Soundness of Inference Rules for New Vague Multivalued Dependencies

Dženana Gušić^{1,*}

¹University of Sarajevo, Faculty of Sciences and Mathematics, Department of Mathematics, Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina

Abstract. In the present paper we give a new definition of vague multivalued dependencies in database relations. The definition is based on application of arbitrary similarity measure on vague values, which is known to be reflexive, symmetric, and max-min transitive. The definition is adapted in order to include the imprecise and precise vague multivalued dependencies. The inference rules for new vague multivalued dependencies are listed, and are shown to be sound.

1 Introduction

Let V be a vague set in U , where U is some universe of discourse. We have,

$$V = \{ \langle u, [t_V(u), 1 - f_V(u)] \rangle : u \in U \},$$

where $t_V, f_V : U \rightarrow [0, 1]$ are some functions such that $t_V(u) + f_V(u) \leq 1$ for all $u \in U$.

The interval $[t_V(u), 1 - f_V(u)] \subseteq [0, 1]$ represents a vague value associated to $u \in U$.

Obviously, if $t_V(u) = 1 - f_V(u) \in (0, 1)$, the vague value $[t_V(u), 1 - f_V(u)]$ becomes the fuzzy value $t_V(u)$. In particular, $[t_V(u), 1 - f_V(u)]$ becomes the crisp value 1 if $t_V(u) = 1 - f_V(u) = 1$. Finally, if $t_V(u) = 1 - f_V(u) = 0$, then we assume that u is not an element of the vague set V .

Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in \{1, 2, \dots, n\} = I$. Suppose that $V(U_i)$ is the family of all vague sets in U_i , $i \in I$. A vague relation instance r on $R(A_1, A_2, \dots, A_n)$ is a subset of the cross product

$$V(U_1) \times V(U_2) \times \dots \times V(U_n).$$

Suppose that

$$t[A_i] = \{ \langle u_i, a_{u_i}^t \rangle : u_i \in U_i \},$$

for all $i \in I$, and all tuples $t = (t[A_1], t[A_2], \dots, t[A_n]) \in r$, where r is some fuzzy relation instance on $R(A_1, A_2, \dots, A_n)$. More precisely, suppose that $a_{u_i}^t \in [0, 1]$ for all $u_i \in U_i$, $i \in I$, and all $t \in r$, where $a_{u_i}^t$ is the membership value of the element $u_i \in U_i$ to the fuzzy set $t[A_i]$. The value $t[A_i]$ of the attribute A_i on the tuple t may be represented as

$$t[A_i] = \{ \langle u_i, [a_{u_i}^t, a_{u_i}^t] \rangle : u_i \in U_i \}.$$

Hence, the fuzzy relation instance r on $R(A_1, A_2, \dots, A_n)$ may be represented as a vague relation instance on $R(A_1, A_2, \dots, A_n)$.

Similarly, if

$$t[A_i] = \{ u_i^t \}$$

for all $i \in I$, and all tuples $t \in r$, where u_i^t is some element in U_i , and r is some relation instance on $R(A_1, A_2, \dots, A_n)$, then, we may write

$$t[A_i] = \langle u_i^t, [1, 1] \rangle \cup \{ \langle u_i, [0, 0] \rangle : u_i \in U_i \setminus \{ u_i^t \} \}.$$

Therefore, the relation instance r on $R(A_1, A_2, \dots, A_n)$ may also be represented as a vague relation instance on $R(A_1, A_2, \dots, A_n)$.

The discussion given above, shows that the vague relational concept generalizes in a natural way both, the classical relational concept as well as the fuzzy relational concept.

For the basic relational concepts, we refer to [12] (see also, [8], [1], [20]).

In [10], we applied the similarity measures defined as follows (see also, [14], [5]-[11], [13], [18], [9]).

Let $x = [a, 1 - b] \subseteq [0, 1]$ and $y = [c, 1 - d] \subseteq [0, 1]$ be some vague values. It is not required these values to be associated to the elements of the same universe of discourse.

The similarity measure $SE(x, y)$ between the vague values x and y is given by

$$SE(x, y) = \sqrt{\left(1 - \frac{|(a - c) - (b - d)|}{2} \right) (1 - |(a - c) + (b - d)|)}.$$

It is known that $SE(x, y) \in [0, 1]$, $SE(x, y) = SE(y, x)$, $SE(x, y) = 1$ if and only if $x = y$, and $SE(x, y) = 0$ if and only if $x = [0, 0]$, $y = [1, 1]$ or $x = [0, 1]$, $y = [q, q]$, $0 \leq q \leq 1$.

