Different applications of RRE on nonsymmetric algebraic Riccati equation in transport theory

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Abstract. This paper proposes three different ways of applying a vector extrapolation method to a slow converging vector sequence to accelerate the convergence. The solution of this vector sequence is used to solve an algebraic Riccati equation in transport theory. A numerical example is presented to compare the three applications.

1 Introduction

The nuclear disaster that occurred in Japan on March 2011 emphasizes the vulnerability of the population at the vicinity of the nuclear power plant. Nonsymmetric algebraic Riccati equations arise in many fields such as transport theory when dealing with particle transfer (or radiative transfer), nuclear physics, control theory, etc. In this paper, we are interested in special Riccati equations that arise in transport theory. An important problem is computing limits of high dimension vector sequences. Many of these sequences converge extremely slowly to their limits. Such vector sequences may arise for example from finite-difference or finite-element discretizations of continuum problems. In such problems, as the mesh sizes decrease, their speed of convergence gets worse. Other vector sequences may result from iterative solution of linear or nonlinear systems of equations. This requires the use of convergence acceleration methods.

Vector extrapolation methods may be applied to accelerate the convergence of vector sequences. These techniques transform a sequence of vectors generated by some process to a new one that converges faster than the initial sequence.

2 Nonsymmetric algebraic Riccati equation

Riccati equations are matrix equations with a quadratic term. These equations were first examined by Jacopo Francesco Riccati in 1723 and still being examined till nowadays. A matrix version of such equations was introduced much later.

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These equations arise in continuous-time and discrete-time linear-quadratic-Gaussian control. Algebraic Riccati equations denote the non-dynamic version of Riccati equations.

This paper is interested in a nonsymmetric algebraic Riccati equation (NARE) of the form

$$XAX - XD - AX + B = 0, \quad (1)$$

with a real square unknown matrix X and real matrices, A, B, C and $D \in \mathbb{R}^{n \times n}$ as matrix coefficients.

Nonsymmetric algebraic Riccati equations (NARE) arise in different fields such as transport theory, engineering, nuclear physics, control theory, applied probability, and much more. These equations play a crucial role in two important applications, namely, the analysis of transport equations [1,2] and the study of fluid queues models [3,4,5].

This paper deals with a nonsymmetric algebraic Riccati equations which occurs in transport theory with specific values of matrix coefficients, that is the one which concerns particle transport or radiative transfer.

Here, the minimal positive solution X^* , that is the one with componentwise minimal entries, is the solution of interest and of physical meaning. In [6], it was seen that the minimal positive solution of a certain vector equation leads to the minimal positive solution of (NARE). Using the method in [7] and the modified version in [8], the minimal positive solution of NARE can be computed as follows

$$X = T \circ (uv^T), \quad (2)$$

where $\circ$ denotes the Hadamard product, u and $v \in \mathbb{R}^{n \times 1}$, and $T \in \mathbb{R}^{n \times n}$ is given.

Then, finding two positive vector u and v leads to the minimal positive solution of (1). These vectors can be computed as in [8] by the following iteration

$$w^{(k+1)} = \begin{bmatrix} u^{k+1} \\ v^{k+1} \end{bmatrix} := \begin{bmatrix} (I_n - \text{diag}(Pv^{(k)}))^{-1} e \\ (I_n - \text{diag}(Qu^{(k+1)}))^{-1} e \end{bmatrix} \quad (3)$$

where P and Q are given and $k=0,1,\dots$

Convergence of vector sequences tend to be very slow, then we will apply a vector extrapolation technique, namely the reduced rank extrapolation (RRE) to accelerate the convergence.

2 Reduced rank extrapolation

The reduced rank extrapolation technique was introduced by Eddy [9] and Mesina [10]. An advantage of this technique is that it does not require an explicit knowledge of the way the sequence is generated. Also it can be applied to the solution of linear and nonlinear systems of equations.

