Heat transport solutions in rectangular shields using harmonic polynomials.

Mariusz Owczarek^{1,*}, and Stefan Owczarek¹

¹Military University of Technology, Faculty of Civil Engineering and Geodesy, gen W. Urbanowicza 2, 00-980 Warszawa, Poland

Abstract. The search for the temperature field in a two-dimensional problem is common in building physics and heat exchange in general. Both numerical and analytical methods can be used to obtain a solution. Here a method of initial functions, the basics of which were given by W.Z. Vlasov i A.Y. Lur'e were adopted. Originally MIF was used for analysis of the loads of a flat elastic medium. Since then it was used for solving concrete beams, plates and composite materials problems. Polynomial half-reverse solutions are used in the theory of a continuous medium. Here solutions were obtained by direct method. As a result, polynomial forms of the considered temperature field were obtained. The Cartesian coordinate system and rectangular shape of the plate were assumed. The governing are the Fourier equation in steady state. Boundary conditions in the form of temperature ($\tau(x), t(y)$) or/and flux ($p(x), q(y)$) can be provided. The solution $T(x, y)$ were assumed in the form of an infinite power series developed in relation to the variable y with coefficients C_n depending on x . The assumed solution were substituted into Fourier equation and after expanding into Taylor series the boundary condition for $y = 0$ and $y=h$ were taken into account. Form this condition a coefficients C_n can be calculated and therefore a closed solution for temperature field in plate.

1 Formulation of two dimensional temperature problem

The search for the temperature field in a two-dimensional problem is common in building physics and heat exchange in general. Both numerical and analytical methods can be used to obtain a solution. Here a Method of Initial Functions, the basics of which were given by [1,2] were adopted. The approach used in MIF allowed to derive the form of harmonic polynomials, which form the basis of solutions in this paper. These polynomials are $4 * \infty$ which is the unique value of this article. An important value of this article is the use of these polynomials to determine the temperature in a rectangular area.

Originally MIF was used for analysis of the loads of a flat elastic medium [3]. Since then it was used for solving concrete beams, plates and composite materials problems [4,5]. A solution in the form of a power series with coefficients depending on x was assumed. Then these coefficients were found by solving the differential equation. Harmonic polynomials were obtained that satisfy the Laplace equation in the area. The coefficients of the linear combination of these functions were determined by approximating the boundary conditions. The values of the approximation function for the edge of considered area are here initial functions. As a result, polynomial forms of the considered temperature function were obtained. Primary concepts are: boundary components, heat transfer flux, governing equation and characteristic operators of solutions.

1.1 Governing equation

The heat transfer equation derived from the energy balance in the infinitesimal volume

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(\lambda_{xx} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda_{yy} \frac{\partial T}{\partial y} \right) + q_v \quad (1.1)$$

Where $T(x, y)$ is temperature, ρ - density, c_p - specific heat capacity, λ - coefficient of thermal conductivity.

In steady state heat exchange, no internal heat sources and isotropic body, equation (1.1) become:

$$\nabla^2 T(x, y) = 0 \quad \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (1.2)$$

In this consideration, expressions were found for the temperature that satisfies the Laplace equation in the area. This solution has the form of a sum of polynomials. These polynomials exist in products with constant coefficients. The

* Corresponding author: mariusz.owczarek@wat.edu.pl

obtained solution was divided into four independent states according to symmetry features of the temperature function.

These are the following :

- symmetry-symmetry SS (x and y even),
- symmetry-antisymmetry SA (x even, y odd),
- antisymmetry-symmetry AS (x odd, y even),
- antisymmetry- antisymmetry AA (x and y odd).

2 Boundary conditions

The coordinate system and geometrical parameters of the medium were assumed as in Figure 1.

Fig. 1. Coordinate system adopted in the analysis.

The variable t is the set temperature at the edge of the shield (approximated), and variable T is the interior temperature (approximating). The following boundary conditions apply:

$$\begin{aligned} 1. \quad y = h \quad Ut(x), \quad 2. \quad y = -h \quad LWt(x), \\ 3. \quad x = b \quad Rt(y), \quad 4. \quad x = -b \quad LFt(y). \end{aligned} \quad (2.1)$$

Preceding the variable t in capital letters U, LW, R, LF, the temperature of the upper, lower, right and left edges respectively was assigned. The task was divided into four independent groups corresponding to four independent thermal states of the symmetry of the shield. The boundary conditions for the entire shield were split over four states specified in the first quadrant of the coordinate system. The following indices have been assigned to shorten the write to these states: The temperature in the SS state is preceded by the letter S, in the SA state with the letter B and the AS with the letter C, and in the state AA with the letter A. The quantities in these states are determined by the initial state with the following formulas:

SS state

$$\begin{aligned} 1.2. \quad y = \pm h, \quad T^S = SUt = SLWt = \frac{1}{4}[Ut(x) + Ut(-x) + LWt(x) + LWt(-x)] \\ 3.4. \quad x = \pm b, \quad T^S = SRt = SLFt = \frac{1}{4}[Rt(y) + Rt(-y) + LFt(y) + LFt(-y)]. \end{aligned} \quad (2.2)$$

