Fractal dimension of concrete mix gradation: a quantitative parameter for some concrete properties

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Abstract. This study is based upon the fractal feature of ideal Particle Size Distributions (PSD) suggested by numerous concrete mix designs, i.e. ideal PSD can be shown to be equivalent to a power law distributions $E_C(\phi_i) \propto \phi_i^{-DF}$, where $E_C(\phi_i)$ is the number of grains of size greater than ϕ_i, and DF is a non-whole number called fractal dimension. This fact allows us to analyze the solid skeleton of a concrete mix (all solid components of the mixture) as a fractal structure, thus to determine some of physical properties of the concrete mixture. For DF ranging from 2.5 to 3, and based upon many parameters of the concrete mix (as the granular range, the volumetric fraction of solids in the concrete mixture…), analytical formulas have been proposed relating DF and some properties of the concrete in the solid state. The required properties are the coarse-to-fine aggregate ratio, the fineness modulus of the sand fraction, the average grain size and volume of the fine fraction in the concrete mix. The focus of this research is to develop formulas by which concrete properties can be predicted knowing the concrete mix gradation, i.e. the mix design method used.

1 Introduction

Fractals can be defined as disordered systems. One of the main properties of fractals is their power law behavior of the form: $N(L > r) \propto r^{-DF}$, such as N is the number of objects in the system with size greater than r, DF is a non-integer number referred to as *Fractal Dimension* and the symbol '$\propto$' stands for '*proportional to*' [1, 2].

Concrete mixtures can be considered as fractal objects because their corresponding solid particle size distributions (PSD), since they must be as close as possible to one of the ideal grading curves, they can be transformed to "grain number E_C" vs "grain size ϕ_i" distributions, following a fractal power law of type $E_C \propto \phi_i^{-DF}$ [1, 3, 4].

To get a better understanding of this, we give an example. First, we transform an ideal grading curve according to Fuller & Thompson [5], on fitted straight line $E_C(\phi \geq \phi_i) = f(\phi_i)$ in log-log scale (see Fig. 1). The data points will be obtained by the following expressions:

$$E_C(\phi \geq \phi_i) = \sum_{j=1}^{i} E_j \qquad (1)$$

Such as:

$$E_j = \frac{1}{\rho\, v_j} R_j \text{ and } v_j = \frac{\pi}{6}\phi_j^3 \qquad (2)$$

Where $E_C(\phi \geq \phi_i)$ denotes the grain number of size greater than ϕ_i; E_j the grain number of one size class

ϕ_j; R_j the weight residue on sieve ϕ_j mesh; v_j the volume per grain of size ϕ_j and ρ the mass density of aggregates. In Fig. 1, using linear regression, fitting results show a good linear fit between $E_C(\phi \geq \phi_i)$ and ϕ_i ($R^2 \approx 1$). The equation of the best linear fit is:

$$\log E_C(\phi_i) = a + (-DF)\log\phi_i$$
$$\Rightarrow E_C(\phi_i) = \mathcal{C}\,\phi_i^{-DF} \qquad (3)$$

$\mathcal{C} = 10^a$ is a parameter depending on some properties of concrete [3], the slope of the best-fitted line representing the relationship between E_C and ϕ_i is the fractal dimension DF.

One can achieve same results whatever the method of concrete composition (see [1,3,4]), that enables us to assume PSDs of real concrete mixtures as fractals, which allows identifying them by knowing two parameters, DF value and the total particle size range d/D.

2 Grading curve of a fractal PSD

The solid concrete mixes will be noted $\mathcal{M}_{d/D}^{DF}$, meaning that the grain size distribution of the mix is fractal and that the grain size is ranging from d to D.

