Effect of mean flow on mutual radiation resistances of a rectangular plate

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Abstract. Very few investigations have considered the effects of fluid convection on the mutual-radiation resistance, which could be important in stationary fluids. In this work, we quantified the effects of a convected fluid on the radiation resistances including self- and mutual radiation resistances. We derived the non-dimensional modal impedance in the presence of mean flow based on an expansion in the panel mode of a simply supported plate. Then we compared the self-radiation resistances with the previous reports and analyzed the effects of flow on mutual radiation resistances. The results show that the radiation efficiency increases with the flow velocity, which compare very well with the previous reports. The flow introduces a strong coupling between modes that weakly coupled in no-flow case and with the increase of flow velocity. This coupling effect increases until Mach number (M) reaches 0.5 and then decreases. For the modes with the same parity indices, the flow shifts coupling lobes toward lower frequency with the increase of flow velocity due to the shift of critical frequency of higher mode radiation resistance, and the degree of coupling change is consistent with the change of critical frequency of the higher mode self-radiation resistance.

1 Introduction

Radiation resistance of a plate is important in vibro-acoustic. It determines the plate velocity, and further determines the radiated acoustic power. Radiation from vibrating structures (particularly for simply geometries) into a static fluid is well understood, and the radiation efficiencies[1-5] as well as the cross-modal coupling [6-12] are well considered. As is reported, the magnitude of the radiation efficiency depends on whether the mode is supersonic or subsonic, and the modal coupling is not negligible both at low frequency and under off-resonant excitation.

However, the sound radiation behaviors of convective fluid-loaded plates could be very different. Maestrello et al.[13] reported one of the earliest acoustic response of a panel in convected fluid and derived the radiation sound pressure in the form of extended Green function. Dowell[14] summarized earlier investigations into the aeroelastic behavior of plates and shells, and demonstrated the significant effects of particular high subsonic and supersonic flow on plate dynamics. Sgard et al.[15] presented that the flow could increase the radiation acoustic power, which is more notable in water than in air, and the coupling with different modal indices parity can become important in the presence of mean flow. Wu et al.[16] also showed that the radiated acoustic pressure are directly related to turbulent boundary layer.

All of these previous works mainly focused on the response of the panel excited by flow and radiation characters, while the radiation impedance of a convected fluid-loaded plates has not been well discussed. Chang et

al.[17] derived the modal radiation impedance, including self- and cross-modal coupling impedance of a simply supported rectangular panel in the presence of a uniform subsonic flow. They claimed that a modal impedance will be zero only if co-directed wavenumber indices of the panel modes in the direction perpendicular to the flow have different parity. Graham[18] and Kou et al.[19] have discussed the averaged self-radiation resistance (the so-called radiation efficiency) of a rectangular plate subjected to turbulent boundary layer fluctuations. They summarized that mean flow results in a significant increase in the modal radiation efficiency. Frampton[20] investigated the influence of mean flow on the frequency-dependent radiation efficiency and obtained the same conclusion. However, the mutual radiation resistances that could be meaningful for no-flow condition[6, 8, 9] have got little attention, due to the belief that the mutual-radiation impedance (coupling terms) can be ignored[18] or the misgivings that the modes of the structure are no longer orthogonal where the fluid is convected[20].

In this work, we focused on the effects of a convected fluid on the mutual radiation resistances based on modal expansion method. We derived the non-dimensional modal impedances for subsonic flow case based on an expansion in the panel mode of a simply supported panel. Then we compared the self-radiation resistances with previous reports and discussed the effects of flow on the mutual radiation resistances.

2 Theoretical analysis

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2.1 Derivation of the impedance

Consider a finite, flexible, rectangular plate, which is embedded in an infinite baffle as depicted in Fig. 1. The plate is subjected to a semi-infinite idealization uniform steady fluid, flowing parallel to the x axis on one side and a vacuum on the other. The fluid is inviscid, with mean density and sound speed c . When flow velocity U is not too close to the speed of sound c , the sound pressure satisfies the convected wave equation, given by

$$\nabla^2 p = \frac{1}{c^2} \frac{D^2 p}{Dt^2} \quad (1)$$

Fig. 1. Schematic of the plate and coordinate system.