SA state

$$\begin{aligned} 1.2. \quad y = \pm h, \quad T^B = BUt = -BLWt = \frac{1}{4}[[Ut(x) + Ut(-x)] - [LWt(x) + LWt(-x)]] \\ 3.4. \quad x = \pm b, \quad T^B = BRt = -BLFt = \frac{1}{4}[Rt(y) - Rt(-y) + [LFt(y) - LFt(-y)]]. \end{aligned} \quad (2.3)$$

AS state

$$\begin{aligned} 1.2. \quad y = \pm h, \quad T^C = CUt = CLWt = \frac{1}{4}[[Ut(x) - Ut(-x)] + [LWt(x) - LWt(-x)]] \\ 3.4. \quad x = \pm b, \quad T^C = CRt = -CLFt = \frac{1}{4}[[Rt(y) + Rt(-y)] - [LFt(y) + LFt(-y)]]. \end{aligned} \quad (2.4)$$

AA state

$$\begin{aligned} 1.2. \quad y = \pm h, \quad T^A = AUt = -ALWt = \frac{1}{4}[[Ut(x) - Ut(-x)] - [LWt(x) - LWt(-x)]] \\ 3.4. \quad x = \pm b, \quad T^A = ARt = -ALFt = \frac{1}{4}[[Rt(y) - Rt(-y)] - [LFt(y) - LFt(-y)]]. \end{aligned} \quad (2.5)$$

For example, boundary conditions for the whole disc on the lower edge will be equal to the sum $SDt(x) + BDt(x) + CDt(x) + ADt(x)$

Example 1. The boundary conditions can be decomposed and described by functions:

$$y = {}^+_h \quad T = Ut = LWT = 5 + \frac{15}{2b}(b+x), \quad (2.6)$$

$$x = {}^+_b \quad T = Rt = 20^\circ C \quad LFT = 5^\circ C$$

Fig. 2. Example of boundary conditions of temperature in the shield

The task is split into four states:

The boundary conditions in the SS and AS states are shown in the Fig. 3 and 4.

SS state

$$y = {}^+_h \quad T^S = SUt = SLWt = 12,5^\circ C \quad (2.7)$$

$$x = {}^+_b \quad T^S = SRt = SLFt = 12,5^\circ C$$

Fig. 3. The symmetrical part of the task. SS state.

AS State

$$y = {}^+_h \quad T^C = CUt = CLWt = \frac{15x}{2b}, \quad (2.8)$$

$$x = {}^+_b \quad T^C = CRt = 7,5^\circ C \quad CLFt = -7,5^\circ C$$

Fig. 4. Antisymmetric part relative to x and symmetric to y . AS state

SA and AA states are identically zero

$$\begin{aligned} y = \pm h \quad T^{SA} = T^{AA} = CUt = CLWt = 0^\circ C \\ x = \pm b \quad T^{SA} = T^{AA} = CRt = CLFt = 0^\circ C \end{aligned} \quad (2.9)$$

Example 2. The boundary conditions can be decomposed and described by functions:

$$\begin{aligned} y = \pm h \quad T = Ut = LWT = 10^\circ C, \\ x = \pm b \quad T = Rt = LFT = 20^\circ C, \end{aligned} \quad (2.10)$$

The picture of boundary conditions is shown in Fig 5.

Fig. 5. Example of boundary conditions of temperature in the shield

The task is divided into four states:

SS state:

$$\begin{aligned} T^S = SUt = SLWt = \frac{1}{4}[10 + 10 + 10 + 10] = 10^\circ C \\ T^S = SRt = SLFt = \frac{1}{4}[20 + 20 + 20 + 20] = 20^\circ C \end{aligned} \quad (2.11)$$

SA state:

$$\begin{aligned} T^B = BUt = -BLWt = \frac{1}{4}[10 + 10 - [10 + 10]] = 0^\circ C \\ T^B = BRt = -BLFt = \frac{1}{4}[20 - 20 + [20 - 20]] = 0^\circ C \end{aligned} \quad (2.12)$$

AS state:

$$\begin{aligned} T^C = CUt = CLWt = \frac{1}{4}[10 - 10 + [10 - 10]] = 0^\circ C \\ T^C = CRt = -CLFt = \frac{1}{4}[20 + 20 - [20 + 20]] = 0^\circ C \end{aligned} \quad (2.13)$$

AA state:

$$T^A = AUt = -ALWt = \frac{1}{4}[10 - 10 - [10 - 10]] = 0 \text{ } ^\circ C \quad (2.14)$$

$$T^A = ARt = -ALFt = \frac{1}{4}[20 - 20 - [20 - 20]] = 0 \text{ } ^\circ C$$

3 Solution of the area problem

The solution functions $T(x, y)$ were assumed in the form of an infinite power series developed in relation to the variable y with coefficients C_n depending on x .