Each mixture will be constituted by particles of n monosized classes, numbered from ϕ_1 for the largest to ϕ_n for the smallest ones (according to a geometric progression of reason $\sqrt[10]{10}$). Such as grains belong to a class 'i' will be the set of particles of number E_i,
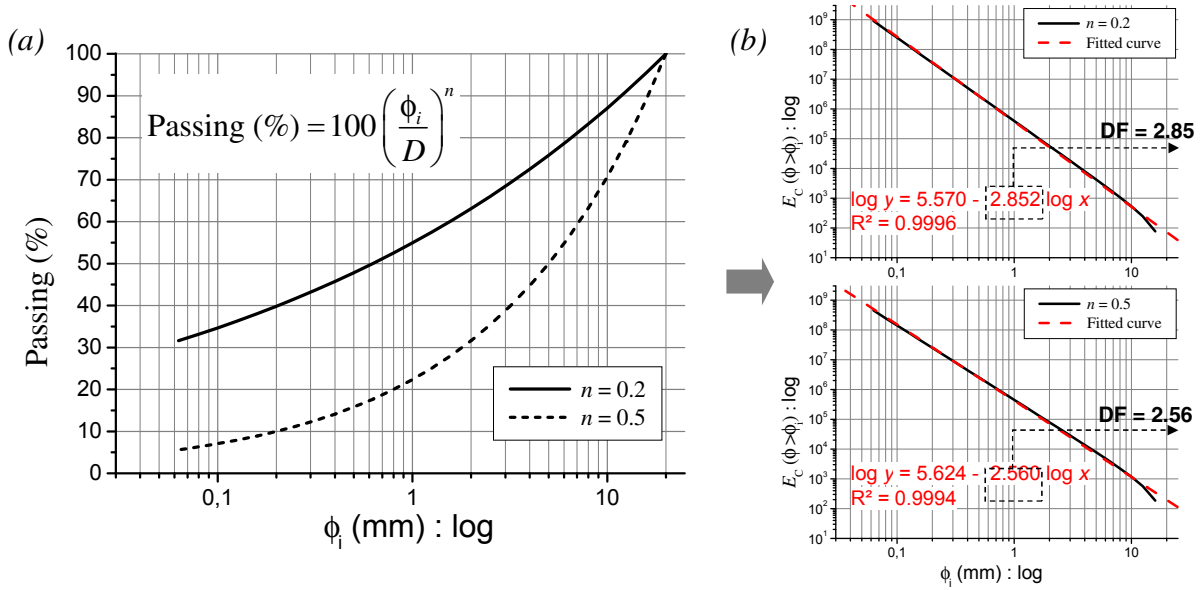

Fig. 1. *(a)* Fuller-Thompson ideal grading curve   *(b)* The corresponding fitted curves: $E_C(\phi \geq \phi_i) = f(\phi_i)$

retained between two successive sieves with a mesh size of ϕ_i and ϕ_{i+1}, as shown in the following expressions:

$$\forall i \in [1,n] : \phi_i > \phi_{i+1}; \quad \text{and:} \quad \frac{\phi_i}{\phi_{i+1}} = \lambda = \sqrt[10]{10} \qquad (4)$$

Such as $\phi_1 = D$; $\phi_n = d$. From the above relations, we can also obtain:

$$\phi_2 = \frac{\phi_1}{\lambda} = \frac{D}{\lambda}; \phi_3 = \frac{\phi_2}{\lambda} = \frac{D}{\lambda^2} \ldots \Rightarrow \phi_i = \frac{D}{\lambda^{i-1}} \qquad (5)$$

From Eq. (3), the grain number of grains of size greater than ϕ_i is given as: $E_C(\phi_i) = \mathcal{C}\,\phi_i^{-DF}$, thus E_i the grain number of one size class ϕ_i can be obtained as follows:

$$E_i = \begin{cases} \mathcal{C}\,\phi_i^{-DF}, & \text{if } i = 1 \\ \mathcal{C}\,(\phi_i^{-DF} - \phi_{i-1}^{-DF}), & \forall i \in [2,n] \end{cases} \qquad (6)$$

By replacing $\phi_i = \frac{D}{\lambda^{i-1}}$, in the above equation, we obtain:

$$E_i = \mathcal{C}D^{-DF} \begin{cases} 1, & \text{if } i = 1 \\ (\lambda^{DF(i-2)})(\lambda^{DF} - 1), & \forall i \in [2,n] \end{cases} \qquad (7)$$

Based on these equations, one can obtain the grading curve of a fractal granular mixture $R_C(\phi_i) = f(\phi_i)$ or $T_C(\phi_i) = f(\phi_i)$ (resp. % cumulative retained or passing), obtained by transformation of the fractal straight line $E_C(\phi_i) = f(\phi_i)$. First, the percentage of material retained $R_i(\%)$ on mesh sieve ϕ_i can be expressed as:

$$R_i(\%) = 100 \frac{R_i}{\sum_{j=1}^{n} R_j} \qquad (8)$$

On another hand, the weight retained on mesh sieve ϕ_i, noted R_i, can be expressed as follows:

$$R_i = E_i m_i = E_i(\rho\, v_i) \qquad (9)$$

Where: $\begin{cases} E_i: \text{number of grains in size class } \phi_i \\ m_i: \text{mass of an individual grain of size } \phi_i \end{cases}$.