*e-mail: dzenang@pmf.unsa.ba

If

$$A = \{\langle u, [t_A(u), 1 - f_A(u)] \rangle : u \in U\},$$

$$B = \{\langle u, [t_B(u), 1 - f_B(u)] \rangle : u \in U\}$$

are two vague sets in some universe of discourse U , then, the similarity measure $SE(A, B)$ between the vague sets A and B is given by

$$SE(A, B) = \frac{1}{|U|} \sum_{u \in U} SE([t_A(u), 1 - f_A(u)], [t_B(u), 1 - f_B(u)])$$

$$= \frac{1}{|U|} \sum_{u \in U} \sqrt{1 - \frac{|(t_A(u) - t_B(u)) - (f_A(u) - f_B(u))|}{2}} \times$$

$$\sqrt{1 - |(t_{VA}(u) - t_B(u)) + (f_A(u) - f_B(u))|},$$

where $|U|$ is the number of the elements in U .

It is easily deduced that $SE(A, B) \in [0, 1]$, $SE(A, B) = SE(B, A)$, $SE(A, B) = 1$ if and only if $A = B$, and $SE(A, B) = 0$ if and only if $[t_A(u), 1 - f_A(u)] = [0, 0]$, $[t_B(u), 1 - f_B(u)] = [1, 1]$ for all $u \in U$ or $[t_A(u), 1 - f_A(u)] = [0, 1]$, $[t_B(u), 1 - f_B(u)] = [q, q]$, for all $u \in U$, where $0 \leq q \leq 1$.

The equality $A = B$ means that $A \subseteq B$ and $B \subseteq A$, where A is contained in B , i.e., $A \subseteq B$ holds true, if and only if $t_A(u) \leq t_B(u)$ and $1 - f_A(u) \leq 1 - f_B(u)$ for all $u \in U$.

So, $A = B$ if and only if $t_A(u) = t_B(u)$, $f_A(u) = f_B(u)$ for all $u \in U$.

Finally, if $R(A_1, A_2, \dots, A_n)$ is a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$, r is a vague relation instance on $R(A_1, A_2, \dots, A_n)$, t_1 and t_2 are any two tuples in r , and $X \subseteq \{A_1, A_2, \dots, A_n\}$ is a set of attributes, then, the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X is given by

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

It is also easily deduced that $SE_X(t_1, t_2) \in [0, 1]$, $SE_X(t_1, t_2) = SE_X(t_2, t_1)$, $SE_X(t_1, t_2) = 1$ if and only if $t_1[A_k] = t_2[A_k]$ for all $A_k \in X$ if and only if

$t_{1[A_k]}(u) = t_{2[A_k]}(u)$, $f_{t_1[A_k]}(u) = f_{t_2[A_k]}(u)$ for all $u \in U_k$, and all $A_k \in X$, and $SE_X(t_1, t_2) = 0$ if and only if there exists $A_k \in X$ such that $[t_{1[A_k]}(u), 1 - f_{t_1[A_k]}(u)] = [0, 0]$, $[t_{2[A_k]}(u), 1 - f_{t_2[A_k]}(u)] = [1, 1]$ for all $u \in U_k$ or $[t_{1[A_k]}(u), 1 - f_{t_1[A_k]}(u)] = [0, 1]$, $[t_{2[A_k]}(u), 1 - f_{t_2[A_k]}(u)] = [q, q]$ for all $u \in U_k$, where $0 \leq q \leq 1$.

In [10], we have proved the assertions:

1) $SE_Y(t_1, t_2) \geq SE_X(t_1, t_2)$ for t_1, t_2 in r , if $Y \subseteq X \subseteq \{A_1, A_2, \dots, A_n\}$,

2) $SE_X(t_1, t_2) \geq \theta$ if $SE(t_1[A], t_2[A]) \geq \theta$ for all $A \in X$,

3) $SE_X(t_1, t_2) \geq \theta$ and $SE_X(t_2, t_3) \geq \theta$ do not necessarily imply that $SE_X(t_1, t_3) \geq \theta$, where t_1, t_2 and t_3 are some mutually distinct tuples in r .

In this paper we introduce the similarity measures in the following way.

Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$.

Denote by $Vag(U_i)$ the set of all vague values associated to the elements $u_i \in U_i$, $i \in I$.

A similarity measure on $Vag(U_i)$ is a mapping

$SE_i : Vag(U_i) \times Vag(U_i) \rightarrow [0, 1]$, such that

$SE_i(x, x) = 1$, $SE_i(x, y) = SE_i(y, x)$, and

$SE_i(x, z) \geq \max_{y \in Vag(U_i)} (\min(SE_i(x, y), SE_i(y, z)))$ for all $x, y, z \in Vag(U_i)$.

Suppose that SE_i is a similarity measure on $Vag(U_i)$, $i \in I$.

Let

$$A_i = \{\langle u, [t_{A_i}(u), 1 - f_{A_i}(u)] \rangle : u \in U_i\}$$

$$= \{a_u^i : u \in U_i\},$$

$$B_i = \{\langle u, [t_{B_i}(u), 1 - f_{B_i}(u)] \rangle : u \in U_i\}$$

$$= \{b_u^i : u \in U_i\}$$

be two vague sets in U_i .