$$T(x, y) = \sum_{n=0}^{\infty} C_n(x) y^n \quad (3.1)$$

Second derivatives with respect to x

$$\frac{\partial^2 T}{\partial x^2} = \sum_{n=0}^{\infty} \frac{\partial^2 C_n(x)}{\partial x^2} y^n, \quad (3.2)$$

and with respect to y

$$\frac{\partial^2 T}{\partial y^2} = \sum_{n=2}^{\infty} (n-1)n C_n(x) y^{(n-2)}, \quad (3.3)$$

were calculated. To sum the series (3.3) from $n = 0$, the substitution $n = n + 2$ was performed where n is a new variable with the same designation and (3.3) was rewritten in the form

$$\frac{\partial^2 T}{\partial y^2} = \sum_{n=0}^{\infty} (n+1)(n+2) C_{n+2}(x) y^n \quad (3.4)$$

After substituting (3.2) and (3.4) in (1.2)

$$\sum_{n=0}^{\infty} \frac{\partial^2 C_n(x)}{\partial x^2} y^n + \sum_{n=0}^{\infty} (n+1)(n+2) C_{n+2}(x) y^n = 0 \quad (3.5)$$

$C_n(x)$ are treated as n -dependent terms of progression and as such can be derived from the differential equation. Putting the sum into a common sign and introducing the operator D :

$$\sum_{n=0}^{\infty} [C_{n+2}(n+2)(n+1) + D^2 C_n] y^n = 0 \quad (3.6)$$

Where

$$D^n = \frac{\partial^n}{\partial x^n}$$

The equation (3.6) will be met if:

$$C_{n+2}(n+2)(n+1) + D^2 C_n = 0 \quad (3.7)$$

Multiplying both sides by $(n!/n!)$ a differential equation on $C_n(x)$ is obtained

$$C_{n+2} \frac{(n+2)!}{n!} + D^2 C_n \frac{n!}{n!} = 0 \quad (3.8)$$

After applying the shift operator

$$E^p C_n = C_{n+p} \quad (3.9)$$

$$[E^2 + D^2] C_n n! = 0 \quad (3.10)$$

Equation (12) can be solved as equation with constant coefficients, the general solution is

$$C_n n! = (-iD)^n A + (iD)^n B \quad (3.11)$$

Variables A and B were determined by substituting $n = 0, 1$ for equation (3.11). In this way expressions were obtained on C_0 and C_1 , although their value is unknown, one can express A and B :

$$\begin{aligned} C_0 &= A + B \\ C_1 &= -iDA + iDB \end{aligned} \quad (3.12)$$

Determining A and B from equations (14), it was obtained:

$$\begin{aligned} A &= \frac{1}{2} [C_0 - (iD)^{-1} C_1] \\ B &= \frac{1}{2} [C_0 + (iD)^{-1} C_1] \end{aligned} \quad (3.13)$$

Substituting (3.13) to (3.11) and introducing divalent functions $j(n), j(n+1)$ with values (0,1)

$$j(n) = \frac{1}{2} [(-1)^n + 1], \quad j(n+1) = \frac{1}{2} [1 + (-1)^{n+1}] \quad (3.14)$$

and trivalent $J(n)$ and $J(n+1)$ with values $(-1, 0, +1)$

$$J(n) = j(n)i^n, \quad J(n+1) = j(n+1)i^{n+1} \quad (3.15)$$

an expression for C_n term is as follows

$$C_n = \frac{1}{n!} [J(n)D^n C_o - J(n+1)D^{n+1} C_i] \quad (3.16)$$

Substituting (3.16) to (3.1)

$$T(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} [J(n)D^n C_o - J(n+1)D^{n+1} C_i] y^n \quad (3.17)$$

The Taylor series expansion of $\sin()$ and $\cos()$ function is:

$$\cos Dy = \sum_{n=0}^{\infty} J(n) \frac{(Dy)^n}{n!}, \quad \sin Dy = -\sum_{n=0}^{\infty} J(n+1) \frac{(Dy)^n}{n!}, \quad (3.18)$$

Equation (3.17) can be written in the form:

$$T(x, y) = \cos Dy C_o + \frac{\sin Dy}{D} C_i \quad (3.19)$$

Separating the expression (3.19) into a symmetrical and antisymmetrical part relative to the variable y , we obtained:

$$XST(x, y) = \cos Dy T(x, 0) = \cos Dy \sum_{i=0}^{\infty} T_i x^i \quad (3.20)$$

$$XAT(x, y) = \frac{\sin(Dy)}{D} C(x, 0) = \frac{\sin(Dy)}{D} \sum_{i=0}^{\infty} C_i x^i \quad (3.21)$$

XS before the function $T(x, y)$ means that the function $T(x, y)$ is arbitrary with respect to x and even with respect to y , likewise the designation XA means that the function $T(x, y)$ is odd in relation to the variable y . Taking into account in (3.20)(3.21) the relationship $j(i) + j(i+1) = 1$ and compounds (3.18) were obtained:

$$XST(x, y) = \sum_{i=0}^{\infty} \left([j(i) + j(i+1)] T_i \sum_{k=0}^i J(k) x^{i-k} y^k \right) \quad (3.22)$$

$$XAT(x, y) = \sum_{i=0}^{\infty} \left([j(i) + j(i+1)] C_i \sum_{k=0}^{i+1} J(k+1) x^{i+1-k} y^k \right) \quad (3.23)$$

Expressions (3.22)(3.23) can be easily separated into states symmetry-symmetry (SS), symmetry-antisymmetry (SA), antisymmetry-symmetry (AS) and antisymmetry-antisymmetry (AA).

