Hydrodynamic structure of laminar flows with oppositely-swirled coaxial layers

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Abstract. The article is devoted to the theoretical study of hydrodynamics of laminar flows with coaxial layers swirled in opposite directions and moving along the pipe. Such flows in a turbulent range have a wide practical application potential in technologies of dissipation of mechanical energy and mixing multiphase and heterogeneous media in microbiology, chemistry, ecology, heat engineering, power engineering, engine and rocket engineering. The article describes the tensor of viscous tangents (τ_{ij}) and normal (σ_{ii}) stresses. The questions of stability of flow according to the Rayleigh (Ra) and Richardson (Ri) criteria are considered. Calculation formulas and graphs of radial-axial distributions of viscous stress components, local stability zones are given, the point of “crisis and decay of the flow” or “vortex breakdown” is indicated. The solutions are obtained in the form of Fourier-Bessel series. The analysis of the hydrodynamic structure of the flow is made.

1 Introduction

This paper describes the viscous stress tensor and local stability zones of liquid flow, in which concurrent coaxial layers swirl in opposite directions (Fig. 1).

Fig. 1. Hydrodynamic structure of a flow with oppositely-swirled layers in a cylindrical pipe: **a** - profiles of the azimuthal (u_θ) and axial (u_x) velocities in a two-layer flow; **b** - components of velocity vectors, vortex of an elementary particle of a fluid and viscous stress tensor.

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Such flows in the Russian-language technical and normative literature are called “contrvihrevie” [1-5]. The work is based on article [6] devoted to the description of the flow kinematic structure. The relevance of the topic is related to the fact that the flows with oppositely swirling coaxial layers in the turbulent range are characterized by intensive mixing (diffusion) and quenching (dissipation) of the mechanical energy of the moving viscous media. Both properties of such flows have a great potential for practical use [4, 7]: the first - in technologies involving the mixing of heterogeneous and multiphase media in microbiology, chemistry, ecology, heat engineering, power engineering, engine and rocket-engineering; the second - to dissipate the mechanical energy of the flow of liquid or gas, for example, in high-pressure hydraulic spillways [1, 2, 8, 9] or for suppressing the noise of aircraft engines, propellers of ships. The effective development of these technologies is impossible without knowledge of the hydrodynamics of such flows. Such flows also arise in cyclones and draft tubes behind Francis hydraulic turbines in non-optimal operation modes [10-14].

2 Method of research

The theoretical investigations performed are based on an approximate analytical solution of the Navier – Stokes’s equations [15]. The solution itself is described in sufficient detail in our previous works, for example in [16-20], so the authors don’t consider it necessary to stop here your attention on it. Let us now turn to the substance of this work.

3 Results and Discussion

3.1 Distributions of flow velocities

Let’s consider initial formulas [3, 6] written in the cylindrical system of coordinate $r - \theta - x$ (see Fig. 1) for the swirl flows symmetric with respect to the pipe axis ($\partial/\partial\theta = 0$) of radial-axial distributions scaled by the average flow velocity (V) of the azimuthal ((u_θ) , axial (u_x) and radial (u_r) velocities, having the form of Fourier – Bessel’s series [21] or the products of the Fourier – Bessel’s series

$$u_\theta(r, x, \text{Re}) = 2 \sum_{n=1}^{\infty} G_n \frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right), \quad (1)$$

$$\begin{aligned} u_x(r, x, \text{Re}) = & 2(1-r^2) - \sum_{i=1}^{\infty} \frac{4}{\lambda_i^2} \left[1 - \frac{J_0(\lambda_i r)}{J_0(\lambda_i)} \right] \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) + \\ & + 2 \left\{ \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[1 + \frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2}r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \right. \\ & - \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & \left. - \sum_{n=1}^{\infty} \frac{G_n^2}{\lambda_n^2} \left[\frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2}r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \right\}, \quad (2) \end{aligned}$$