The similarity measure $SE(A_i, B_i)$ between the vague sets A_i and B_i is given by

$$SE(A_i, B_i) = \min \left\{ \min_{a_u^i \in A_i} \left\{ \max_{b_u^i \in B_i} \left\{ SE_i([t_{A_i}(u), 1 - f_{A_i}(u)], [t_{B_i}(u), 1 - f_{B_i}(u)]) \right\} \right\} \right\},$$

$$\min_{b_u^i \in B_i} \left\{ \max_{a_u^i \in A_i} \left\{ SE_i([t_{B_i}(u), 1 - f_{B_i}(u)], [t_{A_i}(u), 1 - f_{A_i}(u)]) \right\} \right\} \right\}.$$

Now, if r is a vague relation instance on

$R(A_1, A_2, \dots, A_n)$, t_1 and t_2 are any two tuples in r , and X is a subset of $\{A_1, A_2, \dots, A_n\}$, then, the similarity measure $SE_X(t_1, t_2)$ between the tuples t_1 and t_2 on the attribute set X has the same form as before, i.e.,

$$SE_X(t_1, t_2) = \min_{A \in X} \{SE(t_1[A], t_2[A])\}.$$

Note that the assertions 1) and 2) remain valid if we take them with respect to new similarity measures. Namely, it is obvious that the proofs of these assertions do not depend on the choice of function $SE : V(U_i) \times V(U_i) \rightarrow [0, 1]$ (see, [10]).

The assertion 3), however, does not hold anymore. In particular, the following assertion holds true.

Lemma 1 Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$. Let r be a vague relation instance on $R(A_1, A_2, \dots, A_n)$. If $SE_X(t_i, t_j) \geq \theta$ and $SE_X(t_j, t_k) \geq \theta$, where t_i, t_j and t_k are any three tuples in r , and X is a subset of $\{A_1, A_2, \dots, A_n\}$, then $SE_X(t_i, t_k) \geq \theta$.

Proof. Since

$$\begin{aligned} & \min_{A \in X} \{SE(t_i[A], t_j[A])\} \\ &= SE_X(t_i, t_j) \geq \theta, \\ & \min_{A \in X} \{SE(t_j[A], t_k[A])\} \\ &= SE_X(t_j, t_k) \geq \theta, \end{aligned}$$

it follows that $SE(t_i[A], t_j[A]) \geq \theta$, $SE(t_j[A], t_k[A]) \geq \theta$ for all $A \in X$.

Let $A \in X$. Denote by $U \in \{U_1, U_2, \dots, U_n\}$ the corresponding universe of discourse, and by $SE_U : Vag(U) \times Vag(U) \rightarrow [0, 1]$, the corresponding similarity measure on $Vag(U)$. Suppose that $|U| = N$. We shall write $U = \{u_1, u_2, \dots, u_N\}$. We have,

$$\begin{aligned} t_i[A] &= \{\langle u, [t_{t_i[A]}(u), 1 - f_{t_i[A]}(u)] \rangle : u \in U\} \\ &= \{\langle u_1, a_1^i \rangle, \langle u_2, a_2^i \rangle, \dots, \langle u_N, a_N^i \rangle\}, \\ t_j[A] &= \{\langle u, [t_{t_j[A]}(u), 1 - f_{t_j[A]}(u)] \rangle : u \in U\} \\ &= \{\langle u_1, a_1^j \rangle, \langle u_2, a_2^j \rangle, \dots, \langle u_N, a_N^j \rangle\}, \\ t_k[A] &= \{\langle u, [t_{t_k[A]}(u), 1 - f_{t_k[A]}(u)] \rangle : u \in U\} \\ &= \{\langle u_1, a_1^k \rangle, \langle u_2, a_2^k \rangle, \dots, \langle u_N, a_N^k \rangle\}. \end{aligned}$$

Since $SE(t_i[A], t_j[A]) \geq \theta$ and $SE(t_j[A], t_k[A]) \geq \theta$ hold true, it follows that