As aforementioned, for convenience, the mean flow is assumed to move along the x -direction. As a sequence, the material derivative can be expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$$

which measures rate of change following the motion of a fluid particle. The sound pressure is assumed to satisfy the boundary condition in the plane of the panel

$$\left. \frac{\partial p(x, y, z; t)}{\partial z} \right|_{z=0} = \begin{cases} -\rho \frac{D^2 w(x, y; t)}{Dt^2}, & 0 \leq x \leq a, 0 \leq y \leq b \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where $w(x, y; t)$ is the normal displacement of the panel. Applying the Fourier transform on the (x, y) coordinates to the convected wave equation, and assuming that the system is undergoing harmonic motion, Eq.(1) becomes

$$\frac{d^2 \hat{p}(k_x, k_y, z; t)}{dz^2} + \zeta^2 \hat{p}(k_x, k_y, z; t) = 0 \quad (3)$$

where $\zeta^2 = (k - Mk_x)^2 - k_x^2 - k_y^2$ with the Fourier transform variables k_x in x and k_y in y , respectively. $k = \omega / c$ is the wavenumber in the fluid with the angular frequency ω and $M = U / c$ is the Mach number. Similarly, applying the Fourier transform to the boundary condition Eq.(2), resulting in

$$\left. \frac{\partial \hat{p}(k_x, k_y, z; t)}{\partial z} \right|_{z=0} = \rho c^2 (k - Mk_x)^2 \hat{w}(k_x, k_y; t) \quad (4)$$

Combining Eq.(3) and Eq.(4) yields

$$\hat{p}(k_x, k_y, z; t) = \frac{\rho c \omega (k - Mk_x)^2 \hat{w}(k_x, k_y; t)}{jk\zeta} e^{j\zeta z} \quad (5)$$

The solution for the pressure can be obtained by performing inverse Fourier transforms to Eq.(5)

$$p(x, y, 0; t) = -j\rho c \omega \int_0^b \int_0^a G(x, y | x', y') w(x', y'; t) dx' dy' \quad (6)$$

which is similar to the classic Rayleigh's integral, and here we will define

$$G(x, y | x', y') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(k - Mk_x)^2 e^{jk_x(x-x')} e^{jk_y(y-y')}}{k \sqrt{k^2 - 2kMk_x + (M^2 - 1)k_x^2 - k_y^2}} dk_x dk_y \quad (7)$$

For a simply supported plate, the relevant panel mode shape is

$$\phi_{mn}(x, y) = \frac{2}{\sqrt{ab}} \sin \alpha_m x \sin \beta_n y \quad (8)$$

where $\alpha_m = m\pi / a$, $\beta_n = n\pi / b$. Expanding the sound pressure and panel vibration displacement in the panel modal, the sound pressure can be expanded as

$$p_{mn} = \rho c \sum_{pq} Z_{mnpq} (-j\omega w_{pq}(t)) \quad (9)$$

where the non-dimensional modal impedance Z_{mnpq} is defined as

$$Z_{mnpq} = \int_s \int_{s'} \phi_{mn}(s) G_m(s | s') \phi_{pq}(s') ds ds'$$

Eq.(9) shows that each modal pressure would receive multiple contributions from a more general vibration pattern. Eq.(10) can be expressed as

$$Z_{mnpq} = \theta_{mnpq} + j\chi_{mnpq}$$

in which θ_{mnpq} often referred as the non-dimensional modal radiation resistance represents the radiation damping, and the non-dimensional modal radiation reactance χ_{mnpq} represents added mass.

2.2 Derivation of the Green function

Here, we focus on subsonic mean flow ($M < 1$). Applying the following changes in variables

$$\xi = \sqrt{1 - M^2} k_x + \kappa M, \kappa = k / \sqrt{1 - M^2}, \\ u = (x - x') / \sqrt{1 - M^2}, \lambda = y - y'$$

then the expression in Eq.(7) becomes

$$G = \frac{e^{-j\kappa Mu}}{4\pi^2 (1 - M^2)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\kappa - M\xi)^2 e^{j\xi u} e^{jk_y \lambda}}{\kappa \sqrt{\kappa^2 - \xi^2 - k_y^2}} d\xi dk_y \quad (11)$$

