Data Assimilation Method, its Application and Stability

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Abstract. The authors data assimilation method, namely, generalized Kalman filter (GKF) method, its application and stability is considered. The problem of stability of a dynamic system with data assimilation formulated for a sequence of random variables forming a Markov chain is considered. The stability formulation for this problem is suggested as the problem of the convergence of the corresponding Markov chain when the number of its members goes to infinity. Necessary and sufficient conditions of this convergence are proved. A number of numerical experiments with the specific dynamic system, namely with the ocean model circulation HYCOM and the GKF method are conducted and discussed. The stability of the GKF method was proofed.

1 Introduction

The development and application of data assimilation (DA) methods in conjunction with a dynamic system (model) defined by a system of differential equations is one of the popular and relevant problems of modern oceanology, geophysics, meteorology and climatology that have a great practical importance. The goal of these methods is to correct the model solution with the combination of observational data and, improve on this basis both the structure of the model characteristics themselves and the prediction of the characteristics of this system for various space-time scales. At the same time, unlike the data interpolation, the DA methods and schemes involve the changes in the entire dynamic system for both the observable and non-observable variables in order to maintain a balance between model parameters, which usually consists in the mathematical formulation of conservation laws. In the process of data assimilation, it is necessary to preserve the balance conditions and, at the same time, bring the model solution closer to observations in a given metric.

DA schemes and methods in oceanography and meteorology have been quite well represented in the scientific literature since the 1960s. A good review of the methods for solving DA problems is presented in [1]. In the recent years, the main progress in this direction is related to the advance in observational data and computer facilities; the development of powerful computing methods such as parallel programming, and special research programs and experiments, for instance, projects HYCOM + NCODA (USA) [2], REMO (Brazil) [3], etc.

Nevertheless, some important theoretical and practical specific problems in the DA theory remain out of scope of research. There are practically no theoretical studies

stability and reliability of numerical forecast based on the DA theory, except for their direct statistical comparison with observations. It is not clear how the system will behave with quite a long integration process, noticeably perturbed initial conditions and model parameters, as well as with a lack of knowledge (errors) on the observed values.

In geophysical and meteorological models, dynamic equations are very complex, so, their analytical solution cannot be obtained; various examples show that even in fairly simple non-linear models, there are attraction points (attractors) and under close initial conditions, the solutions can converge to different attractors.

When dealing with DA, the task becomes even more complicated. Since the dynamical system is “rebuilt” at the moment of assimilation and begins to integrate with the new initial conditions, it seems to receive a “jolt” and the smoothness of the trajectories in the analytical sense is violated. With sufficiently frequent and long-term assimilation, the system may become unstable simply due to a permanent influx of new observable information and its assimilation. The problem is all the more significant since, as a rule, observational data are irregular in time, inhomogeneous in space, have omissions and, on the contrary, duplicates. In addition, they are often noisy and require preliminary mathematical processing. Because of this, the problem of the stability of a dynamic system with DA is of great importance from the theoretical and practical points of view, especially for the analysis and prediction of real characteristics.

In this paper, we study the general dynamic system with a specific assimilation scheme proposed by the authors in a number of papers, for instance, in [4] as a version of the so-called “generalized Kalman filter (GKF)”. In [5], the GKF method is also compared with

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the Ensemble Optimal Interpolation scheme (EnOI) as a counterpart of the standard Kalman filter method and it is shown that the proposed general method has several advantages.

In the current work, the dynamic system in conjunction with the DA scheme is considered as a Markov chain and the stability problem is reduced to the existence of a stationary distribution of this chain. The goal of this work is to find theoretically the necessary and sufficient conditions for the stability of a dynamical system of general form with DA, using the GKF method. Experiments are also proposed which would allow answering the question of whether the entire model (the dynamic system plus the GKF assimilation) is stable in the aforementioned sense.

2 Data assimilation method

The dynamic system defined by the equations

$$\frac{\partial X}{\partial t} = \Lambda(X, t) \quad (1)$$

on time interval $[t_0, T]$ is considered. Here, X denotes the vector state of the system. The right side of equation (1) is a vector function, which is, generally, non-linear and does not contain explicitly time derivatives. The solution of (1) can be represented as

$$X(t) = X(t_0) + \int_{t_0}^t \Lambda(X(\tau)) d\tau,$$

where $X(t_0)$ is a given value (initial condition).