$$\begin{aligned} & \min \left\{ \min \left\{ \max \{SE_U(a_1^i, a_1^j), \dots, SE_U(a_1^i, a_N^j)\}, \right. \right. \\ & \max \{SE_U(a_2^i, a_1^j), \dots, SE_U(a_2^i, a_N^j)\}, \\ & \vdots \\ & \max \{SE_U(a_N^i, a_1^j), \dots, SE_U(a_N^i, a_N^j)\} \Big\}, \\ & \min \left\{ \max \{SE_U(a_1^j, a_1^k), \dots, SE_U(a_1^j, a_N^k)\}, \right. \\ & \max \{SE_U(a_2^j, a_1^k), \dots, SE_U(a_2^j, a_N^k)\}, \\ & \vdots \\ & \max \{SE_U(a_N^j, a_1^k), \dots, SE_U(a_N^j, a_N^k)\} \Big\} \Big\} \geq \theta, \\ & \min \left\{ \min \left\{ \max \{SE_U(a_1^j, a_1^k), \dots, SE_U(a_1^j, a_N^k)\}, \right. \right. \\ & \max \{SE_U(a_2^j, a_1^k), \dots, SE_U(a_2^j, a_N^k)\}, \\ & \vdots \\ & \max \{SE_U(a_N^j, a_1^k), \dots, SE_U(a_N^j, a_N^k)\} \Big\}, \\ & \min \left\{ \max \{SE_U(a_1^k, a_1^i), \dots, SE_U(a_1^k, a_N^i)\}, \right. \\ & \max \{SE_U(a_2^k, a_1^i), \dots, SE_U(a_2^k, a_N^i)\}, \\ & \vdots \\ & \max \{SE_U(a_N^k, a_1^i), \dots, SE_U(a_N^k, a_N^i)\} \Big\} \Big\} \geq \theta, \end{aligned}$$

$\vdots$

$$\max \{SE_U(a_N^k, a_1^i), \dots, SE_U(a_N^k, a_N^i)\} \Big\} \geq \theta.$$

Hence,

$$\max \{SE_U(a_p^i, a_1^j), \dots, SE_U(a_p^i, a_N^j)\} \geq \theta \quad (1)$$

for all $p \in \{1, 2, \dots, N\}$,

$$\max \{SE_U(a_q^j, a_1^i), \dots, SE_U(a_q^j, a_N^i)\} \geq \theta \quad (2)$$

for all $q \in \{1, 2, \dots, N\}$,

$$\max \{SE_U(a_r^j, a_1^k), \dots, SE_U(a_r^j, a_N^k)\} \geq \theta \quad (3)$$

for all $r \in \{1, 2, \dots, N\}$,

$$\max \{SE_U(a_s^k, a_1^i), \dots, SE_U(a_s^k, a_N^i)\} \geq \theta \quad (4)$$

for all $s \in \{1, 2, \dots, N\}$.

Our aim is to prove that $SE(t_i[A], t_k[A]) \geq \theta$. Thus, we have to prove that

$$\max \{SE_U(a_t^i, a_1^k), \dots, SE_U(a_t^i, a_N^k)\} \geq \theta$$

for all $t \in \{1, 2, \dots, N\}$, and

$$\max \{SE_U(a_w^k, a_1^i), \dots, SE_U(a_w^k, a_N^i)\} \geq \theta$$

for all $w \in \{1, 2, \dots, N\}$.

Let $t \in \{1, 2, \dots, N\}$ and $w \in \{1, 2, \dots, N\}$. It is enough to prove that there exists $u \in \{1, 2, \dots, N\}$ such that $SE_U(a_t^i, a_u^k) \geq \theta$, and $v \in \{1, 2, \dots, N\}$ such that $SE_U(a_w^k, a_v^i) \geq \theta$.

Put $p = t$. By (1), there exists $x \in \{1, 2, \dots, N\}$ such that $SE_U(a_t^i, a_x^j) \geq \theta$. Put $r = x$. By (3), there exists $y \in \{1, 2, \dots, N\}$ such that $SE_U(a_x^j, a_y^k) \geq \theta$. We obtain,

$$\begin{aligned} & SE_U(a_t^i, a_y^k) \\ & \geq \max_{y \in Vag(U)} (\min(SE_U(a_t^i, y), SE_U(y, a_y^k))) \\ & \geq \min(SE_U(a_t^i, a_x^j), SE_U(a_x^j, a_y^k)) \\ & \geq \min(\theta, \theta) = \theta. \end{aligned}$$

Thus, putting $u = y$, we conclude that $SE_U(a_t^i, a_u^k) \geq \theta$.

Put $s = w$. By (4), there exists $e \in \{1, 2, \dots, N\}$ such that $SE_U(a_w^k, a_e^j) \geq \theta$. Put $q = e$. By (2), there exists $f \in \{1, 2, \dots, N\}$ such that $SE_U(a_e^j, a_f^i) \geq \theta$. We deduce,

$$\begin{aligned} & SE_U(a_w^k, a_f^i) \\ & \geq \max_{y \in Vag(U)} (\min(SE_U(a_w^k, y), SE_U(y, a_f^i))) \\ & \geq \min(SE_U(a_w^k, a_e^j), SE_U(a_e^j, a_f^i)) \\ & \geq \min(\theta, \theta) = \theta. \end{aligned}$$

Hence, putting $v = f$, we obtain that $SE_U(a_w^k, a_v^i) \geq \theta$.

