

# Meshfree method for large deformation in dynamic problems

H. Karim Serroukh, M. Mabssout

Faculté des Sciences et Techniques, Laboratoire de Mécanique et Génie Civil, Tanger, Maroc

## Abstract

We present a new meshless method, Taylor-SPH, for the numerical analysis of large deformation dynamic problems. This method is based on the previous work developed by the authors to solve solid dynamics problems within the framework of small deformation theory. The governing equations are given in terms of stress and velocity using the updated Lagrangian approach. The Jaumann rate of the Cauchy stress is used to get an objective stress rate tensor. The Taylor-SPH method is based on two sets of particles resulting on avoiding the classical tensile instability. In order to assess the accuracy of the proposed method, numerical examples based on elastic material involving large deformation are solved.

**Keywords:** *Taylor-SPH; Meshfree method; Large deformation; Dynamic*

## 1. Introduction

The Finite Element Method (FEM) has been successfully used for solving partial differential equations (PDEs) in solid and fluid mechanics. However, the FEM presents a few drawbacks that have to be improved for modelling large deformations and failure problems. One of the main drawbacks is the remeshing procedure. The presence of large deformations is accompanied by severe mesh distortion. The remeshing technique is generally used to avoid the mesh distortions. However, the remeshing is too much time consuming and reduces the accuracy of the numerical solutions. To overcome difficulties related to the mesh, various meshfree methods have been developed in the last two decades and have been used in many areas with considerable success. In the meshfree methods, the approximation of variables is constructed based on scattered points without mesh connectivity. Therefore, meshfree methods can deal in a straightforward manner with large deformation and failure problems without the difficulties encountered in mesh-based methods. A wide variety of meshfree methods have been proposed over the past decades and has been successfully applied to many problems in computational mechanics, for more details see e.g. [1,2]. In this paper, we present a new meshfree method, Taylor-SPH (TSPH), for the numerical analysis of large deformation problems under dynamic conditions. This method is based on the previous work developed by Mabssout et al. [3-6] to solve solid dynamics problems within the framework of small deformation theory. The equations are written in the

form of a system of first order hyperbolic PDEs. The principal variables are stress and velocity. The proposed algorithm is based on two sets of particles resulting on avoiding the classical tensile instability. To illustrate the performance of the proposed method, numerical examples involving large deformation are solved using the Taylor-SPH method.

## 2. Governing equations

Consider a problem defined in domain  $\Omega$  bounded by  $\Gamma_v$  and  $\Gamma_\sigma$  such that  $\Gamma = \Gamma_v \cup \Gamma_\sigma$ . Using the updated Lagrangian approach and the Jaumann rate of the Cauchy stress, the governing equations with boundary conditions are written in terms of stress and velocity as follows

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{D}^e : \mathbf{d} + \boldsymbol{\omega} \boldsymbol{\sigma} - \boldsymbol{\sigma} \boldsymbol{\omega} \quad (1a)$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{b} \quad (1b)$$

$$\mathbf{v} = \bar{\mathbf{v}} \quad \text{on } \Gamma_v \quad (1c)$$

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} \quad \text{on } \Gamma_\sigma \quad (1d)$$

Here  $\mathbf{v}$  represents the velocity vector,  $\boldsymbol{\sigma}$  is the Cauchy stress tensor,  $\mathbf{b}$  is the body forces per unit mass;  $\nabla$  is the gradient operator and  $\rho$  is the density,  $\mathbf{d}$  is the rate of deformation tensor,  $\boldsymbol{\omega}$  is the spin tensor and  $\mathbf{D}^e$  is the elastic tensor.  $\bar{\mathbf{v}}$  and  $\bar{\boldsymbol{\sigma}}$  are prescribed velocity and stress respectively;  $\mathbf{n}$  is the outward normal to the domain.

