

Interfacial instability of two inviscid fluid layers under quasi-periodic oscillations

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Abstract

We investigate the effect of horizontal quasi-periodic oscillations on the stability of two immiscible fluids of different densities. The two fluid layers are confined in a cavity of infinite extension in the horizontal directions. We show in the inviscid theory that the linear stability analysis leads to the quasi-periodic Mathieu equation, with damping, which describes the evolution of the interfacial amplitude. Thus, we examine the effect of horizontal quasi-periodic vibration, with two incommensurate frequencies, on the stability of the interface. The numerical study shows the existence of two types of instability: the Kelvin-Helmholtz instability and the quasi-periodic resonances. The numerical results show also that an increase of the frequency ratio has a destabilizing effect on the Kelvin-Helmholtz instability and curves converge towards those of the periodic case.

Keywords: *Interfacial instability, quasi-periodic oscillation, Floquet's theory.*

1 Introduction

Numerous theoretical and experimental works studying the horizontal periodic forcing of two superimposed fluids filling a rectangular cavity were realized in the last decades [1, 2], leading to a thorough understanding on the interfacial instability between the two fluid layers. The oscillations can lead to significant and sometimes unexpected effects on the fluid interface. The oscillation can also be used to delay instabilities, for instance, Wolf [3] has shown that the horizontal modulation allows to suppress instabilities.

The studies cited above have examined the flow in a rectangular cavity subject to periodic oscillation. In this study, we focus attention on the influence of a horizontal quasi-periodic oscillation on the stability of the interface between two inviscid immiscible and incompressible fluid layers of different densities filling a rectangular cavity of infinite extension in the horizontal directions. In this situation, we determine the quasi-periodic basic flow. Hereafter, we perform an inviscid linear stability analysis and we examine the effect of the frequency ratio on the threshold of the interfacial instability.

2 Problem formulation

Consider two inviscid immiscible fluid layers filling a rectangular cavity of height $H = h_1 + h_2$ and infinite extension in the horizontal directions. The heavy fluid of density ρ_1 occupies the bottom region of height h_1 , and the light fluid of density ρ_2 occupies the upper region of height h_2 . Both fluids are in a stable gravitational configuration. The cavity is submitted to a horizontal quasi-periodic oscillation according to the law : $(a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t))\mathbf{i}$.

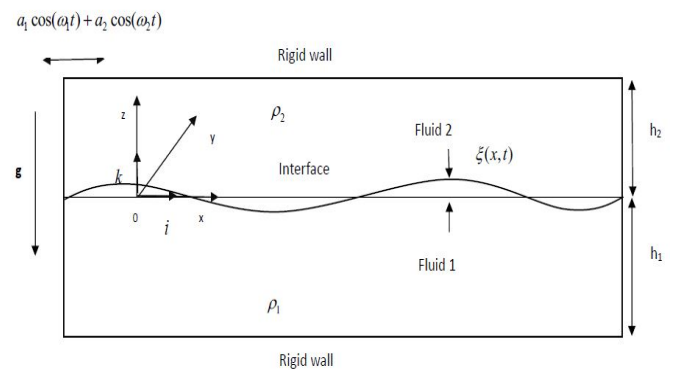


Figure 1: Two superimposed fluid layers in a rectangular cavity

Under these assumptions, the physical problem is governed by the following Navier-Stokes equations written in a relative frame:

$$\frac{\partial \mathbf{V}_\beta}{\partial t} + (\mathbf{V}_\beta \cdot \nabla) \mathbf{V}_\beta = -\frac{1}{\rho_\beta} \nabla P_\beta - g\mathbf{k} + (a_1 \omega_1^2 \cos(\omega_1 t) + a_2 \omega_2^2 \cos(\omega_2 t))\mathbf{i}, \quad (1)$$

$$\nabla \cdot \mathbf{V}_\beta = 0, \quad (2)$$

where P_β is the pressure, $\mathbf{V}_\beta(u, w)$ the velocity in each fluid layer and $\beta = 1, 2$. The corresponding boundary conditions are :