The temperature in individual states is expressed in the following formulas:

SS state

$$SST(x, y) = \sum_{i=0}^{\infty} \left(j(i) T_i \sum_{k=0}^i J(k) \binom{i}{k} x^{i-k} y^k \right) \quad (3.24)$$

AS state

$$AST(x, y) = \sum_{i=0}^{\infty} \left(j(i+1) T_i \sum_{k=0}^i J(k) \binom{i}{k} x^{i-k} y^k \right) \quad (3.25)$$

SA state

$$SAT(x, y) = -\sum_{i=0}^{\infty} \left(j(i) C_i \sum_{k=0}^{i+1} J(k+1) \binom{i+1}{k} x^{i+1-k} y^k \right) \quad (3.26)$$

AA state

$$AAT(x, y) = -\sum_{i=0}^{\infty} \left(j(i+1) C_i \sum_{k=0}^{i+1} J(k+1) \binom{i+1}{k} x^{i+1-k} y^k \right) \quad (3.27)$$

The polynomials specified in expressions (3.24-3.27) are harmonics, i.e. they satisfy the Laplace equations. By accepting a sufficient number of polynomials and their corresponding constant factors, boundary conditions can be met precisely enough.

In this work, the method of approximating the temperature at the edge of the shield was used to determine the constant coefficients.

4 Expressions on temperature in case of limitation to the first consecutive polynomials of rank 10.

Expressions (3.24)-(3.27) contain an infinite number of constant factors and an infinite number of polynomials corresponding to these coefficients. In specific calculations, the number of these coefficients and the corresponding degrees of polynomials can be limited. The values of functions $j(n)$, $J(n)$ necessary to write the equations are summarized in the Table 1.

Table 1. Divalent and trivalent functions used in expressions (3.24)-(3.27)

n	0	1	2	3	4	5	6
j(n)	1	0	1	0	1	0	1
J(n)	1	0	-1	0	1	0	-1
j(n+1)	0	1	0	1	0	1	0
J(n+1)	0	-1	0	1	0	-1	0

The following are given in individual states of temperature symmetry: five polynomials corresponding to the following "n" rank: SS -8, SA-9, AS-9, AA-10.

SS state

$$\begin{aligned} T(x, y) = & T_0 + T_2(x^2 - y^2) + T_4(x^4 - 6x^2y^2 + y^4) + \\ & + T_6(x^6 - 15x^4y^2 + 15x^2y^4 - y^6) + \\ & + T_8(x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8), \end{aligned} \quad (4.1)$$

AS state

$$\begin{aligned} T(x, y) = & T_1x + T_3(x^3 - 3xy^2) + T_5(x^5 - 10x^3y^2 + 5xy^4) + \\ & + T_7(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6) + \\ & + T_9(x^9 - 36x^7y^2 + 126x^5y^4 - 84x^3y^6 + 9xy^8) \end{aligned} \quad (4.2)$$

SA state

$$\begin{aligned} T(x, y) = & C_1y + C_3(3x^2y - y^3) + C_5(5x^4y - 10x^2y^3 + y^5) + \\ & + C_7(7x^6y - 35x^4y^3 + 21x^2y^5 - y^7) + \\ & + C_9(9x^8y - 84x^6y^3 + 126x^4y^5 - 36x^2y^7 + y^9) \end{aligned} \quad (4.4)$$

AA state

$$\begin{aligned} T(x, y) = & C_2xy + C_4(x^3y - xy^3) + C_6(x^5y - 10x^3y^3 + 3xy^5) + \\ & + C_8(x^7y - 7x^5y^3 + 7x^3y^5 - xy^7) + \\ & + C_{10}(5x^9y - 60x^7y^3 + 126x^5y^5 - 60x^3y^7 + 5xy^9) \end{aligned} \quad (4.5)$$

As can be seen, these polynomials can be easily written, because their coefficients are equivalents from the Pascal triangle.

5 Solution to the examples formulated in point 2.

We solve the task of meeting boundary conditions assuming fixed temperatures at the edges $x=b$, $T=t_z$, $y=h$, $T=\tau$ Fig 3. This is symmetric part of Example 1, thus SS state.