Therefore, $SE(t_i[A], t_k[A]) \geq \theta$ for all $A \in X$. Now,

$$\begin{aligned} & SE_X(t_i, t_k) \\ &= \min_{A \in X} \{SE(t_i[A], t_k[A])\} \geq \theta, \end{aligned}$$

This completes the proof. $\square$

2 Vague multivalued dependencies

Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$. Suppose that r is a relation instance on $R(A_1, A_2, \dots, A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, \dots, A_n\}$, and $Z = \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$.

Relation instance r is said to satisfy the multivalued dependency $X \twoheadrightarrow Y$, if for every pair of tuples t_1 and t_2 in r , $t_1[X] = t_2[X]$ implies that there exists a tuple t_3 in r , such that $t_3[X] = t_1[X]$, $t_3[Y] = t_1[Y]$, and $t_3[Z] = t_2[Z]$.

Note the following facts:

Multivalued dependencies are introduced by Fagin [7], $t_1[X] = t_2[X]$ means that $t_1[A] = t_2[A]$ for all $A \in X$, $t[A_i] \in U_i$ for all $i \in I$, and all $t \in r$,

there exists the identity relation $i_j : U_j \times U_j \rightarrow \{0, 1\}$, $j \in I$, such that $i_j(t_k[A_j], t_l[A_j]) = 1$ if and only if $t_k[A_j] = t_l[A_j]$, and $i_j(t_k[A_j], t_l[A_j]) = 0$ if and only if $t_k[A_j] \neq t_l[A_j]$, where $t_k, t_l \in r$.

If we put $\emptyset \neq t[A_i] \subseteq U_i$ for all $i \in I$, and all $t \in r$, then the relation instance r becomes a fuzzy relation instance on $R(A_1, A_2, \dots, A_n)$. In this setting we are able to determine how similar (or how conformant) $t_1[X]$ and $t_2[X]$ are. More precisely, we calculate the conformance $\varphi(X[t_1, t_2])$ of the attribute set X on tuples t_1 and t_2 as

$$\varphi(X[t_1, t_2]) = \min_{A_k \in X} \{\varphi(A_k[t_1, t_2])\},$$

where the conformance $\varphi(A_k[t_1, t_2])$ of the attribute A_k on tuples t_1 and t_2 is given by

$$\begin{aligned} \varphi(A_k[t_1, t_2]) &= \min \left\{ \min_{x \in t_1[A_k]} \left\{ \max_{y \in t_2[A_k]} \{s_k(x, y)\} \right\}, \right. \\ &\quad \left. \min_{x \in t_2[A_k]} \left\{ \max_{y \in t_1[A_k]} \{s_k(x, y)\} \right\} \right\}. \end{aligned}$$

Here, $s_k : U_k \times U_k \rightarrow [0, 1]$ is a similarity relation on U_k , $k \in I$, i.e., $s_k(x, x) = 1$, $s_k(x, y) = s_k(y, x)$, and $s_k(x, z) \geq \max_{y \in U_k} (\min(s_k(x, y), s_k(y, z)))$ for all $x, y, z \in U_k$.

For the similarity-based fuzzy relational database approach, see, [2]-[4].

Now, it would be natural to state that some fuzzy relation instance r on $R(A_1, A_2, \dots, A_n)$ satisfies the fuzzy multivalued dependency $X \twoheadrightarrow_F Y$, if for every pair of tuples t_1 and t_2 in r , there exists a tuple t_3 in r , such that

$$\begin{aligned} \varphi(X[t_3, t_1]) &\geq \varphi(X[t_1, t_2]), \\ \varphi(Y[t_3, t_1]) &\geq \varphi(X[t_1, t_2]), \\ \varphi(Z[t_3, t_2]) &\geq \varphi(X[t_1, t_2]). \end{aligned} \quad (5)$$

However, it is not so hard to select both, a fuzzy relation instance r on $R(A_1, A_2, \dots, A_n)$, and a fuzzy multivalued dependency $X \twoheadrightarrow_F Y$, $X, Y \subseteq \{A_1, A_2, \dots, A_n\}$, such that r satisfies $X \twoheadrightarrow_F Y$ in reality, and $X \twoheadrightarrow_F Y$ makes perfectly sense by itself, but there are tuples t_1 and t_2 in r , such that (5) fails for every t_3 in r . In other words, the scenario where some of the elements $\varphi(X[t_3, t_1])$, $\varphi(Y[t_3, t_1])$ and $\varphi(Z[t_3, t_2])$ are slightly smaller than $\varphi(X[t_1, t_2])$ for

all $t_3 \in r$, may occur. Therefore, the condition (5) is not adequate for determining if some fuzzy relation instance satisfies some fuzzy multivalued dependency. In particular, if (5) holds true, the instance r satisfies $X \twoheadrightarrow_F Y$. Otherwise, r may or may not satisfy $X \twoheadrightarrow_F Y$.

Note that several authors, including Tripathy-Saxena [19] and Nakata [16], have been taken attempts in order to express the fuzzy multivalued dependencies in various fuzzy relational database models.