Expanding the square terms, Eq.(11) can be expressed as

$$G = \kappa G_m + 2jMG_d - \frac{M^2}{\kappa} G_q \quad (12)$$

where

$$G_m = \frac{e^{-j\kappa Mu}}{4\pi^2 (1 - M^2)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{j\xi u} e^{jk_y \lambda}}{\sqrt{\kappa^2 - \xi^2 - k_y^2}} d\xi dk_y \quad (13)$$

$$G_d = \frac{e^{-j\kappa Mu}}{4\pi^2 (1 - M^2)^2} \frac{\partial}{\partial u} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{j\xi u} e^{jk_y \lambda}}{\sqrt{\kappa^2 - \xi^2 - k_y^2}} d\xi dk_y \right) \quad (14)$$

$$G_d = \frac{e^{-j\kappa Mu}}{4\pi^2(1-M^2)^2} \frac{\partial}{\partial u} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{j\xi u} e^{jk_y \lambda}}{\sqrt{\kappa^2 - \xi^2 - k_y^2}} d\xi dk_y \right) \quad (15)$$

Accordingly, Eq.(13) is given by[13]

$$G_m = \frac{-je^{j\kappa(-Mu+\sqrt{u^2+\lambda^2})}}{2\pi(1-M^2)^2\sqrt{u^2+\lambda^2}} \quad (16)$$

then,

$$G_d = \frac{e^{j\kappa(-Mu+\sqrt{u^2+\lambda^2})}}{2\pi(1-M^2)^2} \left[\frac{\kappa u}{u^2+\lambda^2} + \frac{ju}{\sqrt{(u^2+\lambda^2)^3}} \right]$$

$$G_q = \frac{e^{j\kappa(-Mu+\sqrt{u^2+\lambda^2})}}{2\pi(1-M^2)^2} \left[\frac{\kappa}{u^2+\lambda^2} - \frac{3\kappa u^2}{(u^2+\lambda^2)^2} + \frac{j(1+\kappa^2 u^2)}{\sqrt{(u^2+\lambda^2)^3}} - \frac{j3u^2}{\sqrt{(u^2+\lambda^2)^5}} \right]$$

2.3 Transformation of the resistance

We can rewrite the non-dimensional modal impedance as

$$Z_{mnpq} = \kappa \int_s \int_{s'} \phi_{mn}(s) G_m(s|s') \phi_{pq}(s') ds ds' + 2jM \int_s \int_{s'} \phi_{mn}(s) G_d(s|s') \phi_{pq}(s') ds ds' - \frac{M^2}{\kappa} \int_s \int_{s'} \phi_{mn}(s) G_q(s|s') \phi_{pq}(s') ds ds'$$

This is important because it gives some insight into the sources of sound: The first term often refers to as the monopole sources caused by panel vibration; The second term describes the inherent dipole sources of loading noise; and the third term is given by the quadrupole sources which depends on the viscous stress due to the flow.

Since sound radiation is associated with radiation resistance, we restrict attention to θ_{mnpq} . The non-dimensional modal resistance is defined by

$$\theta_{mnpq} = \Re \left\{ \int_s \int_{s'} \phi_{mn}(s) G_m(s|s') \phi_{pq}(s') ds ds' \right\}$$

where $\Re\{\}$ indicates the real part of the argument. The integrals are usually evaluated via precisely the contour deformations and asymptotic estimation techniques[18]. In order to lessen the computational burden, a co-ordinate transformation technique can be used to recast the quadruple integral into several double integrals[3, 9, 21]. The self- and mutual radiation resistances can be generally expressed as:

If $(m+p)$ is even, accordingly,

(1) for $m=p$ and $n=q$

$$\theta_{mmnn} = \frac{4}{ab} \left\{ \frac{1}{\alpha_m \beta_n} I_1^{mn} + I_2^{mn} + \frac{1}{\alpha_m} I_3^{mn} + \frac{1}{\beta_n} I_4^{mn} \right\}$$

(2) for $m=p$ and $q-n=\pm 2, \pm 4, \pm 6, \dots$

$$\theta_{mmnq} = \frac{4}{ab(\beta_n^2 - \beta_q^2)} \left\{ \beta_n I_4^{mq} - \beta_q I_4^{mn} + \frac{\beta_n}{\alpha_m} I_1^{mq} - \frac{\beta_q}{\alpha_m} I_1^{mn} \right\}$$