Given $\{u^{(k)}\}_{k \in \mathbb{N}}$ a sequence of vectors in $\mathbb{R}^n$. Define the first and second forward differences of $u^{(k)}$ by

$$\Delta u^{(k)} = u^{(k+1)} - u^{(k)} \quad \text{and} \quad \Delta^2 u^{(k)} = \Delta u^{(k+1)} - \Delta u^{(k)}, \quad k = 0, 1, \dots$$

When RRE is applied to the vector sequence $\{u^{(k)}\}_{k \in \mathbb{N}}$, this technique produces an approximation $s^{(k)}$ of the limit of $\{u^{(k)}\}_{k \in \mathbb{N}}$. The approximation $s^{(k)}$ is defined by

$$s^{(k)} = \sum_{j=0}^k \beta_j u^{(j)},$$

where $\sum_{j=0}^k \beta_j = 1$ and $\sum_{j=0}^k \gamma_{i,j} \beta_j = 0$, for $i = 0, 1, \dots, k-1$.

The scalars $\gamma_{i,j} \in \mathbb{R}$ are defined for $i = 0, 1, \dots, k-1$ and $j = 0, 1, \dots, k$ by

$$\gamma_{i,j} = (\Delta^2 u^{(i)}, \Delta u^{(j)}).$$

Using the expression $\Delta^i U^{(k)} = [\Delta^i u^{(0)}, \dots, \Delta^i u^{(k-1)}]$ and Schur complements, $s^{(k)}$ can be written as

$$s^{(k)} = u^{(0)} - \Delta U^{(k)} (\Delta^2 U^{(k)})^+ \Delta u^{(0)},$$

where $(\Delta^2 U^{(k)})^+$ denotes the Moore-Penrose generalized inverse of $\Delta^2 U^{(k)}$ defined by

$$(\Delta^2 U^{(k)})^+ = ((\Delta^2 U^{(k)})^T \Delta^2 U^{(k)})^{-1} (\Delta^2 U^{(k)})^T.$$

2.1 Application of RRE

The steps of application of the basic RRE technique are presented in table 1.

Table 1. RRE algorithm.

Input	Vectors $u^{(0)}, u^{(1)}, \dots, u^{(k+1)}$.
Step 1	Calculate $\Delta u^{(i)} = u^{(i+1)} - u^{(i)}$, for $i = 0, 1, \dots, k$
	Set $\Delta U^{(k+1)} = [\Delta u^{(0)}, \Delta u^{(1)}, \dots, \Delta u^{(k)}]$. Compute the QR factorization of $\Delta U^{(k+1)}$, namely, $\Delta U^{(k+1)} = Q^{(k)} R^{(k)}$.
Step 2	Solve the linear system $(R^{(k)})^T R^{(k)} d^{(k)} = e$; where $d^{(k)} = [d_0^{(k)}, d_1^{(k)}, \dots, d_k^{(k)}]^T$ and $e = [1, 1, \dots, 1]^T$. Set $\lambda = (\sum_{i=0}^k d_i^{(k)})^{-1}$, $\lambda \in \mathbb{R}^+$. Set $\beta_i^{(k)} = \lambda d_i^{(k)}$, for $i = 0, 1, \dots, k$.
Step 3	Compute $\xi^{(k)} = [\xi_0^{(k)}, \xi_1^{(k)}, \dots, \xi_{k-1}^{(k)}]^T$, where $\xi_0^{(k)} = 1 - \beta_0^{(k)}$ and $\xi_j^{(k)} = \xi_{j-1}^{(k)} - \beta_j^{(k)}$, $1 \leq j \leq k-1$. Compute $s^{(k)} = u^{(0)} + Q^{(k-1)} (R^{(k-1)} \xi^{(k)})$.

When RRE is applied, as k increases this technique becomes increasingly expensive because of the work requirement which grows quadratically. Also, the storage requirement grows linearly with k . Then, RRE should be restarted periodically every a number of steps r where r is an integer greater than 1. The choice of r is analyzed in [8] and it has been seen that a small integer r is sufficient.

Here, three different ways of applying the restarted RRE to iteration (3) are listed in Tables 2,3 and 4. First, the restarted RRE is applied to the vector sequence $\{w^{(k)}\}_{k \in \mathbb{N}}$. Then, it is applied to the vector sequences $\{u^{(k)}\}_{k \in \mathbb{N}}$ and $\{v^{(k)}\}_{k \in \mathbb{N}}$ separately. Finally, it is applied only on $\{v^{(k)}\}_{k \in \mathbb{N}}$ to get a better approximation $s^{(k)}$, then compute $u^{(k)}$ using the following equation

$$u^{(k)} = \left(I_n - \text{diag}(Ps^{(k)}) \right)^{-1} e \quad (4)$$

Table 2. Application of restarted RRE to $\{w^{(k)}\}$.