Sozat and Yazici [17], adapted (5) in the following way.

Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$. Suppose that r is a fuzzy relation instance on $R(A_1, A_2, \dots, A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, \dots, A_n\}$, $Z = \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$, and $\theta \in [0, 1]$. Fuzzy relation instance r is said to satisfy the fuzzy multivalued dependency $X \twoheadrightarrow_F^\theta Y$, if for every pair of tuples t_1 and t_2 in r , there exists a tuple t_3 in r , such that

$$\begin{aligned} \varphi(X[t_3, t_1]) &\geq \min(\theta, \varphi(X[t_1, t_2])), \\ \varphi(Y[t_3, t_1]) &\geq \min(\theta, \varphi(X[t_1, t_2])), \\ \varphi(Z[t_3, t_2]) &\geq \min(\theta, \varphi(X[t_1, t_2])). \end{aligned} \quad (6)$$

Thus, if it happens that for some t'_1 and t'_2 in r , some of the elements $\varphi(X[t_3, t'_1])$, $\varphi(Y[t_3, t'_1])$ and $\varphi(Z[t_3, t'_2])$ are smaller than $\varphi(X[t'_1, t'_2])$, and the elements in

$$\{\varphi(X[t_3, t'_1]), \varphi(Y[t_3, t'_1]), \varphi(Z[t_3, t'_2])\}$$

that are smaller than $\varphi(X[t'_1, t'_2])$ are larger than θ for all $t_3 \in r$, then, the condition (6) will be fulfilled, so the instance r will satisfy $X \twoheadrightarrow_F^\theta Y$ (assuming that (6) is fulfilled for $(t_1, t_2) \in r \times r$, $(t_1, t_2) \neq (t'_1, t'_2)$).

The value $\theta \in [0, 1]$ that appears in the notation $X \twoheadrightarrow_F^\theta Y$ is called the linguistic strength of the fuzzy multivalued dependency. If $\theta = 1$, the fuzzy multivalued dependency $X \twoheadrightarrow_F^\theta Y$ becomes $X \twoheadrightarrow_F Y$.

Now, reasoning as in the fuzzy case, we first state that some vague relation instance r on $R(A_1, A_2, \dots, A_n)$ satisfies the vague multivalued dependency $X \twoheadrightarrow_V Y$, if for every pair of tuples t_1 and t_2 in r , there exists a tuple t_3 in r , such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq SE_X(t_1, t_2), \\ SE_Y(t_3, t_1) &\geq SE_X(t_1, t_2), \\ SE_Z(t_3, t_2) &\geq SE_X(t_1, t_2). \end{aligned} \quad (7)$$

Then, we adapt (7) to the following form.

Let $R(A_1, A_2, \dots, A_n)$ be a relation scheme on domains $U_1, U_2, \dots, U_n$, where A_i is an attribute on the universe of discourse U_i , $i \in I$. Suppose that r is a vague relation instance on $R(A_1, A_2, \dots, A_n)$. Furthermore, let X and Y be subsets of $\{A_1, A_2, \dots, A_n\}$, $Z = \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$, and $\theta \in [0, 1]$. Vague relation instance r is said to satisfy the vague multivalued dependency $X \twoheadrightarrow_V^\theta Y$, if for every

pair of tuples t_1 and t_2 in r , there exists a tuple t_3 in r , such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min(\theta, SE_X(t_1, t_2)), \\ SE_Y(t_3, t_1) &\geq \min(\theta, SE_X(t_1, t_2)), \\ SE_Z(t_3, t_2) &\geq \min(\theta, SE_X(t_1, t_2)). \end{aligned}$$

If $\theta = 1$, the vague multivalued dependency $X \xrightarrow{\theta}_V Y$ becomes $X \rightarrow_V Y$.

For yet another definition of vague multivalued dependency, called α -vague multivalued dependency, see [15].

Note that by [10], r satisfies the vague functional dependency $X \xrightarrow{\theta}_V Y$, if for every pair of tuples t_1 and t_2 in r ,

$$SE_Y(t_1, t_2) \geq \min(\theta, SE_X(t_1, t_2)).$$

$X \xrightarrow{\theta}_V Y$ becomes $X \rightarrow_V Y$ if $\theta = 1$.

3 Soundness of inference rules for vague multivalued dependencies

The following rules are the inference rules for vague multivalued dependencies (VMVDs).

VM1 Inclusive rule for VMVDs: If $X \xrightarrow{\theta_1}_V Y$ holds, and $\theta_1 \geq \theta_2$, then $X \xrightarrow{\theta_2}_V Y$ holds.

VM2 Complementation rule for VMVDs: If $X \xrightarrow{\theta}_V Y$ holds, then $X \xrightarrow{\theta}_V Q$ holds, where $Q = \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$.