$$\begin{aligned}
 u_r(r, x, \text{Re}) = & -\frac{2}{\text{Re}} \sum_{i=1}^{\infty} \left[r - \frac{2J_1(\lambda_i r)}{\lambda_i J_0(\lambda_i)} \right] \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) + \\
 & + \frac{1}{\text{Re}} \left\{ \sum_{n=1}^{\infty} G_n \left[r + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\
 & + \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[r + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \lambda_k^2 \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\
 & - 2 \sum_{n=1}^{\infty} \frac{G_n \lambda_n J_1(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \left[\sum_{k=1}^{n-1} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\
 & + \sum_{k=n+1}^{\infty} \frac{G_k J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \left. \right] - 2 \sum_{n=1}^{\infty} \frac{G_n J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \times \\
 & \times \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k^2 J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k^2 J_0(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \right] + \\
 & + 2 \sum_{n=1}^{\infty} \frac{G_n J_0(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\
 & + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \left. \right] + 2 \sum_{n=1}^{\infty} \frac{G_n J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \times \\
 & \times \left[\sum_{k=1}^{n-1} \frac{G_k \lambda_k^3 J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \sum_{k=n+1}^{\infty} \frac{G_k \lambda_k^3 J_1(\lambda_k r)}{(\lambda_n^2 - \lambda_k^2) J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) \right] - \\
 & - 2 \sum_{n=1}^{\infty} G_n^2 \left[\frac{rJ_0^2(\lambda_n r) + rJ_1^2(\lambda_n r)}{J_0^2(\lambda_n)} + \frac{rJ_0(\lambda_n \sqrt{2})}{J_2(\lambda_n \sqrt{2})} - \frac{\sqrt{2}J_1(\lambda_n r \sqrt{2})}{\lambda_n J_2(\lambda_n \sqrt{2})} \right] \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \Big\}, \quad (3)
 \end{aligned}$$

where r and x are radial and longitudinal coordinates referred to the pipe radius R ; Re is Reynolds number, $\text{Re} = VR/\nu$; V is average flow velocity across the pipe section, $V = Q/\pi R^2$; Q is discharge; ν is kinematic fluid viscosity; $J_0(\dots)$, $J_1(\dots)$ and $J_2(\dots)$ is the Bessel function of the first kind of zero, first and second orders; λ_n and λ_k are real zeros of the Bessel function of the first kind of the first order ($J_1(\lambda_n) = 0$, $J_1(\lambda_k) = 0$, $J_1(\lambda_k) = 0$; λ_i are real zeros of the Bessel function of the first kind of the second order ($J_2(\lambda_i) = 0$); G_n and G_k are constants of n and k partial solutions

$$G_n = G_0 \left[\frac{1}{J_0(\lambda_n)} - 1 \right] - \Omega_0 - \frac{A_0 J_1(\mu)}{1 - (\mu/\lambda_n)^2}, \quad G_k = G_0 \left[\frac{1}{J_0(\lambda_k)} - 1 \right] - \Omega_0 - \frac{A_0 J_1(\mu)}{1 - (\mu/\lambda_k)^2}, \quad (4)$$

Ω_0 , G_0 , A_0 and μ (where $\mu \neq \lambda_n$ and $\mu \neq \lambda_k$) are the given normalized parameters of the radial distribution of azimuthal velocities at the entrance to the active zone with $x = 0$

$$u_\theta|_{x=0} = \Omega_0 r + \frac{G_0}{r} + A_0 J_1(\mu r).$$

Here and below, under the “active zone” is meant the area of intensive interaction of concurrent oppositely swirling layers.

Fig. 2. Standardized profiles of azimuthal, axial and radial velocities in flows with oppositely-swirled coaxial layers: above – mode 1, below – mode 2.

Fig. 2 shows radial-longitudinal distribution of azimuthal, axial and radial velocities in laminar flows with oppositely-swirled layers. The calculations of the graphs were made

using formulas (1) – (4) with Reynolds numbers equal to $Re = 500$ [6]. For comparison purposes, two flows are defined at the entrance to the active zone: the first (mode 1) two-layered with the parameters $\Omega_0 = 4.241$, $G_0 = -1.216$, $A_0 = 0$; the second (mode 2) is a fourlayered one with the parameters $G_0 = \Omega_0 = 0$, $A_0 J_1(\mu) = 0.1584$, $\mu = 13.3$. The distances from the entry section of the active zone to the cross-sections, in which the velocity profiles were calculated, are shown in Fig. 2 in fractions of the pipe radius ($x = 5R, 10, 20, 40, 80$)