Step 1	For $k = 0, u^{(0)} = v^{(0)} = 0$. Set $r=4$. Then, $w^{(0)} = 0$.
Step 2	For $k = 1, 2, \dots$ For $j = 0, 1, \dots, r - 1$ $g^{(0)} = u^{(k-1)}, h^{(0)} = v^{(k-1)}; s^{(0)} = (g^{(0)}, h^{(0)})^T$; $g^{(j+1)} = \left(I_n - \text{diag}(Ph^{(j)}) \right)^{-1} e$; $h^{(j+1)} = \left(I_n - \text{diag}(Qg^{(j+1)}) \right)^{-1} e$; $s^{(j+1)} = (g^{(j+1)}, h^{(j+1)})^T$.
Step 3	Calculate the approximations $t^{(r-1)}$ via applying the RRE algorithm of table 1 on the vectors $(s^{(0)}, s^{(1)}, \dots, s^{(r)})$.
Step 4	If $t^{(r-1)}$ satisfies accuracy test, stop. Else, $w^{(k)} = t^{(r-1)}$.

Table 3. Application of restarted RRE to $\{u^{(k)}\}$ and $\{v^{(k)}\}$.

Step 1	For $k = 0, u^{(0)} = v^{(0)} = 0$. Set $r=4$.
Step 2	For $k = 1, 2, \dots$ For $j = 0, 1, \dots, r - 1$ $g^{(0)} = u^{(k-1)}, h^{(0)} = v^{(k-1)}$; $g^{(j+1)} = \left(I_n - \text{diag}(Ph^{(j)}) \right)^{-1} e$; $h^{(j+1)} = \left(I_n - \text{diag}(Qg^{(j+1)}) \right)^{-1} e$;
Step 3	Calculate the approximations $t_1^{(r-1)}$ and $t_2^{(r-1)}$ via applying the RRE algorithm of table 1 on the vectors $[(g^{(0)}, h^{(0)})^T, \dots, (g^{(r)}, h^{(r)})^T]$
Step 4	If $t_1^{(r-1)}$ and $t_2^{(r-1)}$ satisfies accuracy test, stop. Else, $u^{(k)} = t_1^{(r-1)}$ and $v^{(k)} = t_2^{(r-1)}$.

Now, we will apply the restarted RRE on $\{v^{(k)}\}_{k \in \mathbb{N}}$ only. Suppose that for all $t \in \mathbb{R}^n$, K and $L \in \mathbb{R}^{n \times n}$, there exist two functions f_K and $f_{K,L}$ in $\mathbb{R}^n$ defined by

$$f_K(t) = (I_n - \text{diag}(Kt))^{-1}e \quad \text{and} \quad f_{K,L}(t) = f_K(f_L(t)). \tag{5}$$

Equations (5) will be used in implementing the final application presented in Table 4.

Table 4. Application of restarted RRE to $\{v^{(k)}\}$.

Step 1	For $k = 0, u^{(0)} = v^{(0)} = 0$. Set $r=4$.
Step 2	For $k = 1, 2, \dots$ For $j = 0, 1, \dots, r - 1$ $g^{(0)} = v^{(k-1)}$; ; $g^{(j+1)} = f_{Q,P}(v^j)$;
Step 3	Compute the approximations $t^{(r-1)}$ by applying the RRE algorithm of table 1 on the vectors $(g^{(0)}, g^{(1)}, \dots, g^{(r)})$.
Step 4	If $t^{(r-1)}$ satisfies accuracy test, stop. Else, $v^{(0)} = t^{(r-1)}$;
Step 5	$u^{(k)} = f_P(v^{(k)})$.

Fig. 1. Non critical case

Fig. 2. Critical case

3 Conclusion

In this paper, different ways of application of the restarted RRE technique were presented. Figures 1 and 2 show the behaviour of the three applications both in the critical case, where the convergence rate is at its worst, and outside it. It has been seen that the application to

$\{w^{(k)}\}$ as in Table 2 was the best in the critical case. On the other hand, applying the restarted RRE only to $\{v^{(k)}\}$ as in Table 4 worked the best in the non critical case.

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