We adopt harmonic functions

$$H_0 = 1, \quad H_2 = x^2 - y^2 \quad (5.1)$$

Assuming an approximating function in the form:

$$T = a_1 + a_2(x^2 - y^2) \quad (5.2)$$

This function on the right edge takes the form:

$$\begin{aligned} x=b \quad T_p &= a_1 + a_2(b^2 - y^2) \\ y=h \quad T_g &= a_1 + a_2(x^2 - h^2) \end{aligned} \quad (5.3)$$

We will set parameters a_1, a_2 from the approximation condition; meeting the minimum deviation on the right and upper edges. The minimum deviation condition has the form:

$$\begin{aligned} \min_{a_1, a_2} \delta &= \int_0^h dy [a_1 + a_2(b^2 - y^2) - t]^2 + \int_0^b dx [a_1 + a_2(x^2 - h^2) - \tau]^2 = \\ &= \int_0^h dy \{ [a_1 + a_2(b^2 - y^2)]^2 - 2[a_1 + a_2(b^2 - y^2)]t + t^2 \} + \\ &+ \int_0^b dx \{ [a_1 + a_2(x^2 - h^2)]^2 - 2[a_1 + a_2(x^2 - h^2)]\tau + \tau^2 \} = \\ &= \int_0^h dy \{ [a_1^2 + 2a_1a_2(b^2 - y^2) + a_2^2(b^2 - y^2)^2] - 2[a_1 + a_2(b^2 - y^2)]t + t^2 \} + \\ &+ \int_0^b dx \{ [a_1^2 + 2a_1a_2(x^2 - h^2) + a_2^2(x^2 - h^2)^2] - 2[a_1 + a_2(x^2 - h^2)]\tau + \tau^2 \} \end{aligned} \quad (5.4)$$

By calculating derivatives relative to constant approximations and then integrating, we get :

$$\begin{aligned} \min_{\delta_{a_1, a_2}} &= a_1^2 h + 2a_1 a_2 \left(b^2 h - \frac{h^3}{3} \right) + \\ &+ a_2^2 \left(b^4 h - 2b^2 \frac{h^3}{3} + \frac{h^5}{5} \right) - 2 \left(a_1 h + a_2 \left(b^2 h - \frac{h^3}{3} \right) \right) t + t^2 h + \\ &+ a_1^2 b + 2a_1 a_2 \left(\frac{b^3}{3} - h^2 b \right) + \\ &+ a_2^2 \left(\frac{b^5}{5} - 2 \frac{b^3}{3} h^2 + h^4 b \right) - 2 \left(a_1 b + a_2 \left(\frac{b^3}{3} - h^2 b \right) \right) \tau + \tau^2 b \end{aligned} \quad (5.4a)$$

By calculating derivatives relative to a_1 and a_2 , we obtained:

$$\begin{aligned} \frac{\partial \delta}{\partial a_1} &= a_1 \cdot \left[2a_2 [hb(b-h) + \frac{1}{3}(b^3 - h^3)] - 2(ht - b\tau) \right] \\ \frac{\partial \delta}{\partial a_2} &= a_1 \cdot \left[t^2 b + \frac{1}{3}(b^3 - h^3) \right] + 2a_2 [b^4 h - bh^4] + \\ &- \frac{2}{3}(b^2 h^3 + b^3 h^2) + \frac{1}{5}(b^5 + h^5) - 2[hb^2 - \frac{1}{3}h^3]t + 2[bh^2 - \frac{1}{3}b^3]\tau \end{aligned} \quad (5.5)$$

By equating derivatives to zero, 2 equations were obtained to determine a_1 , a_2 constants :

$$\begin{aligned} a_1 \cdot & \quad \cdot \\ a_1 \cdot & \quad \cdot \end{aligned} \quad (5.6)$$

Where

$$\begin{aligned} A_{11} &= 2(h+b), \quad A_{12} = A_{21} = 2[hb(b-h) + \frac{1}{3}(b^3 - h^3)] \\ A_{22} &= 2[(b^4 h + bh^4) - \frac{2}{3}(b^2 h^3 + b^3 h^2) + \frac{1}{5}(b^5 + h^5)] \\ P_1 &= 2(ht + b\tau), \quad P_2 = 2[hb^2 - \frac{1}{3}h^3]t - 2[bh^2 - \frac{1}{3}b^3]\tau \\ W_1 &= A_{22}P_1 - A_{12}P_2 = 4[-\frac{2}{15}b^5 h + \frac{4}{9}b^3 h^3 + \frac{2}{3}bh^5 + \frac{4}{45}h^6]t + \\ &+ 4[\frac{2}{3}b^5 h + \frac{4}{9}b^3 h^3 - \frac{2}{15}bh^5 + \frac{4}{45}b^6]\tau \\ W &= A_{11}A_{22} - A_{12}A_{21} = 4[\frac{4}{45}b^6 + \frac{8}{15}b^5 h + \frac{8}{9}b^3 h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6] \end{aligned} \quad (5.7)$$