A brief analysis of the results obtained (for details, see article [6]) for profiles of azimuthal velocities shows that both of the given laminar flows with oppositely swirling layers within the active zone are transformed into longitudinal flows without swirling. The length of the active zone - the zones of intensive viscous dissipation of the swirl of interacting layers in a two-layer flow is equal to 40 pipe radii; with a four-layer current, the length of the active zone reduces twice - up to 20 pipe radii. Concerning the distribution of axial velocities, it should be noted that the common for the two calculated modes is the strong return flow with significant negative velocities observed in the axial section in the active zone. Further lengthwise the pipe, return currents decrease and disappear, transforming into velocity deficiency characteristic for the flow behind a poorly-streamlined body. Similar velocity deficiency is observed in the zones between oppositely swirling coaxial layers. Beyond the limits of the return near-axis flow the velocity in the flow stratum may significantly exceed the average flow rate ($V = 1$). In the process of reforming the profile of axial velocities to the site, 80 pipe radii distant from the entrance to the active zone, it acquires a parabolic profile of the Poiseuille's flow. In Fig. 2 it can be seen that the values of the radial velocities are one or two orders lower than the azimuthal and axial velocities with radial velocities being lower, the higher is the Reynolds number in the denominator (3). In the whole, the kinematic structure of the flows with oppositely-swirling layers features high radial and longitudinal gradients of azimuthal and axial velocities. This is due to the action of internal force fields, determined by viscous tangential (τ_{ij}) and normal (σ_{ii}) stresses (see Fig. 1,b).

3.2 Tensor of viscous stresses

Consistent with [15], in the flows symmetric with respect to the longitudinal axis of the pipe ($\partial/\partial\theta = 0$) in accordance with velocity distributions (1) – (3), we obtain the following distributions of the normalized tensor of viscous tangential stresses

$$\begin{aligned} \frac{\tau_{r\theta}(r, x, Re)}{\mu} &= \frac{\tau_{\theta r}(r, x, Re)}{\mu} = r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) = -2 \sum_{n=1}^{\infty} G_n \frac{J_2(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{Re}\right), \\ \frac{\tau_{\theta x}(r, x, Re)}{\mu} &= \frac{\tau_{x\theta}(r, x, Re)}{\mu} = \frac{\partial u_\theta}{\partial x} = -\frac{2}{Re} \sum_{n=1}^{\infty} G_n \frac{\lambda_n J_1(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{Re}\right), \\ \frac{\tau_{xr}(r, x, Re)}{\mu} &= \frac{\tau_{yx}(r, x, Re)}{\mu} = \frac{\partial u_x}{\partial r} = -4 \left[r + \sum_{i=1}^{\infty} \frac{J_1(\lambda_i r)}{\lambda_i J_0(\lambda_i)} \exp\left(-\lambda_i^2 \frac{x}{Re}\right) \right] + \\ &+ 2 \left[\sum_{n=1}^{\infty} G_n \frac{\sqrt{2} J_1(\lambda_n \sqrt{2} r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \exp\left(-\lambda_n^2 \frac{x}{Re}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{Re}\right) + \right. \\ &+ \sum_{n=1}^{\infty} G_n \frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{Re}\right) \sum_{k=1}^{\infty} G_k \frac{J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{Re}\right) + \\ &+ \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{Re}\right) \sum_{k=1}^{\infty} G_k \frac{\lambda_k J_1(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{Re}\right) - \\ &\left. - \sum_{n=1}^{\infty} G_n^2 \frac{\sqrt{2} J_1(\lambda_n \sqrt{2} r)}{\lambda_n J_2(\lambda_n \sqrt{2})} \exp\left(-2\lambda_n^2 \frac{x}{Re}\right) \right] \end{aligned}$$