Developments of the above equations yields to the equations of a fractal grading curves (for further details see reference [6]):

$$R_{C_i}(\%) \approx 100 \begin{cases} \dfrac{D^{3-DF} - \phi_i^{3-DF}}{D^{3-DF} - d^{3-DF}} & ; \text{if } DF < 3 \\[2ex] \dfrac{\log D - \log \phi_i}{\log D - \log d} & ; \text{if } DF = 3 \end{cases} \qquad (10)$$

3 Physical properties of concrete

Using equations in (10) we can determine grading curves for some granular mixes (dry state) $\mathcal{M}_{d/D}^{DF}$. That will enable us to determine some physical properties of concrete mixes. The fractal dimension considered in these mixes will be in the range: $2.5 \leq DF \leq 3$ [4,6].

3.1 Ratio of fine-to-coarse fractions

The ratio of fine-to-coarse fractions of the dry concrete mix, noted G/F, can be determined by the use of the following expression, where ϕ_{GF} represents the particle size cut between fine and coarse fractions:

$$G/F = \frac{\text{weight of coarse fraction} \left(\phi_i > \phi_{GF}\right)}{\text{weight of fine fraction} \left(\phi_i \leq \phi_{GF}\right)}$$
$$= \frac{M_T\, R_{C_{GF}}(\%)}{M_T\left[100 - R_{C_{GF}}(\%)\right]} = \frac{R_{C_{GF}}(\%)}{T_{C_{GF}}(\%)} \qquad (11)$$

By replacing $R_{C_i}(\%)$ (Eq. (10)); Eq. (11) can be rewritten as follows, such as $d < \phi_{GF} < D$:

$$G/F = \begin{cases} \frac{D^{3-DF} - \phi_{GF}^{3-DF}}{\phi_{GF}^{3-DF} - d^{3-DF}} & ; \text{if } DF < 3 \\ \frac{\log D - \log \phi_{GF}}{\log \phi_{GF} - \log d} & ; \text{if } DF = 3 \end{cases} \quad (12)$$

Fig. 2 illustrates an example of calculation of the ratio G/F for some dry concrete mixes, by using Eqs in (12). In this example, variables are d, D and DF.
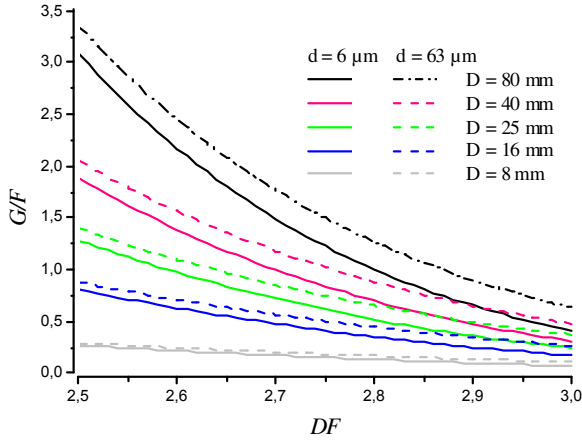

Fig. 2. G/F ratio related to d, D and DF $(\phi_{GF} = 5 \text{ mm})$

3.2 Average diameter of grains

Characteristic diameters of grains as the average grain diameter are relevant parameters involved in the description of size distribution of granular mixtures. The average particle diameter, noted D_{50}, is defined as the size for which 50% of the material of larger particles, i.e. the cumulative weight percent of material passing a sieve size of D_{50} noted $P_{50\%}$ is 50% [7]. According to this definition, the size D_{50} of a dry concrete mix $\mathcal{M}_{d/D}^{DF}$, can be expressed as follows (by using Eqs in (10)):

$$P_{50\%} = 100 \begin{cases} \frac{D_{50}^{3-DF} - d^{3-DF}}{D^{3-DF} - d^{3-DF}} & ; \text{if } DF < 3 \\ \frac{\log D_{50} - \log d}{\log D - \log d} & ; \text{if } DF = 3 \end{cases} \quad (13)$$