(3) for $p-m=\pm 2, \pm 4, \pm 6, \dots$ and $n=q$

$$\theta_{mnpn} = \frac{4}{ab} \frac{2}{(\alpha_m^2 - \alpha_p^2)} \left\{ \alpha_m I_3^{pn} - \alpha_p I_3^{mn} + \frac{\alpha_m}{\beta_n} I_1^{pn} - \frac{\alpha_p}{\beta_n} I_1^{mn} \right\}$$

(4) for $p-m=\pm 2, \pm 4, \pm 6, \dots$ and

$q-n=\pm 2, \pm 4, \pm 6, \dots$

$$\theta_{mnpq} = \frac{4}{ab} \frac{4}{(\alpha_m^2 - \alpha_p^2)(\beta_n^2 - \beta_q^2)} \left\{ \alpha_m \beta_n I_1^{pq} - \alpha_m \beta_q I_1^{pn} - \alpha_p \beta_n I_1^{mq} + \alpha_p \beta_q I_1^{mn} \right\}$$

(5) for $q-n=\pm 1, \pm 3, \pm 5, \dots$,

$$\theta_{mnmq} = 0$$

If $(m+p)$ is odd, the radiation resistances are no longer zero because the wave number is no longer centered on $k=0$. Then

(6) for $n=q$

$$\theta_{mnmq} = \frac{4}{ab} \frac{2}{(\alpha_p^2 - \alpha_m^2)} \left[\alpha_m J_3^{pn} + \alpha_p J_3^{mn} + \frac{\alpha_m}{\beta_n} J_1^{pn} + \frac{\alpha_p}{\beta_n} J_1^{mn} \right]$$

(7) for $n \neq q = \pm 2, \pm 4, \dots$

$$\theta_{mnmq} = \frac{4}{ab} \frac{4}{(\alpha_p^2 - \alpha_m^2)(\beta_n^2 - \beta_q^2)} \left[\alpha_m \beta_n J_1^{pq} - \alpha_m \beta_q J_1^{pn} + \alpha_p \beta_n J_1^{mq} - \alpha_p \beta_q J_1^{mn} \right]$$

Where

$$\begin{pmatrix} I_1^{mn} / J_1^{mn} \\ I_2^{mn} / J_2^{mn} \\ I_3^{mn} / J_3^{mn} \\ I_4^{mn} / J_4^{mn} \end{pmatrix} = \int_0^b \int_0^a dx dy \begin{Bmatrix} 1 \\ (a-x)(b-y) \\ (b-y) \\ (a-x) \end{Bmatrix} \times \begin{Bmatrix} \sin \alpha_m x \sin \beta_n y \\ \cos \alpha_m x \cos \beta_n y \\ \sin \alpha_m x \cos \beta_n y \\ \cos \alpha_m x \sin \beta_n y \end{Bmatrix} \begin{Bmatrix} even[\Re(G)] / odd[\Re(G)] \end{Bmatrix}$$

$even[\Re(G)]$ means even function part of $\Re(G)$ and $odd[\Re(G)]$ means odd function part of $\Re(G)$. These equations are similar with those at Mach zero[10] except for the difference in the cross-coupling resistances with the different indices parity of the panel modes in the flow direction.

3 Numerical results and discussions

The example of a thin elastic plate with isotropic properties will serve to illustrate the effects of flow on the radiation resistances. The self- and mutual radiation resistances for some lower order modes are plotted. The acoustic wave number have all been normalized by the

structural wave number for a simply supported mode shape, where

$$k_{mn} = \sqrt{(m\pi/a)^2 + (n\pi/b)^2} \quad (22)$$

The structural parameters used here are listed in table 1.

Table 1. Structural parameters used in examples.

Structural parameters	Values
Plate length, $a(m)$	0.82
Plate width, $b(m)$	0.82
Fluid density, $\rho(kg/m^3)$	1.2
Sound speed, $c(m/s)$	340
Mach number, M	0, 0.2, 0.5, 0.7

3.1 Self-radiation resistance

Fig. 2. Self-radiation resistance of the (1,1) mode at various Mach numbers for the case of flow.

Fig. 3. Self-radiation resistance of the (3,1) mode at various Mach numbers for the case of flow.

Fig. 4. Self-radiation resistance of the (1,3) mode at various Mach numbers for the case of flow.