No-slip condition at the rigid walls

$$\mathbf{V}_1 \cdot \mathbf{n}_2 = 0, \quad z = h_1 \quad (3)$$

$$\mathbf{V}_2 \cdot \mathbf{n}_2 = 0, \quad z = h_2 \quad (4)$$

Kinematic and normal stress conditions at the interface

$$\frac{\partial \xi}{\partial t} = -\frac{\partial \xi}{\partial x} u_\beta + w_\beta \quad (5)$$

Normal stress balance:

$$P_1 - P_2 = \alpha \operatorname{div}(\mathbf{n}) \quad (6)$$

where α is the surface tension. The integral condition of balance of the displaced volume of both fluids is:

$$\int_{-h_1}^{\xi} \mathbf{V}_1 \cdot \mathbf{x} dz + \int_{\xi}^{h_2} \mathbf{V}_2 \cdot \mathbf{x} dz = 0 \quad (7)$$

The governing equations of the linear stability (1)-(7) are reduced to a quasi-periodic Mathieu equation for the amplitude $\xi(x, t)$ describing the interface displacement from its quasi-equilibrium horizontal position:

$$\begin{aligned} & \frac{d^2 \tilde{\xi}}{dt^2} (S_1 + S_2)(S_1 + S_2) + 2ik \frac{d\tilde{\xi}}{dt} (S_1 U_1^b + S_2 U_2^b) \\ & + \tilde{\xi} (ik(S_1 \frac{dU_1^b}{dt} + S_2 \frac{dU_2^b}{dt}) - k^2 (S_1 (U_1^b)^2 + S_2 (U_2^b)^2) \\ & + \alpha k^3 + gk(\rho_1 - \rho_2)) = 0 \end{aligned} \quad (8)$$

where k the wavenumber, $S_1 = \rho_1 \coth(kh_1)$, $S_2 = \rho_2 \coth(kh_2)$ and

$$U_1^b = a_1 \omega_1 h_2 \frac{\rho_1 - \rho_2}{\rho_2 h_1 + \rho_1 h_2} \sin(\omega_1 t) \quad (9)$$

$$+ a_2 \omega_2 h_2 \frac{\rho_1 - \rho_2}{\rho_2 h_1 + \rho_1 h_2} \sin(\omega_2 t)$$

$$U_2^b = -a_1 \omega_1 h_1 \frac{\rho_1 - \rho_2}{\rho_2 h_1 + \rho_1 h_2} \sin(\omega_1 t) \quad (10)$$

$$- a_2 \omega_2 h_1 \frac{\rho_1 - \rho_2}{\rho_2 h_1 + \rho_1 h_2} \sin(\omega_2 t)$$

The dimensionless form of equation (8) is obtained using the following reference quantities for time and length: $\tau = \omega_1 t$, $l_c = \left(\frac{\alpha}{g(\rho_1 - \rho_2)}\right)^{\frac{1}{2}}$ [1]. The expression (8) is then written as follows:

$$\begin{aligned} & \frac{d^2 \tilde{\xi}}{d\tau^2} + 2i\delta [\sin(\tau) + A\Omega \sin(\Omega\tau)] \frac{d\tilde{\xi}}{d\tau} \\ & + [i\delta (\cos(\tau) + A\Omega^2 \cos(\Omega\tau)) \\ & - \beta_1 (\sin(\tau) + A\Omega \sin(\Omega\tau))^2 + \beta_2] \tilde{\xi} = 0 \end{aligned} \quad (11)$$

with:

$$\delta = n \sqrt{\frac{4Bv}{We} \frac{(\rho-1)}{(H_1+\rho H_2)} \frac{\rho \coth(nH_1)H_2 - \coth(nH_2)H_1}{\rho \coth(nH_1) + \coth(nH_2)}} \quad (12)$$