From which we can obtain D_{50} in simplified form as follows, such as $P_{50} = \frac{1}{100} P_{50\%}$ and $\overline{P_{50}} = 1 - P_{50}$:

$$D_{50} = \begin{cases} \left(P_{50} D^{3-DF} + \overline{P_{50}} \, d^{3-DF} \right)^{\frac{1}{3-DF}} & ; \text{if } DF < 3 \\ D^{P_{50}} \, d^{\overline{P_{50}}} & ; \text{if } DF = 3 \end{cases} \quad (14)$$

Fig. 3 illustrates an example of calculation of D_{50} for some dry concrete mixes by using Eqs in (14). In this example, considered variables are d, D and DF.
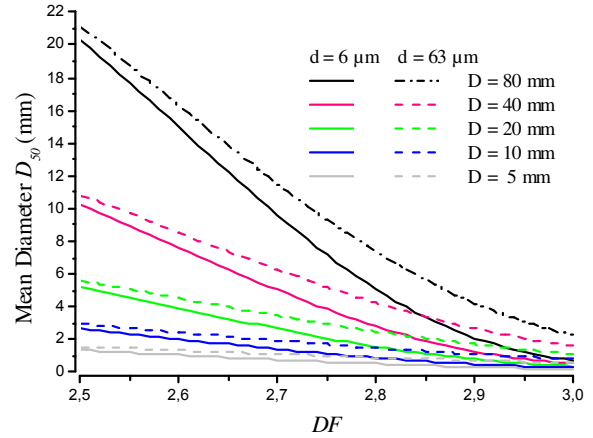

Fig. 3. Mean diameter D_{50} related to d, D and DF

3.3 Amount of fines in a concrete mix

Fines are beneficial in a concrete mix, because they fill voids, reduce cement and improve workability [8-10]. Fines in the concrete mix must fill the voids of the bigger aggregates; hence, the optimum content of fines is related to the granular extent d/D and to the PSD of the mix, i.e. DF.

In this work, the amount of fines will be considered consisting of aggregate fines, mineral addition(s) and cement with most particles passing through 63 µm sieve, this sieve opening will be denoted ϕ_{fines}. In this section, we will derive equations to determine the amount of fines ($\leq \phi_{fines}$), noted A_{fines} (in %), in a granular mix $\mathcal{M}_{d/D}^{DF}$, as follows:

$$A_{fines}(\%) = 100 \begin{cases} \frac{(\phi_{fines}/d)^{3-DF} - 1}{(D/d)^{3-DF} - 1} & ; \text{if } DF < 3 \\ \frac{\log(\phi_{fines}/d)}{\log(D/d)} & ; \text{if } DF = 3 \end{cases} \quad (15)$$

M_T is the total mass of the concrete ingredients (except water), hence M_{fines}, the mass of fines in this mix that can be composed of sand fines (G_1), cement (G_2) and mineral addition (G_3), can be given as follows:

$$\begin{aligned} M_{fines} &= M_T \frac{A_{fines}(\%)}{100} \\ &= G_1 + G_2 + G_3 = M_T \left(P_1 + P_2 + P_3 \right) \end{aligned} \quad (16)$$

Such as P_1, P_2 & P_3 are the volumetric proportions of fines and $P_1 + P_2 + P_3 = 1$. By using Eq. (16), we can determine the fines content, denoted V_{fines}, as follows:

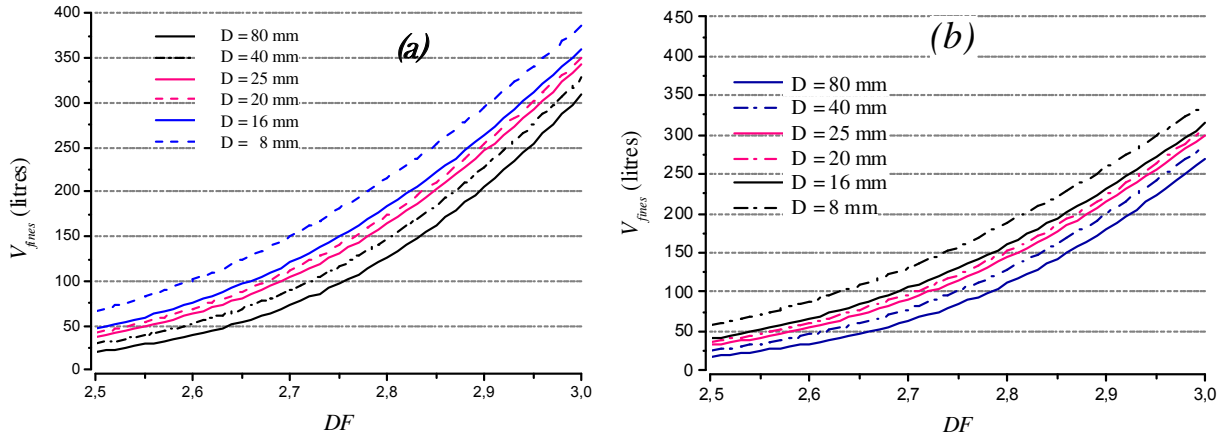$$V_{fines} = \frac{G_1}{\rho_{G1}} + \frac{G_2}{\rho_{G2}} + \frac{C}{\rho_C} \quad (17)$$

Fig. 4. Amount of fines of concrete mixes, such as $M_T = 2350\ \text{kg}/1\ \text{m}^3$ of concrete : *(a)* Fines = 100% fines of aggregates ($\rho_G = 2.5\ \text{g/cm}^3$) *(b)* Fines = 100% cement ($\rho_C = 3.2\ \text{g/cm}^3$).

ρ_{G1}, ρ_{G2} & ρ_C are respectively the absolute densities of each type of fines in concrete: aggregates, mineral addition and cement. By replacing Eq. (16) in (17), we obtain:

$$V_{fines} = M_T\ \frac{A_{fines}(\%)}{100}\left(\frac{P_1}{\rho_{G1}} + \frac{P_2}{\rho_{G2}} + \frac{P_3}{\rho_C}\right) \qquad (18)$$

However, for m types of concrete fines, of the absolute densities: $\rho_1, \ldots, \rho_m$; V_{fines} can be expressed as follows:

$$V_{fines} = M_T\ \frac{A_{fines}(\%)}{100}\left(\frac{P_1}{\rho_1} + \frac{P_2}{\rho_2} + \ldots + \frac{P_m}{\rho_m}\right) \qquad (19)$$

The mean density of all concrete fines ρ_{fines}, can be expressed as follows:

$$\rho_{fines} = \frac{M_{fines}}{V_{fines}} = \frac{1}{\frac{P_1}{\rho_1} + \frac{P_2}{\rho_2} + \ldots + \frac{P_m}{\rho_m}} \qquad (20)$$

By combining Eqs (15)-(20), we obtain the following equations, allowing the determination of the amount of fines by unit mass of concrete mix:

$$V_{fines} = \frac{M_{fines}}{\rho_{fines}} = \frac{P\ M_T}{100}\begin{cases} \frac{(\phi_{fines}/d)^{3-DF} - 1}{(D/d)^{3-DF} - 1} & ;\text{if } DF < 3 \\ \frac{\log(\phi_{fines}/d)}{\log(D/d)} & ;\text{if } DF = 3 \end{cases} \qquad (21)$$

Such that: $P = \frac{P_1}{\rho_1} + \frac{P_2}{\rho_2} + \ldots + \frac{P_m}{\rho_m}$

Fig. 4 illustrates graphically the obtained results of the calculation of the amount of fines for some dry concrete mixes by using Eqs in (21). In this example, the variables are D and DF.

3.4 Fineness modulus of sand fraction

The fineness modulus (FM) is an empirical factor used to estimate the proportions of fine and coarse aggregates in concrete mixtures. According to NF P18-541 standard [11], FM can be obtained by adding the cumulative percentages of aggregate retained on each of the standard sieves ranging from 80 to 0.16 mm and dividing this sum by 100. Generally, FM of sand shall not be less than 1.8 or more than 3.2 to make good concrete.