For the sake of accuracy, the self-radiation resistance calculations are established through comparison with results previously published by Graham[18] and Frampton[20]. The effects of mean flow on the radiation efficiencies for several modes are demonstrated in Fig.2-4. It can be simply pointed out that good agreements are found for the subsonic flow with those of Frampton. The effects of flow are negligible for small Mach numbers ($M < 2$). As for the (1,1) plate mode shown in Fig.2, the notable change is that the critical frequency shifts toward low frequency with the increase of flow velocity. The resultant effect is a significant increase in the total radiation efficiency. Compared Fig.3 with Fig.4, the asymmetry between (1,3) and (3,1) plate modes in the radiation resistance also have been observed. The increase in radiation resistance for (3,1) mode is more prominent than that for (1,3) mode. Note that, there exists minor difference in (3,1) modal radiation resistance at critical frequency with Frampton's results. The increase of the peak at critical frequency is more significant shown in Fig.3, which may be more accurate because of no approximation in this paper.

3.2 Mutual-radiation resistance

The effects of flow on the mutual-radiation resistances are plotted in Fig.5-9 for some possible cross-couplings of the modes. It should be noted that all the curves have been normalized with the self-radiation resistance of the lowest mode in that group.

Fig. 5. Mutual-radiation resistance of the (1,1) x (2,1) mode at various Mach numbers in the presence of mean flow.

In the no-flow case, it is well known that the cross-modal coupling occurs only between a pair of modes having the same parity indices. While in the presence of flow, a strong coupling appears between different parity indices of panel modes in the flow direction as shown in Fig.5-7. As the frequency extends to near the coincidence frequency, the coupling disappears. As the increase of flow velocity, the effect of coupling increases in going from $M=0.2$ to $M=0.5$ and then decreases. That is because the supersonic wave number region gradually envelops the spectrum peak and then deviates from it. Compared Fig.5 and Fig.6, the effects of flow on the (1,1) x (2,1) and (2,1) x (1,1) mode mutual radiation resistance is antisymmetric. This coupling between higher modes is stronger but decreases rapidly, for example, the coupling between (3, 3) and (2, 3) modes in Fig.7.

Fig. 6. Mutual-radiation resistance of the $(2,1) \times (1,1)$ mode at various Mach numbers in the presence of mean flow.

Fig. 7. Mutual-radiation resistance of the $(3,3) \times (2,3)$ mode at various Mach numbers in the presence of mean flow.

Fig. 8. Mutual-radiation resistance of the $(1,1) \times (1,3)$ mode at various Mach numbers in the presence of mean flow.

Fig.8,9 plot some typical mutual-radiation resistances at various Mach numbers in the presence of subsonic flow. In stationary fluids, the mutual-radiation resistance is quite significant in a quite wide frequency that well extends to the coincidence frequency of the higher mode. After the coincidence frequency, the mutual-radiation resistance oscillatory decays with frequency[9]. In flow case, the coupling lobes move toward lower frequency with the increase of flow velocity due to the shift in coincidences frequency. The influence of flow on the mutual radiation resistance depends on the effect of flow on the higher mode self-radiation resistance. In Fig.8, for the mutual radiation resistance of $(1,1) \times (1,3)$ mode, the coupling lobes exhibit a slightly shift to lower frequency and the mutual-radiation resistance oscillatory decays slightly with the increase of flow. While, in Fig.9, it is easy to see that the shift of coupling lobes in the $(1,1) \times (3,1)$ modal radiation resistance is more signi-

ficant than that on the mutual-radiation resistance of the $(1,1) \times (1,3)$ mode. The oscillatory peaks increase remarkably due to the notable effect of flow on $(1,3)$ self-radiation resistance.

Fig. 9. Mutual-radiation resistance of the $(1,1) \times (3,1)$ mode at various Mach numbers in the presence of mean flow.

4 Conclusions

Based on the modal expansion approach, the effect of subsonic flow on the self- and mutual- radiation resistance is investigated. The flow causes a decrease of the critical frequency when the modal becomes an efficient radiator, which results in a notable increase in the mid-wave number domain, and further increases the self-modal radiation resistance, which is in consistent with the previous reports. The flow introduces a strong coupling even between modes that were weakly coupled in the no-flow case. This coupling increases with Mach number(M) until M reaches 0.5, and then decreases with the increase of flow velocity. For the modes with the same parity indices, the flow shifts coupling lobes toward lower frequency with the increase of flow velocity due to the shift of critical frequency of higher order modal radiation resistance, and the degree of coupling change is consistent with the change of critical frequency of the higher mode self-radiation resistance.

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