$$\beta_1 = n^2 \frac{4Bv}{We} \frac{(\rho-1)^2}{(H_1+\rho H_2)^2} \frac{\rho \coth(nH_1)H_2^2 + \coth(nH_2)H_1^2}{\rho \coth(nH_1) + \coth(nH_2)} \quad (13)$$

$$\beta_2 = \frac{n(n^2+1)(\rho-1)}{We_1} \frac{1}{\rho \coth(nH_1) + \coth(nH_2)} \quad (14)$$

where, $n = kl_c$ is the dimensionless wavenumber, $\rho = \frac{\rho_2}{\rho_1}$ is the density ratio, $A = \frac{a_2}{a_1}$ is the amplitude ratio, $\Omega = \frac{\omega_2}{\omega_1}$ is the frequency ratio, $We = \frac{\omega_1^2 l_c}{g}$ is the Weber number based on the capillary length and $Bv = \frac{a_1^2 \omega_1^2}{4} \left(\frac{\rho_1 - \rho_2}{g\alpha}\right)^{\frac{1}{2}}$ is the Bond number characterizing the vibration intensity. Hereafter, we use the Floquet theory combined with the numerical method of Runge -Kutta for the numerical resolution, and we determine the boundaries of the instability regions in terms of the two parameters, B_v and n .

3 Results and discussion

We present in figure (2) the neutral stability diagram, $B_v(n)$ for $We = 10$, $H_1 = H_2 = 1$, $\rho = 2$, $A = 0.1$ and for the frequency ratio $\Omega = 0$ (periodic case) and $\Omega = \sqrt{2}$.

For $\Omega = 0$, we obtain the marginal stability curves for the periodic case (figure 2.a), where the first region corresponds to the Kelvin-Helmholtz instability and the other regions correspond to the quasi-periodic resonances located at $n = 3$, $n = 5.81$. For $\Omega = \sqrt{2}$ (quasi-periodic oscillation), we have the Kelvin-Helmholtz instability and quasi-periodic resonances located at $n = 1.58$, $n = 2.08$, $n = 3$, $n = 3.87$ and $n = 5.55$. We can see that the quasiperiodic oscillation gives rise to more resonances than the periodic case.

Ω	0	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{11}$	$\sqrt{37}$
B_{vo}	1.722	1.681	1.680	1.520	1.260

Table 1: Threshold of Kelvin-Helmholtz instability.

The effect of the frequency ratio, Ω , on the marginal stability curves is illustrated in figure 3. The marginal stability curves are plotted for $We = 10$, $H_1 = H_2 = 1$, $\rho = 2$, $A = 0.1$ and for various values of frequency ratios, Ω . By increasing the frequency ratio, Ω , the threshold of Kelvin-Helmholtz instability decreases slightly and keeps the same wavenumber $n = 0.001$ (see Table 1). Also, when Ω increases, the resonance zones shift to the right. For example, when $\Omega = \frac{1}{\sqrt{2}}$, the first region of the parametric instability takes place at $n = 1.26$, while for $\Omega = \sqrt{2}$, it takes place at $n = 1.58$. Beyond $\Omega = \sqrt{37}$, the first region of the parametric instability takes place at $n = 3$ as in the periodic case. This behavior was also observed in the recent works [4].

4 Conclusion

In this paper, we have presented a study of the interfacial instability of two immiscible, incompressible and inviscid fluid layers filling a rectangular cavity of infinite extent in the horizontal directions. The cavity is submitted to horizontal quasi-periodic oscillations. The linear stability analysis leads to the quasi-periodic Mathieu equation, with damping and describes the evolution of the amplitude of the interface. We have used the floquet's theory combined with the method of Runge-Kutta to solve numerically this equation. The numerical results shows the existence of two types of instability: The Kelvin-Helmholtz instability and the quasi-periodic resonances. We have examined the influence of the frequency ratio, Ω . We have shown that the increase of the frequency ratio has a destabilizing effect on the region of Kelvin-Helmholtz instability and allows to delete the parametric resonances, and we turn to the main resonance with wavenumber, $n = 3$.