VM3 Augmentation rule for VMVDs: If $X \xrightarrow{\theta}_V Y$ holds, and $W \supseteq Z$, then $W \cup X \xrightarrow{\theta}_V Y \cup Z$ also holds.

VM4 Transitivity rule for VMVDs: If $X \xrightarrow{\theta_1}_V Y$ and $Y \xrightarrow{\theta_2}_V Z$ hold true, then $X \xrightarrow{\min(\theta_1, \theta_2)}_V Z$ holds true.

VM5 Replication rule: If $X \xrightarrow{\theta}_V Y$ holds, then $X \xrightarrow{\theta}_V Y$ holds.

VM6 Coalescence rule for VFDs and VMVDs: If $X \xrightarrow{\theta_1}_V Y$ holds, $Z \subseteq Y$, and for some W disjoint from Y , we have that $W \xrightarrow{\theta_2}_V Z$ holds true, then $X \xrightarrow{\min(\theta_1, \theta_2)}_V Z$ also holds true.

In [10], we listed the inference rules for vague functional dependencies. There, we proved that the rules are sound, and that the set of these rules, i.e., the set $\{VF1, VF2, VF3, VF4\}$ is complete set. Additional inference rules (labeled as $VF5, VF6$ and $VF7$) are also proved to be sound.

Since the proofs of the corresponding theorems in [10] do not depend on the choice of similarity measures between vague values and vague sets, these theorems remain valid in the present setting, i.e., for the choice: SE_i , $i \in I$, SE , and SE_X .

Theorem 2 The inference rules: VM1, VM2, VM3, VM4, VM5 and VM6 are sound.

Proof. (proof for VM1) It is enough to prove that r satisfies $X \xrightarrow{\theta_2}_V Y$, if r satisfies $X \xrightarrow{\theta_1}_V Y$, where r is any vague relation instance on $R(A_1, A_2, \dots, A_n)$.

Assume that r satisfies $X \xrightarrow{\theta_1}_V Y$.

Furthermore, assume that r does not satisfy $X \xrightarrow{\theta_2}_V Y$.

Since r does not satisfy $X \xrightarrow{\theta_2}_V Y$, we know that there are tuples t_1 and t_2 in r , such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min(\theta_2, SE_X(t_1, t_2)), \\ SE_Y(t_3, t_1) &\geq \min(\theta_2, SE_X(t_1, t_2)), \\ SE_Z(t_3, t_2) &\geq \min(\theta_2, SE_X(t_1, t_2)) \end{aligned}$$

don't hold at the same time for any $t_3 \in r$.

Since t_1 and t_2 belong to r , and r satisfies $X \xrightarrow{\theta_1}_V Y$, it follows that there exists a tuple t_3 in r such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min(\theta_1, SE_X(t_1, t_2)), \\ SE_Y(t_3, t_1) &\geq \min(\theta_1, SE_X(t_1, t_2)), \\ SE_Z(t_3, t_2) &\geq \min(\theta_1, SE_X(t_1, t_2)). \end{aligned}$$

The fact that $\theta_1 \geq \theta_2$ yields that

$$\min(\theta_1, SE_X(t_1, t_2)) \geq \min(\theta_2, SE_X(t_1, t_2)).$$

Hence,

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min(\theta_2, SE_X(t_1, t_2)), \\ SE_Y(t_3, t_1) &\geq \min(\theta_2, SE_X(t_1, t_2)), \\ SE_Z(t_3, t_2) &\geq \min(\theta_2, SE_X(t_1, t_2)). \end{aligned}$$

This is a contradiction.

We conclude, r satisfies $X \xrightarrow{\theta_2}_V Y$.

(proof for VM3) Suppose that r is a vague relation instance on $R(A_1, A_2, \dots, A_n)$, such that r satisfies $X \xrightarrow{\theta}_V Y$, and violates $W \cup X \xrightarrow{\theta}_V Y \cup Z$.

Since r violates $W \cup X \xrightarrow{\theta}_V Y \cup Z$, it follows that there exist tuples t_1 and t_2 in r , such that

$$\begin{aligned} SE_{W \cup X}(t_3, t_1) &\geq \min(\theta, SE_{W \cup X}(t_1, t_2)), \\ SE_{Y \cup Z}(t_3, t_1) &\geq \min(\theta, SE_{W \cup X}(t_1, t_2)), \\ SE_{\{A_1, A_2, \dots, A_n\} \setminus (W \cup X \cup Y)}(t_3, t_2) &\geq \min(\theta, SE_{W \cup X}(t_1, t_2)) \end{aligned}$$

don't hold at the same time for any tuple $t_3 \in r$.