Solution of equations

$$a_1 = \frac{W_1}{W} = \frac{[-\frac{2}{15}b^5 h + \frac{4}{9}b^3 h^3 + \frac{2}{3}bh^5 + \frac{4}{45}h^6]t + [\frac{2}{3}b^5 h + \frac{4}{9}b^3 h^3 - \frac{2}{15}bh^5 + \frac{4}{45}b^6]\tau}{[\frac{4}{45}b^6 + \frac{8}{15}b^5 h + \frac{8}{9}b^3 h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6]} \quad (5.8)$$

$$a_2 = \frac{W_2}{W} = \frac{[\frac{2}{3}b^3 h + \frac{2}{3}bh^3]t + [-\frac{2}{3}b^3 h - \frac{2}{3}bh^3]\tau}{[\frac{4}{45}b^6 + \frac{8}{15}b^5 h + \frac{8}{9}b^3 h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6]} \quad (5.9)$$

We make an example

$$\begin{aligned} \frac{h}{b} &= 6 \quad t = \tau = 12,5^\circ C \\ a_1 &= \frac{W_1}{W_2} = \frac{\frac{47132}{5}t - \frac{42152}{45}t}{\frac{382036}{45}} = \frac{\frac{382036}{45}t}{\frac{382036}{45}} = t = \tau \\ a_2 &= \frac{W_2}{W} = \frac{[\frac{4}{3}b^3 h + \frac{4}{3}bh^3]t + [-\frac{4}{3}b^3 h - \frac{4}{3}bh^3]\tau}{[\frac{4}{45}b^6 + \frac{8}{15}b^5 h + \frac{8}{9}b^3 h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6]} = 0 \\ y &=_{-}^{+} h \quad T^C = CGt = CDt = 12,5^\circ C \\ x &=_{-}^{+} b \quad T^C = CPt = CLt = 12,5^\circ C \end{aligned} \quad (5.10)$$

The temperature field in this case is the constant temperature equal to that shown in Figure 3.

AS state

$$y =_+^+ h \quad T^C = CGt = CDT = \frac{15x}{2b}, \quad (5.11)$$

$$\begin{aligned} x =_+^+ b \quad T^C = CPt = 7,5^\circ C \quad CLT = -7,5^\circ C \\ T(x, y) = T_1x + T_3(x^3 - 3xy^2) + T_5(x^5 - 10x^3y^2 + 5xy^4) + \\ + T_7(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6) + \\ + T_9(x^9 - 36x^7y^2 + 126x^5y^4 - 84x^3y^6 + 9xy^8) \end{aligned} \quad (5.12)$$

In general, we can write for the AS state

$$T(x, y) = T_1H_1 + T_3H_3 + T_5H_5 + T_7H_7 + T_9H_9 \quad (5.13)$$

$$\begin{aligned} H_1 &= x, \\ H_3 &= x^3 - 3xy^2 \\ H_5 &= x^5 - 10x^3y^2 + 5xy^4 \\ H_7 &= x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6 \\ H_9 &= x^9 - 36x^7y^2 + 126x^5y^4 - 84x^3y^6 + 9xy^8 \end{aligned}$$

We accept for the solution of this example the solving function is:

$$T(x, y) = a_1H_1 + a_2H_3 = a_1x + a_2(x^3 - 3y^2x) \quad (5.14)$$

The boundary conditions are described by the equations:

$$\begin{aligned} y =_+^+ h \quad T^C = CGt = CDT = \frac{15x}{2b} = \frac{Fx}{2b}, \\ x =_+^+ b \quad T^C = CPt = 7,5^\circ C = t \quad CLT = -7,5^\circ C = -t, \quad F = 2t. \end{aligned} \quad (5.15)$$

Where F, t are fixed numbers

The square of deviation on the right edge is equal to:

$$\delta^2(p) = \int_0^h dy [(a_1b + a_2(b^3 - 3by^2) - t]^2 \quad (5.16)$$

The deviation square on the upper edge is:

$$\delta^2(g) = \int_0^b dx [(a_1x + a_2(x^3 - 3xh^2) - \frac{Fx}{2b})^2] \quad (5.17)$$

As a criterion of approximation, we accept the minimum sum of deviations:

$$\begin{aligned} \min_{a_1, a_2} \delta^2 = \delta^2(p) + \delta^2(g) = \int_0^h dy [(a_1b + a_2(b^3 - 3by^2) - t]^2 + \\ + \int_0^b dx [(a_1x + a_2(x^3 - 3xh^2) - \frac{Fx}{2b})^2] \end{aligned} \quad (5.18)$$