The above system (1a, 1b) can alternatively be written in a concise manner as

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S} \quad (2)$$

$$\mathbf{U} = \begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{v} \end{pmatrix} \quad \text{vector of unknowns}$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_x \\ \mathbf{F}_y \end{pmatrix} \quad \text{flux vector}$$

$$\mathbf{S} = \begin{pmatrix} \boldsymbol{\omega} \boldsymbol{\sigma} - \boldsymbol{\sigma} \boldsymbol{\omega} \\ \mathbf{b} \end{pmatrix} \quad \text{source term}$$

## 3. Numerical methods: Taylor-SPH

The Taylor-SPH method is used to discretize the above equation (2). The TSPH is a collocation meshfree method which consists of applying first the time discretization using a Taylor series expansion in two steps and the space discretization using the corrected SPH method. The application of the TSPH on the equation (2) leads to

- *First step: Vector  $\mathbf{U}$  at time  $(n + \frac{1}{2})$*

$$\mathbf{U}_I^{n+1/2} = \mathbf{U}_I^n + \frac{\Delta t}{2} \left[ \sum_{J=1}^{N_R} \frac{m_J}{\rho_J} \mathbf{S}_J^n \tilde{W}_{IJ} - \sum_{J=1}^{N_R} \frac{m_J}{\rho_J} (\mathbf{F}_I^n - \mathbf{F}_J^n) \tilde{V}W_{IJ} \right] \quad (3)$$

$N_R$  is the set of real particles J such that  $\|\mathbf{x}_I - \mathbf{x}_J\| \leq 2h$ .  $\tilde{W}_{IJ}$  and  $\tilde{V}W_{IJ}$  are the normalized kernel and the corrected gradient respectively. The parameter  $h$  represents the smoothing length that defines the size of the kernel support.

- Second step: Vector  $\mathbf{U}$  at time  $(n + 1)$

$$\mathbf{U}_I^{n+1} = \mathbf{U}_I^n + \Delta t \left[ \sum_{J=1}^{N_V} \frac{m_J}{\rho_J} \mathbf{S}_J^{n+1/2} \tilde{W}_{IJ} - \sum_{J=1}^{N_V} \frac{m_J}{\rho_J} (\mathbf{F}_I^{n+1/2} - \mathbf{F}_J^{n+1/2}) \tilde{V}W_{IJ} \right] \quad (4)$$

$N_V$  is the set of virtual particles J such that  $\|\mathbf{x}_I - \mathbf{x}_J\| \leq 2h$ .

## 4. Numerical results

### 4.1 1D elastic bar: Test of stability

The aim of this example is to verify the stability of the new algorithm on the smoothing length parameter. The problem consists of a shock wave that propagates in 1D elastic bar. The bar has 1m length with unit section. The material properties are: density  $\rho = 2000 \text{ kg/m}^3$  and elastic modulus  $E = 8 \times 10^7 \text{ Pa}$ . The boundary conditions are  $v(1, t) = v_o(t)$  and  $v(0, t) = 0$ .  $v_o(t)$  is given by a rectangular function between 0 and  $t_o=2.5\text{ms}$ . The bar is discretized by 101 real particles. The distance between two consecutive real particles  $\Delta x = 0.01$ . The time step used in the computation is  $\Delta t = \Delta x \cdot \sqrt{\rho/E} = 0.05 \text{ ms}$ .

It is well known that the smoothing length  $h$  is an important parameter in the SPH method. First, we can write the smoothing length as:  $h = \alpha \cdot \Delta x$ ; where  $\alpha$  is a factor that defines the radius of the kernel function. To investigate the influence of  $h$ , three values of  $\alpha$  have been used in the numerical analysis. The error is computed using the L2 norm of the velocity. The results are summarized in Table 1.

$\alpha=h/\Delta x$	$\Delta x$ (m)	Radius of compact support $r = 2h$ (m)	Number of real particles in compact domain	Error <sub>L2</sub> (%)
0.6	0.01	0.012	2	$10^{-3}$
1.2		0.024	4	$10^{-2}$
1.6		0.032	6	10

Table 1 Sensitivity of the solution on smoothing length

It can be observed that when  $\alpha = h/\Delta x$  is within the range of 0.6–1.2, the numerical solution is in very good agreement with the analytical solution. No dispersion no diffusion appears in the numerical result. As  $\alpha$  is increased ( $\geq 1.6$ ), the number of particles increases and the oscillations appear in the numerical solution. The error increases and the solution loses its accuracy. This example shows that the Taylor-SPH

method avoids the SPH tension instability and it can be used for the propagation of shock wave in elastic media provided to take  $0.6 \leq \alpha \leq 1.2$ .