System (1) is considered on a probability space, so that the vectors $X(t)$ and the initial condition $X(t_0)$ are assumed to be random and measurable with respect to a given probability measure P . In this case, the vector function Λ is not assumed to be random.

Along with the dynamic system (1), the adjoint dynamic system

$$\frac{\partial X}{\partial t} = \Lambda(X, t) + K(Y - HX) \quad (2)$$

is considered, where Y is a random variable independent of X , measurable with respect to P , and considered on the same probability space; H is a linear non-random projection operator that transfers the random variable X to the space defined by the values of the quantity Y ; and K is a given weight matrix, which, generally speaking, depends on X and Y , and, of course, on time. Its meaning is to correct the value of the solutions of the dynamic system (1) using the observed value. It should be noted that, using formula (2), we change the entire vector rather than only its observed part HX , that is, the matrix K converts the observed vector to the model space (of the dynamic system (1)). All model variables are considered on the Euclidian space with the dimension r . Let the dimension of observational vector Y be m .

Remark. Since random functions are considered in (1) and (2), the corresponding time derivatives are determined in the sense of the mean square.

Further, we will consider the system (2) at discrete equidistant moments of time, $0 = t_0 < t_1 < \dots < t_{N_T} = T$,

$t_{n+1} = t_n + \Delta t_n$ are the so-called assimilation moments. The requirement of equidistance is insignificant and is introduced only to simplify calculations. So, let the dynamic system be given in the form

$$X_{n+1} = X_n + \Lambda(X_n, t_n) \Delta t + K_{n+1} (Y_{n+1} - HX_n) \Delta t \quad (3)$$

and assimilation scheme GKF produces the following matrix [4].

$$K_{n+1} = \frac{(\Lambda_{n+1} - C_{n+1})(H\Lambda_{n+1})^T Q_n^{-1}}{(H\Lambda_{n+1})^T Q_n^{-1} (H\Lambda_{n+1})}, \quad (4)$$

where $C_{n+1} = E(\hat{Y}_{n+1} - X_n)$, $\hat{Y}_{n+1}$ is a vector which coincides with observations on observational space and is extended to for entire model phase-space. The matrix Q is defined as $Q_n = E(\tilde{Y}_{n+1} - HX_n)(\tilde{Y}_{n+1} - HX_n)^T$, $\tilde{Y}_{n+1} = Y_{n+1} - HC_{n+1}$ and it is a covariance matrix of the observational errors with respect to the model state at moment n .

3 Numerical experiments

As an example of applications of the DA scheme described above, the following experiments are considered. The ocean circulation model HYCOM (HYbrid Circulation Ocean Model, University of Miami, USA) is used as a given dynamic system according to (1). This model produces all parameters of ocean state, such as temperature, salinity, sea level and others as well as their variability in time. The model domain is shown in Fig 1. Since this model is used only as an example, no detailed descriptions for the model equations, numerical configuration and boundary conditions are presented; full information can be found in [6].

For the assimilation according to equations (3),(4), the AVISO (Archiving, Validating and Interpolating Satellite Observations) database was used. This archive contains the sea level data for several years, we used the data for 10 years: from January, 2001 to December, 2010. The data assimilation has been performed daily for one year; the results of integration for one month, January, 2010, is discussed below. Before the DA, the model forced by the climatology data from the NCEP/NCAR archive atlas runs for 40 years [7].

Figures 2 present the results of calculation after the one-month integration for January, 2010. Figure 2a is the direct observations taken from archive AVISO on this day; Fig. 2b is the control field that is the model calculation without any assimilation; and Fig. 2c presents the analysis. i.e., the model field with DA according to scheme (3), (4). It is clearly seen that the field in Fig. 2c realizes some intermediate position between those in Figs. 2a and 2b. Indeed, Fig. 2a has many synoptical eddies, dynamically active and chaotic structures. On the contrary, Fig. 2b contains much less synoptical eddies and it is less dynamical. Figure 2c captures the observed dynamic structures, however, it is substantially smoothed by the model itself.

Fig. 1. The model domain and the trace of one of the satellite.

Fig. 2. The SSLA fields on 31 Jan 2010: (a) – observations, (b) – control, (c) – analysis.