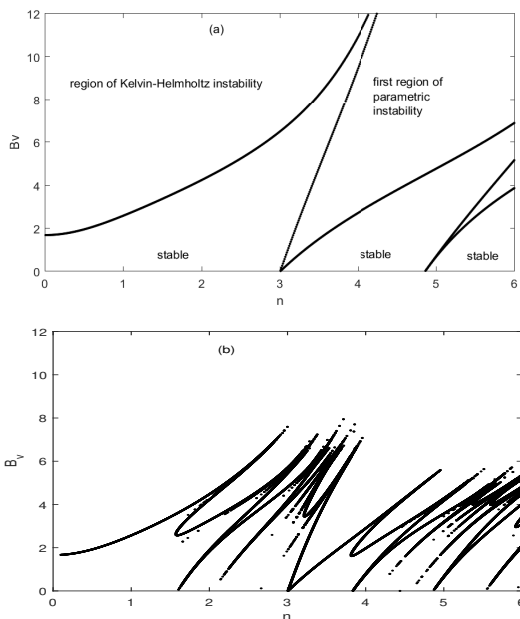


Figure 2: Neutral stability diagram for $We = 10$, $H_1 = H_2 = 1$, $\rho = 2$, $A = 0.1$ and for different frequency ratios: (a) $\Omega = 0$ (Periodic case), (b) $\Omega = \sqrt{2}$

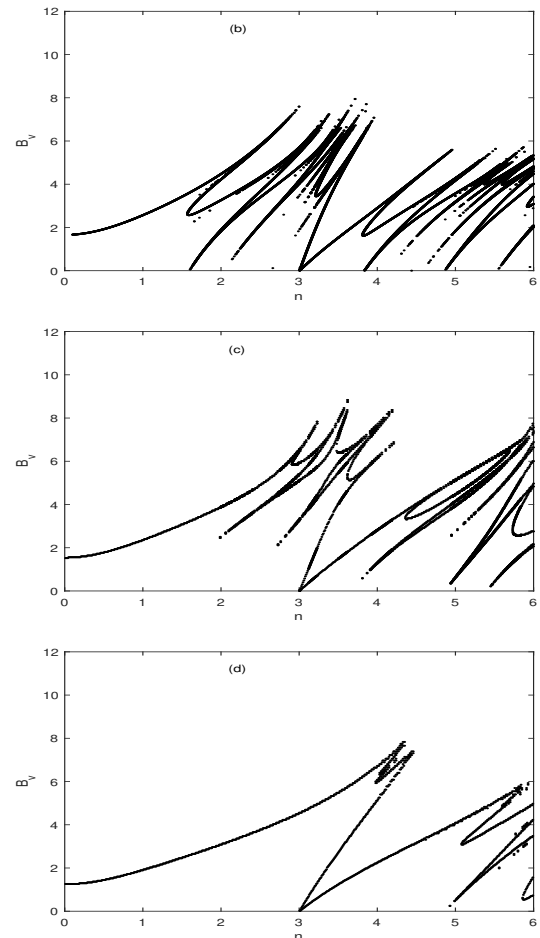
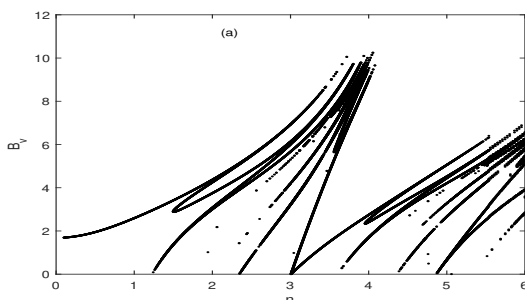


Figure 3: Neutral stability diagram for $We = 10$, $H_1 = H_2 = 1$, $\rho = 2$, $A = 0.1$ and for different frequency ratios: (a) $\Omega = \frac{1}{\sqrt{2}}$, (b) $\Omega = \sqrt{2}$, (c) $\Omega = \sqrt{11}$, (d) $\Omega = \sqrt{37}$

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