### 4.2 Cantilever beam

In this example, the Taylor-SPH method is used to solve a 2D bending problem. The problem consists of a cantilever beam subjected to a vertical load P at its free end. The beam has dimensions L x D and a unit thickness. The exact solution of this problem is given by:

- Vertical displacement

$$u_y = \frac{P}{6EI} \left[ 3vy^2(L-x) + (4+5v)\frac{D^2}{4}x + (3L-x)x^2 \right] \quad (5)$$

- Normal stress

$$\sigma_x = -\frac{P(L-x)y}{I} \quad (6)$$

The parameters used for the computation are: L=1m; D=0.1m; the material is elastic with Young's modulus  $E = 8 \times 10^7 \text{ Pa}$ . Poisson's ratio  $\nu = 0.3$  and density  $\rho = 2000 \text{ kg/m}^3$ . The applied shear force  $P = 1000 \text{ N}$ .

A structured arrangement of 101 x 11=1111 real particles is used for the computation. Fig.1 depicts the vertical displacement of the beam obtained with the Taylor-SPH method. Fig. 2 plots the exact solution and the numerical result for the vertical displacement along the beam axis. The results show an excellent agreement between the exact and numerical solutions.

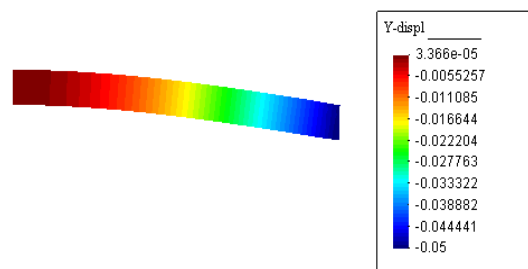


Fig. 1 Vertical displacement

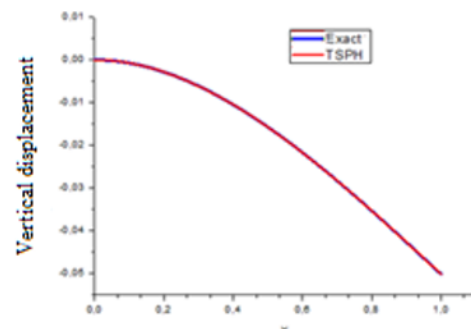


Fig.2 Vertical displacement along the beam

## 5. Conclusion

The Taylor-SPH meshfree method for large deformation in dynamic problems has been presented. The main advantage of the proposed method is the best accuracy because of (i) using stress and velocity as main variables in the PDEs written with the updated Lagrangian approach (ii) and using two types of particles in the time discretization at time steps  $(n + \frac{1}{2})$  and  $(n + 1)$  resulting on avoiding the classical tensile instability. From the numerical examples presented above, we can conclude that the Taylor-SPH method:

- avoids numerical instabilities,
- achieves an excellent convergence with small number of particles,
- provides accurate results for bending problems and shock wave propagation.

## References

- [1] G.R. Liu, M.B. Liu. Smoothed Particle Hydrodynamics: A meshfree particle method. World Scientific Publishing: Singapore, 2003.
- [2] V. P. Nguyen, T. Rabczuk, S. Bordas, Marc Duflot. Meshless methods: A review and computer implementation aspects. *Mathematics and Computers in Simulation* 79 (2008) 763–813.
- [3] M. Mabssout, M.I. Herreros, H. Idder. Predicting dynamic fracture in viscoplastic materials using Taylor-SPH. *Int. J. Impact Engrg.*, 87 (2016) 95-107.
- [4] M. Mabssout, M.I. Herreros. Runge-Kutta vs Taylor-SPH. Two time integration schemes for SPH with application to Soil Dynamics. *App. Math. Modelling*, 37 (2013) 3541-3563
- [5] M.I. Herreros, M. Mabssout. A two-steps time discretization scheme using the SPH method for shock wave propagation. *Comput. Methods Appl. Mech. Engrg.*, 200 (2011) 1833-1845.
- [6] H. Idder. Une nouvelle approche pour la modélisation des problèmes dynamiques : Taylor-SPH. *Faculté des Sciences et Techniques de Tanger. Thèse soutenue en novembre 2015.*