and normal stresses

$$\begin{aligned} \frac{\sigma_{xx}(r, x, \text{Re})}{\mu} = & -2 \frac{\partial u_x}{\partial x} = -\frac{8}{\text{Re}} \sum_{i=1}^{\infty} \left[1 - \frac{J_0(\lambda_i r)}{J_0(\lambda_i)} \right] \exp\left(-\lambda_i^2 \frac{x}{\text{Re}}\right) + \\ & + \frac{4}{\text{Re}} \left\{ \sum_{n=1}^{\infty} G_n \left[1 + \frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2}r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) + \right. \\ & + \sum_{n=1}^{\infty} \frac{G_n}{\lambda_n^2} \left[1 + \frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2}r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \lambda_k^2 \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & - \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & - \sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{\lambda_n^2 J_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{x}{\text{Re}}\right) \sum_{k=1}^{\infty} G_k \frac{\lambda_k^2 J_0(\lambda_k r)}{J_0(\lambda_k)} \exp\left(-\lambda_k^2 \frac{x}{\text{Re}}\right) - \\ & \left. - 2 \sum_{n=1}^{\infty} G_n^2 \left[\frac{J_0(\lambda_n \sqrt{2}) - J_0(\lambda_n \sqrt{2}r)}{J_2(\lambda_n \sqrt{2})} \right] \exp\left(-2\lambda_n^2 \frac{x}{\text{Re}}\right) \right\}, \\ \frac{\sigma_{\theta\theta}(r, x, \text{Re})}{\mu} = & -2 \frac{u_r(r, x, \text{Re})}{r}, \\ \frac{\sigma_{rr}(r, x, \text{Re})}{\mu} = & -\frac{\sigma_{xx}(r, x, \text{Re})}{\mu} - \frac{\sigma_{\theta\theta}(r, x, \text{Re})}{\mu}, \end{aligned}$$

here μ is dynamic viscosity coefficient of fluid, $\mu = \rho\nu$; ρ is fluid density.

The profiles of the viscous stress tensor in the calculated sections of the active zone are shown in Fig. 3 for a flow with two oppositely-swirling layers (mode 1) and in Fig. 4 for the four-layer one (mode 2). The most significant stresses can be observed at the beginning of the active zone at the $x = 5R$ site. This is caused by the process of intensive viscous interaction of oppositely swirling layers and transformation of velocity profiles. As a result, at the beginning of the active zone in swirl currents the flow loses the most significant part of its mechanical energy due to high internal viscous stresses. This principally differs the flow with oppositely-swirled layers from circular-longitudinal currents [22-24], where significant stresses are concentrated in the near-axial core and longitudinal-axial flows [15], where just tangential stresses $\tau_{xr} = \tau_{rx}$, whose values are ignorant, are non-zero. For example, in the Poiseuille's flow they vary along the radius according to linear law reaching specified value ± 4

$$\frac{\tau_{xr}}{\mu} = \frac{\tau_{rx}}{\mu} = \frac{\partial u_x}{\partial r} = \frac{\partial}{\partial r} [2(1 - r^2)] = -4r,$$

at the solid flow boundary with $r = \pm 1$.

The performed calculations show that at the beginning of the active zone the alternating stresses in the stratum of the flows with oppositely-swirling layers reach the values of tens and hundreds of normalized units. The radial stress distributions $\tau_{xr} = \tau_{rx}$, corresponding to the Poiseuille's flow, are realized in a given two-layer swirl flow with $\text{Re} = 500$ to the $x = 40R$ site (Fig. 3), and in the four-layer swirl flow to the $x = 20R$ site (Fig. 4). Comparing the two- and four- oppositely-swirled layered flows, one can observe in the latter a multiple

increase in sign change for all components of the tensor of viscous stresses (except for $\sigma_{\theta\theta}$) and absolute values of tangential stresses $\tau_{r\theta} = \tau_{\theta r}$ and $\tau_{\theta x} = \tau_{x\theta}$. This intensifies the process of viscous dissipation (suppression) of swirling of interacting layers in multilayer flows resulting in proportional reduction in the length of the active zone. These same factors contribute to the tendency of losing the stability of the laminar flow with oppositely-swirling layers and its transition to a turbulent flow with a high intensity of energy and mass exchange processes between the interacting layers.

Fig. 3. Components of normalized viscous stress tensor in two-layered flow with oppositely-swirled layers (mode 1).

Fig. 4. Components of normalized viscous stress tensor in four-layered flow with oppositely-swirled layers (mode 2).