Hence the deviation derivatives relative to a_1 and a_2 are equal to zero:

$$\frac{\partial W}{\partial a_1} = 0, \quad \frac{\partial W}{\partial a_2} = 0, \quad (5.19)$$

$$W = \int_0^h dy [(a_1b + a_2(b^3 - 3by^2) - t]^2 + \int_0^b dx [(a_1x + a_2(x^3 - 3xh^2) - \frac{Fx}{2b})^2]$$

$$\frac{\partial W}{\partial a_1} = 0,$$

$$W = \int_0^h dy [(a_1b + a_2(b^3 - 3by^2) - t]^2 + \int_0^b dx [(a_1x + a_2(x^3 - 3xh^2) - \frac{Fx}{2b})^2] \quad (5.20)$$

$$\frac{\partial W}{\partial a_1} = \int_0^h dy 2[(a_1b + a_2(b^3 - 3by^2) - t)b] + \int_0^b dx 2[(a_1x + a_2(x^3 - 3xh^2) - \frac{Fx}{2b})x] = 0$$

$$\frac{\partial W}{\partial a_1} = 2[a_1bh + a_2(b^3h - 3b\frac{h^3}{3}) - th]b + 2[a_1\frac{b^3}{3} + a_2(\frac{b^5}{5} - 3\frac{b^3}{3}h^2) - \frac{Fb^3}{6}] = 0$$

$$\frac{\partial W}{\partial a_1} = a_1[b^2h + \frac{b^3}{3}] + a_2(b^4h - 3b^2\frac{h^3}{3} - 3\frac{b^3}{3}h^2 + \frac{b^5}{5}) - thb + [-\frac{Fb^2}{6}] = 0$$

$$\begin{aligned}\frac{\partial W}{\partial a_2} &= 0, \\ W &= \int_0^h dy [(a_1 b + a_2 (b^3 - 3by^2) - t)^2] + \int_0^b dx [(a_1 x + a_2 (x^3 - 3xh^2) - \frac{Fx}{2b}]^2 \\ \frac{\partial W}{\partial a_2} &= 2 \int_0^h dy [(a_1 b + a_2 (b^3 - 3by^2) - t)(b^3 - 3by^2) + \\ &\quad + \int_0^b dx [(a_1 x + a_2 (x^3 - 3xh^2) - \frac{Fx}{2b})(x^3 - 3xh^2) \\ &= a_1 (\frac{b^5}{5} + b^4 h - b^3 h^2 - b^2 h^3) + a_2 (\frac{b^7}{7} + b^6 h - \frac{6}{5} b^5 h^2 - 2b^4 h^3 + 3b^3 h^4 + \frac{9}{5} b^2 h^5) + \\ &\quad - t(b^3 h - bh^3) - F(\frac{b^4}{10} - \frac{b^2 h^2}{2})\end{aligned}\tag{5.21}$$

The equations determining a_1 and a_2 have the form:

$$\begin{aligned}\frac{\partial W}{\partial a_1} &= a_1 (b^2 h + \frac{b^3}{3}) + a_2 (b^4 h - 3b^2 \frac{h^3}{3} - 3 \frac{b^3}{3} h^2 + \frac{b^5}{5}) - thb + [-\frac{Fb^2}{6}] = 0 \\ \frac{\partial W}{\partial a_1} &= a_1 (\frac{b^5}{5} + b^4 h - b^3 h^2 - b^2 h^3) + a_2 (\frac{b^7}{7} + b^6 h - \frac{6}{5} b^5 h^2 - 2b^4 h^3 + 3b^3 h^4 + \frac{9}{5} b^2 h^5) + \\ &\quad - t(b^3 h - bh^3) - F(\frac{b^4}{10} - \frac{b^2 h^2}{2}) = 0\end{aligned}\tag{5.22}$$

or in an alternative form:

$$\begin{aligned}a_1 [b^2 h + \frac{b^3}{3}] + a_2 (b^4 h - 3b^2 \frac{h^3}{3} - 3 \frac{b^3}{3} h^2 + \frac{b^5}{5}) &= thb + \frac{Fb^2}{6} \\ a_1 (\frac{b^5}{5} + b^4 h - b^3 h^2 - b^2 h^3) + a_2 (\frac{b^7}{7} + b^6 h - \frac{6}{5} b^5 h^2 - 2b^4 h^3 + 3b^3 h^4 + \frac{9}{5} b^2 h^5) &= \\ &= t(b^3 h - bh^3) + F(\frac{b^4}{10} - \frac{b^2 h^2}{2})\end{aligned}\tag{5.22a}$$

We write set of equations in the form:

$$\begin{aligned}A_{11}a_1 + A_{12}a_2 &= P_1 \\ A_{21}a_1 + A_{22}a_2 &= P_2 \\ A_{11} &= b^2 h + \frac{b^3}{3} \quad A_{12} = b^4 h - 3b^2 \frac{h^3}{3} - 3 \frac{b^3}{3} h^2 + \frac{b^5}{5}, \quad P_1 = thb + \frac{Fb^2}{6} \\ A_{21} &= \frac{b^5}{5} + b^4 h - b^3 h^2 - b^2 h^3 \quad A_{22} = \frac{b^7}{7} + b^6 h - \frac{6}{5} b^5 h^2 - 2b^4 h^3 + 3b^3 h^4 + \frac{9}{5} b^2 h^5, \\ P_2 &= t(b^3 h - bh^3) + F(\frac{b^4}{10} - \frac{b^2 h^2}{2}) \\ W_1 &= P_1 A_{22} - P_2 A_{12} = t \frac{2}{35} b^3 h (-b^5 + 35bh^4 + 14h^5) + \\ &\quad + F \frac{1}{1050} b^4 (4b^5 + 70b^4 h + 280b^3 h^2 - 210h^5) \\ W_2 &= A_{11}P_2 - A_{21}P_1 = \frac{b^4 h}{15} (b^2 + 5h^2)(2t - F) \\ W &= A_{11}A_{22} - A_{12}A_{21} = \frac{4b^4}{525} (b^6 + 10b^5 h + 70b^3 h^3 + 210bh^5 + 105h^6)\end{aligned}\tag{5.23}$$