Now, $t_1, t_2 \in r$, and r satisfies $X \xrightarrow{\theta}_V Y$, hence, there is $t_3 \in r$, such that

$$\begin{aligned} SE_X(t_3, t_1) &\geq \min(\theta, SE_X(t_1, t_2)), \\ SE_Y(t_3, t_1) &\geq \min(\theta, SE_X(t_1, t_2)), \\ SE_{\{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)}(t_3, t_2) &\geq \min(\theta, SE_X(t_1, t_2)). \end{aligned}$$

Since $X \subseteq W \cup X$, we have that $SE_X(t_1, t_2) \geq SE_{W \cup X}(t_1, t_2)$. Therefore,

$$\min(\theta, SE_X(t_1, t_2)) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)).$$

Hence,

$$SE_X(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)),$$

$$SE_Y(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)),$$

$$SE_{\{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)}(t_3, t_2) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)).$$

The last inequality and the fact that $W \setminus (X \cup Y) \subseteq \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$ yield that

$$SE_{W \setminus (X \cup Y)}(t_3, t_2) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \quad (8)$$

Obviously,

$$SE_{W \cup X}(t_2, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)).$$

Hence, $W \setminus (X \cup Y) \subseteq W \cup Y$ implies that

$$SE_{W \setminus (X \cup Y)}(t_2, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \quad (9)$$

Now, (8) and (9) give us

$$SE_{W \setminus (X \cup Y)}(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)).$$

This inequality, and the inequalities:

$$SE_X(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)),$$

$$SE_Y(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2))$$

yield that

$$\begin{aligned} & SE_{W \cup X \cup Y}(t_3, t_1) \\ &= \min_{A \in W \cup X \cup Y} \{SE(t_3[A], t_1[A])\} \\ &= \min \left(\min_{A \in W \setminus (X \cup Y)} \{SE(t_3[A], t_1[A])\}, \right. \\ & \quad \min_{A \in X} \{SE(t_3[A], t_1[A])\}, \\ & \quad \left. \min_{A \in Y} \{SE(t_3[A], t_1[A])\} \right) \\ &= \min(SE_{W \setminus (X \cup Y)}(t_3, t_1), SE_X(t_3, t_1), SE_Y(t_3, t_1)) \\ &\geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \end{aligned}$$

Hence, $W \cup X \subseteq W \cup X \cup Y$ gives us

$$SE_{W \cup X}(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \quad (10)$$

Moreover, $Y \cup Z \subseteq W \cup X \cup Y$ gives us

$$SE_{Y \cup Z}(t_3, t_1) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \quad (11)$$

Finally,

$$SE_{\{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)}(t_3, t_2) \geq \min(\theta, SE_{W \cup X}(t_1, t_2)),$$

and the fact that $\{A_1, A_2, \dots, A_n\} \setminus (W \cup X \cup Y) \subseteq \{A_1, A_2, \dots, A_n\} \setminus (X \cup Y)$ imply that

$$\begin{aligned} & SE_{\{A_1, A_2, \dots, A_n\} \setminus (W \cup X \cup Y)}(t_3, t_2) \\ &\geq \min(\theta, SE_{W \cup X}(t_1, t_2)). \end{aligned} \quad (12)$$

Now, the inequalities (10), (11) and (12) taken together lead to contradiction.

We conclude, r satisfies $W \cup X \xrightarrow{\theta}_V Y \cup Z$.

The cases VM2, VM4-VM6 are discussed similarly.

This completes the proof. $\square$

4 Soundness of additional inference rules for vague multivalued dependencies

The following inference rules are additional inference rules for vague multivalued dependencies.

VM7 Union rule for VMVDs: If $X \xrightarrow{\theta_1}_V Y$ and $X \xrightarrow{\theta_2}_V Z$ hold true, then $X \xrightarrow{\min(\theta_1, \theta_2)}_V Y \cup Z$ holds true.

VM8 Pseudo-transitivity rule for VMVDs: If $X \xrightarrow{\theta_1}_V Y$ and $W \cup Y \xrightarrow{\theta_2}_V Z$ hold true, then $W \cup X \xrightarrow{\min(\theta_1, \theta_2)}_V Z \setminus (W \cup Y)$ holds also true.

VM9 Decomposition rule for VMVDs: If $X \xrightarrow{\theta_1}_V Y$ and $X \xrightarrow{\theta_2}_V Z$ hold true, then $X \xrightarrow{\min(\theta_1, \theta_2)}_V Y \cap Z$, $X \xrightarrow{\min(\theta_1, \theta_2)}_V Y \setminus Z$, and $X \xrightarrow{\min(\theta_1, \theta_2)}_V Z \setminus Y$ hold also true.

VM10 Mixed pseudo-transitivity rule: If $X \xrightarrow{\theta_1}_V Y$ and $X \cup Y \xrightarrow{\theta_2}_V Z$ hold true, then $X \xrightarrow{\min(\theta_1, \theta_2)}_V Z \setminus Y$ holds true.

Theorem 3 The inference rules: VM7, VM8, VM9 and VM10 are sound.

Proof. Follows by successive application of the inference rules: VF1-VF4, and VM1-VM6. $\square$

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