According to the above definition, FM of the sand fraction ($\phi_m < \phi_i < \phi_{sand}$) in a granular mix $\mathcal{M}_{d/D}^{DF}$, can be written as follow:

$$FM = \frac{1}{100}\sum_{i=1}^{t}\overline{R}_C\left(\phi_i \atop FM\right)$$
$$\text{and: } \phi_i \atop FM \in \begin{cases} \phi_m/\phi_{sand} \\ \left[FM_j\right] \end{cases} \qquad (22)$$

Such as: $\left[FM_j\right] = [0.16; \ldots; 80\ \text{mm}]$ = sieves considered in FM definition; t = number of sieves $\phi_i \atop FM$; $\overline{R}_C\left(\phi_k\right)$ the cumulative % passing of the sand fraction in the mix given by Eq. (24):

$$\overline{R}_C\left(\phi_k\right) = 100\begin{cases} \frac{\phi_{sand}^{3-DF} - \phi_k^{3-DF}}{\phi_{sand}^{3-DF} - \phi_m^{3-DF}} & ;\text{if } DF < 3 \\ \frac{\log\phi_{sand} - \log\phi_k}{\log\phi_{sand} - \log\phi_m} & ;\text{if } DF = 3 \end{cases} \qquad (23)$$

By substituting the above expressions in Eq. (22), we obtain the following expressions allowing the determination of FM :

$$FM = \begin{cases} \frac{t\phi_{sand}^{3-DF} - \sum_{i=1}^{t}\phi_i^{3-DF} \atop FM}{\phi_{sand}^{3-DF} - \phi_m^{3-DF}} & ;\text{if } DF < 3 \\ \frac{t\log\phi_{sand} - \sum_{i=1}^{t}\log\phi_i \atop FM}{\log\phi_{sand} - \log\phi_m} & ;\text{if } DF = 3 \end{cases} \qquad (24)$$

Fig. 5 illustrates an example of calculation of FM for some dry concrete mixes by using Eqs in (24). In this example, the considered variables are DF and the sand fraction sizes: ϕ_{sand} and ϕ_m.
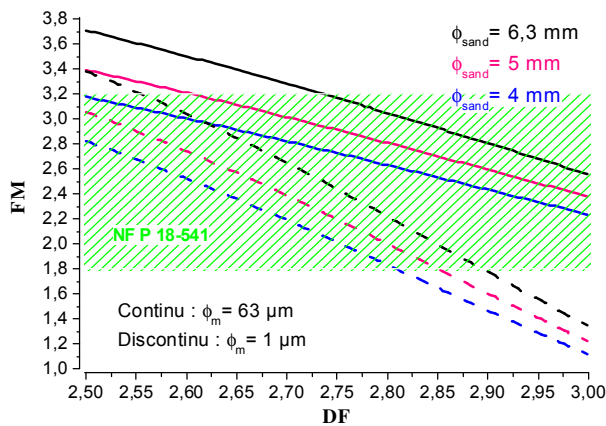
Fig. 5. *FM* of sand fraction of granular mixes, related to *DF* and the limit sizes ϕ_{sand} & ϕ_m. In green, recommended *FM* values $(1.8 \leq FM \leq 3.2)$ according to NF P 18-541 standard.

4 Analysis and conclusion

The key point of this study is to consider ideal size gradations for concrete, hence concrete mix gradations as almost fractal. This allowed us first, to propose a general fractal gradation curve, which would describe any concrete mix gradation with the help of two parameters: the granular extent d/D and the fractal dimension DF. Second, to propose simple analytical formulas allowing the calculation of some physical properties of the dry concrete mixes.

According to our analytical results, DF values varying from 2.5 to 3 indicate that the concrete mix gradation is coarser for small DF values and finer for more DF values, thus we can see decreasing G/F ratio, fineness modulus FM and average particle size D_{50}; and a significant increase of the amount of fines V_{fines} in the concrete mix. These results are in agreement with others in the literature, for instance, some researchers have proposed an optimal gradation curve $\%\text{Passing} = 100\left(\frac{\phi_i}{D}\right)^q$, such as $q = 0.37$ (corresponding to $DF = 2.63$) to achieve maximum compactness for ordinary concrete and $q = 0.30$ (corresponding to $DF = 2.70$) for Self-Compacting Concrete that the mix must be finer [12].

Therefore, the fractal dimension can be a good characterization parameter to be adjusted to obtain some required properties of concrete.

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