$$\begin{aligned}a_1 &= \frac{W_1}{W} = \frac{60t^2 b^3 h (-b^5 + 35bh^4 + 14h^5) + Fb^3 (4b^5 + 70b^4 h + 280b^3 h^2 - 210h^5)}{8(b^6 + 10b^5 h + 70b^3 h^3 + 210bh^5 + 105h^6)} \\ a_2 &= \frac{W_2}{W} = \frac{35b^3 h (b^2 + 5h^2)(2t - F)}{4(b^6 + 10b^5 h + 70b^3 h^3 + 210bh^5 + 105h^6)}\end{aligned}\tag{5.25}$$

We assume: $b=h$:

$$\begin{aligned}A_{11} &= \frac{4h^3}{3} \quad A_{12} = -\frac{4h^5}{5} \quad A_{22} = \frac{96}{35} \quad W = \frac{4}{3} \frac{96}{35} - \frac{16}{25} = \frac{528}{175} = 3,017, \\ W_1 &= 22,62857, \quad W_2 = 0, \quad a_1 = 7,5 \quad a_2 = 0\end{aligned}\tag{5.26}$$

The temperature field has the form:

$$T(x, y) = a_1 H_1 = a_1 x = 7,5x\tag{5.27}$$

Fig 6. Temperature map in the AS state calculated in example 1. Side ratio $b = h = l$ Drawing made in the Matlab software.

In case we consider: $b=0,25$ $h=1,5$

$$\begin{aligned} A_{11} &= 0,098958 & A_{12} &= -0,24004 & A_{22} &= 1,062973 \\ P_1 &= 2,96875 & P_2 &= -7,20117 & W &= \frac{4}{3} \frac{96}{35} - \frac{16}{25} = \frac{528}{175} = 0,047571 \\ W_1 &= 1,427137 & W_2 &= 0, & a_1 &= 30 & a_2 &= 0 \end{aligned}$$

The temperature field has the form:

$$T(x,y) = a_1 H_1 = a_1 x = 30x \tag{5.28}$$

Fig 6. Temperature map in the AS state calculated in example 1. Dimensions of sides Drawing made in the Matlab software.

Example 2: Data ratio $\frac{h}{b}=1$, $t=20$, $\tau=10$

We analyze according to the state of SS
Solution of equations

$$a_1 = \frac{W_1}{W} = \frac{[-\frac{2}{15}b^5h + \frac{4}{9}b^3h^3 + \frac{2}{3}bh^5 + \frac{4}{45}h^6]t + [\frac{2}{3}b^5h + \frac{4}{9}b^3h^3 - \frac{2}{15}bh^5 + \frac{4}{45}b^6]\tau}{[\frac{4}{45}b^6 + \frac{8}{15}b^5h + \frac{8}{9}b^3h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6]} \quad (5.29)$$

$$a_2 = \frac{W_2}{W} = \frac{[\frac{2}{3}b^3h + \frac{2}{3}bh^3]t + [-\frac{2}{3}b^3h - \frac{2}{3}bh^3]\tau}{[\frac{4}{45}b^6 + \frac{8}{15}b^5h + \frac{8}{9}b^3h^3 + \frac{8}{15}bh^5 + \frac{4}{45}h^6]} \quad (5.30)$$

For the case $\frac{h}{b} = 1$, $t = 20$, $\tau = 10$

Fig.8. Temperature image in the area of the shield (temperature field); violet - lowest temperature, highest green Side ratio $b = h = 1$. Drawing made in the Matlab software.

6 Conclusions

In the first example, there are continuous boundary conditions along the edge. In the second example there is a discontinuity between the right and the upper edge. Such tasks are useful in all aspects of building physics in rectangular areas. Additionally, the examples show how to decompose boundary tasks into four tasks based on symmetry features. In this way, we reduce the number of unknowns in the problem. The paper should not be viewed from the point of view of only need to provide a numerical solution to specific tasks, but as a interpretation of the concepts and methods of solving detailed tasks in building physics, such as:

1. Solution classes (polynomial solutions).
2. Breaking down the task into smaller ones with fewer constants (expressiveness of the task)
3. Forms of description of boundary tasks. (number of conditions on the edges)
4. Simplicity of the form of solutions received.

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