4 Stability of the data assimilation method

The stability of the method will be proved on the theory of Markov chains. As it is seen from (3) the sequents of random variables $X_n, n=0, \dots, N$ forms the Markov chain with discrete time-steps and continuous, in general, set of states and let x be their values. Let $\{X_n\}$ be the set of states at time-step $n, n=0, \dots, N$ and $\{X_N\} = \bigcup_{n=0}^N \{X_n\}$. Denote $p^s(x, y)$ the conditional probability of event, that the Markov chain (3) starting at state x arrives at state y after s time-steps. If there exist two time-steps s_1, s_2 such as two states x, y belonging to $\{X_N\}$ satisfy the condition $p^{s_1}(x, y) > 0$ and $p^{s_2}(x, y) > 0$ these states are called (mutual) recurrent. It is easy to prove that entire state space $\{X_N\}$ is divided by this relation into disjoint classes, $K_s, s \geq 1$, $\{X_N\} = \bigcup_s K_s$, within each class any two states (mutually) recurrent. Initial condition(s) are belonged to the same class according to definition.

Definition. Let random variable Y be consistent with model $\Lambda(X, t)$ if there is a variable $X \in \{X_n\}$ such as $Y = HX$ with P – non zero probability.

Lemma. Let the following conditions be fulfilled: 1) the value y_{N+1} is consistent with the model at step N , 2) the function $\Lambda(X, t)$ is continuous with respect to both variables and bounded for any t ; 3) the interval Δt is small enough such the matrix $I - KH\Delta t$ is invertible, where I is the unique matrix of r -dimension. Then the solutions of system (3) at step $N+1$ belong to the same class as at steps $n = 0, \dots, N$.

Proof. It is sufficient to consider the statement of Lemma only for $s=1$, where s is the time-interval of transition of the studied Markov chain. Let x, y are two states, $x \in \{X_N\}$ and $y \in \{X_{N+1}\}$. The condition $p(x, y) > 0$ means that the solution of equation

$y = x + \Lambda(x, t_N)\Delta t + K_{N+1}(Y_{N+1} - Hx)\Delta t$ exists with P – non-zero probability. From this equation one can obtain that $x - K_{N+1}Hx\Delta t = y - \Lambda(x, t_N)\Delta t - K_{N+1}Y_{N+1}\Delta t$ or

$$(I - (K_{N+1}H\Delta t)^{-1})x = y - \Lambda(x, t_N)\Delta t - K_{N+1}Hx\Delta t \quad (5)$$

for some $z \in \{X_{N+1}\}$.

From the Lemma conditions the matrix $(I - K_{N+1}H\Delta t)^{-1}$ exists and the function $\Lambda(x, t_N)\Delta t$ can be replaced as $\Lambda(x^*, t_N)$, where x^* is some mean value, because of its continuity and limitation. Therefore the equation (5) has a solution with non -zero P -probability, which leads to the condition $p(x, y) > 0$. This proves the lemma.

Corollary. If any two states $x, y \in \{X_\infty\} = \bigcup_{n=0}^\infty \{X_n\}$ belong to the same class as x_0 they are recurrent, so that

there is a number s , such as $p^s(x, y) > 0$.

Using the properties of irreducible Markov chains we can conclude that the following statement is valid

Theorem. If the conditions of Lemma hold there exist the limit distribution of Markov chain (3)

$$p(X_n = x) \rightarrow \pi(x), n \rightarrow \infty.$$

If, additionally, the model solution of the system (1) is a priori bounded, that is, there is the constant C , such as

$$\left\| \int_{t_0}^t \Lambda(X(\tau))d\tau \right\| < C \text{ for any } t, \text{ then } \int_{-\infty}^{\infty} \pi(x)dx = 1.$$

The proof of the theorem follows from the known properties of Markov chains [8].

As a consequence of the presented result we can assert that if all conditions hold, the equations (3), (4) becomes stable if the number of time-steps unlimitedly increase.

5 Conclusions

This paper states that the presented earlier and studied in the current work DA method has several advantages. It is not only practically feasible and numerical effective, which is shown before but it is also theoretically consistent. It is proved that under reasonable conditions this DA scheme is stable in sense of long-term integrations and its limit behavior is realistic and can be assessed numerically. In the further development it is possible to obtain the characteristics of the limit distribution and directly calculate their average value and variance. Physically it means that if the model scenario is realistic and is consistent with the observations the resulted model fields after assimilations have physical sense and can be analyzed.

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