = \sum_{n=1}^{\infty}\left(B_n\cos\bar{\omega}_n\tau + C_n\sin^2\frac{\bar{\omega}_n}{2}\tau\right)W_n(\xi), \tag{23}$$

where

$$C_n = \frac{2\bar{q}}{\tilde{\omega}_n^2}\frac{\int_0^1 W_n(\xi)d\xi}{\int_0^1 W_n^2(\xi)d\xi}.$$

Using similar transformation, we obtain the series for bending moments

$$M_{dyn} = w''_{dyn} = \sum_{n=1}^{\infty}\left(B_n\cos\bar{\omega}_n\tau + C_n\sin^2\frac{\bar{\omega}_n}{2}\tau\right)W''_n(\xi).$$

Figure 3 and 4 present a diagram of the bending moment in the cross section $\xi = 0.43$ at the beginning of dynamic process after sudden foundation destruction of a half of the beam ($\nu = 0.5$, generalized stiffness of the system "beam – foundation" is $\bar{\omega}_0 = 18$ $(\alpha = 3)$) in

Figure 3 and a diagram of stationary oscillations at $\tau > 14$ in Figure 4. The maximum value of bending moment reaches $M_{\max}^{dyn} = 0.389$. Figure 5 demonstrates the fact that the maximum bending moment by quasistatic process of destruction appears to be approximately twice lesser than by sudden destruction.

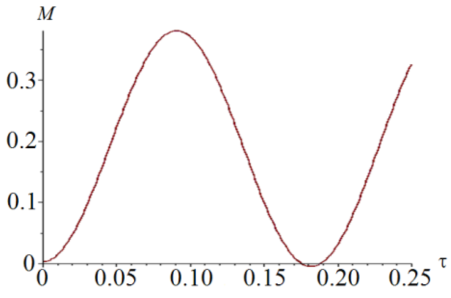


Fig. 3. Bending moment in the cross section ξ=0.43 at the beginning of the vibrational process.

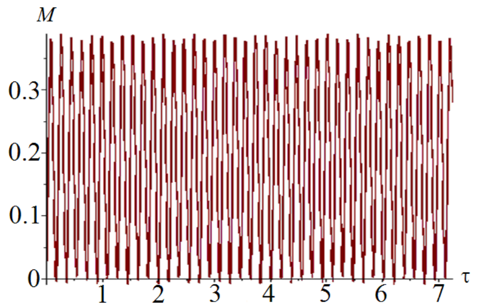


Fig. 4. Stationary oscillations of bending moment in the same cross section.

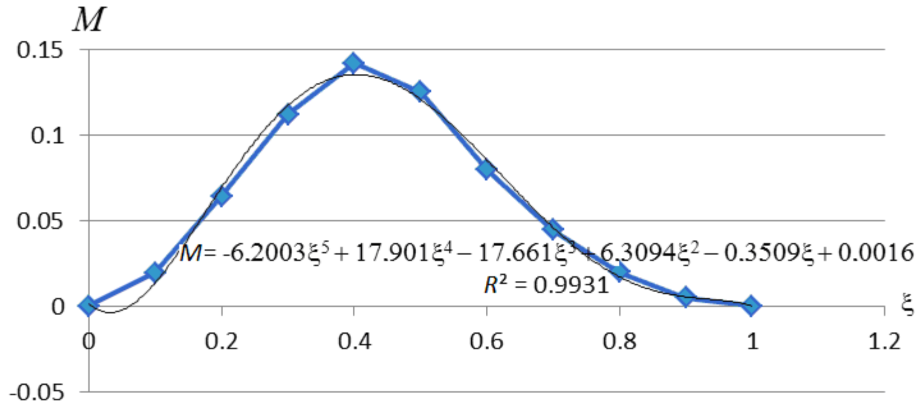


Fig. 5. Bending moment distribution caused by quasistatic destruction of foundation

5 Conclusion

An analytical solution for the problem on determination of forces, modes, and frequencies (both natural and forced) of transversal oscillations for a beam resting on elastic two-parametrical foundation is obtained. This solution can be used for testing of mathematical models describing static-dynamic and quasi-static deformation of a complex non-linear "erection – foundation" system under special crash impacts associated with sudden destruction of foundation's segments.

This analytical solution can also be applied in numerical analysis of building and erection defense against a progressive destruction when an additional loading of a "band foundation – erection" system is caused by sudden subsidence of foundation in correspondence with a possible scenario of crash impact. For example, it may be actual for objects build on slopes in case of foundation destruction due to displacement of foundation.

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References

1. P. L. Pasternak, *On a new method of analysis of an elastic foundation by means of two foundation constants* (State Press of Literature on Civil Engineering and Architecture, Moscow, USSR, 1954)
2. V. Gordon, O. Pilipenko, *Proc. 22nd Intern. Congress on Sound and Vibration, Florence, Italy*, ICSV (2015)
3. V. Gordon, O. Pilipenko, T. Gasimov, *Proc. European Congress on Computational Methods in Applied Science and Engineering, Crete, Greece*, 5533-5549 (2016)
4. V. Gordon, O. Pilipenko, *Proc. 6th Intern. Conf. on Computational Methods in Stuctural Dynamics and Earthquake Engineering*, *Rhodes, Greece*, **2**, 3847-3860 (2017)
5. V. Gordon, O. Pilipenko O, V. Trifonov, *5th Intern. Conf. on Engineering Against Failure , Chios, Greece*, **188**, 03008 (2018)
6. V. Travush, V. Gordon, V. Kolchunov, E. Leontiev, IOP Conf. Series: Mat. Sc. Eng. **456**, 012130 (2018)
7. Mossakovsky, *Strength rocked design: A Textbook for Engineering Universities* (Higher School, Moscow, 1990)
8. T. F. Fwa, X. P. Shi, S. A. Tan, J. Transp. Eng., **122**, 323-328 (1996)
9. I. B. Teodoru, V. Musat, M. Vrabie, Buletinul Institutului Politehnic din Iasi **LII**, Fasc. **3-4**, 7-20 (2006)
10. H. Elhuni, D. Basu, *Proc. 19th Intern. Conf. on Soul Mechanics and Ceotechnical Engineering, Seoul*, 729-732 (2017)
11. M. Chiwanga, A. J. Valsangkar, J. Struct. Eng., **114**, 1414-1430 (1988)
12. R. U. A. Uzzal, R. B. Bhat, W. Ahmed, Shock and Vibration, **19**, 205-220 (2012)
13. G. Onu, J. Struct. Eng, **126**, 1104-1107 (2000)
14. A. Valsangcar, R. J. Pradhanang, Earthquake Eng. and Struct. Dyn. **16**, 217-225 (1988)