3.3 Hydrodynamic stability of the flow

We will consider hydrodynamic stability as the ability of a flow in the local area to maintain the stability, suppressing random perturbations that arise in it, or to lose it, passing into a more stable form of motion. The transition from one form of flow to another can be

manifested in the form of a laminar-turbulent transition, or in the form of a decay of the flow with oppositely-swirled layers with its transition to a longitudinal-axial flow. To find the solution, we use the Rayleigh method [3, 24-28], which allows us to evaluate the stability of a particular fluid motion in its local areas. The essence of the method consists in the following. If we assume that the mass exchange between the layers moving in the cylindrical flow channel is determined by the radial displacements of the liquid particles, and the azimuthal and axial displacements as a whole do not lead to the diffusion of particles from the initial layer, then the local stability condition can be formulated as follows: if the elementary liquid particle (see Fig. 1,b) with mass $\rho \cdot dr \cdot r d\theta \cdot dx$ for random reasons is displaced from the original trajectory of its motion on radius r_0 to the new one on radius r , and this displacement ($\Delta r = r - r_0$) is small, the sum of forces effecting the particle determined by the difference between the forces of pressure P and the inertial mass centrifugal force with acceleration j

$$P \cdot r d\theta \cdot dx - (P + \frac{\partial P}{\partial r} dr) \cdot r d\theta \cdot dx + j \cdot \rho \cdot dr \cdot r d\theta \cdot dx$$

may: a) strive to return it to the original trajectory, in this case the current in the local area will remain stable, and random perturbations will be suppressed; b) contribute to further displacement of an elementary particle resulting in local loss of stability by the current. If the unstable area extends over a considerable stratum of the flow, then the current can lose its stability in the whole to “vortex breakdown”. Thus, the local stability condition is written in the form

$$\frac{\partial P}{\partial r} - \rho j \begin{cases} > 0 & \text{with } \Delta r = r - r_0 > 0, \\ < 0 & \text{with } \Delta r = r - r_0 < 0. \end{cases}$$

Further, it is necessary to determine which feature of the particle motion can be assumed to be transported lengthwise the random displacement Δr . It is known that by analyzing the spatial flows to which the flow with oppositely-swirled layers is related, the Taylor’s theory of vorticity transport [29, 30] is more preferable than the Prandtl’s momentum diffusivity theory [15]. Assuming, according to the Taylor’s theory, that the vorticity of an elementary liquid particle lengthwise the displacement path to interaction with the surrounding particles is constant

$$\text{rot}_x U = \frac{\partial(r u_\theta)}{r \partial r} = \text{const},$$

where $\text{rot}_x U$ is the vortex axial component on initial radius r_0 , in article [28] the local flow stability criterion by Rayleigh has been obtained in the form

$$\text{Ra} = -\frac{\partial}{r \partial r} [r u_\theta \frac{\partial(r u_\theta)}{r \partial r}] = 4 \left\{ \sum_{n=1}^{\infty} G_n \frac{J_1(\lambda_n r)}{\lambda_n J_0(\lambda_n)} \exp(-\lambda_n^2 \frac{x}{\text{Re}}) \times \right. \\ \left. \times \sum_{k=1}^{\infty} G_k \frac{\lambda_k J_1(\lambda_k r)}{J_0(\lambda_k)} \exp(-\lambda_k^2 \frac{x}{\text{Re}}) - \left[\sum_{n=1}^{\infty} G_n \frac{J_0(\lambda_n r)}{J_0(\lambda_n)} \exp(-\lambda_n^2 \frac{x}{\text{Re}}) \right]^2 \right\} > 0.$$

Thus, the local stability criterion (Ra) is defined by the sign of partial derivative by circulation period radius ($G = r u_\theta$), times axial component of vortex ($\text{rot}_x U$). For a negatively signed derivative, the Rayleigh number is above zero ($\text{Ra} > 0$), then the centrifugal force will try to suppress random turbulences and drive the flow in the local area back to stability; but for a positively signed derivative, the Rayleigh criterion is negative ($\text{Ra} < 0$) – the flow loses stability. Rayleigh’s local stability criterion helps to isolate areas that generate and other that suppress random turbulences in the flow with oppositely-swirling layers, but it fails to assess stability of such flows respective to such random

turbulences. Yet, to analyze influence of azimuthal circulation on flows structure, the Richardson number (Ri) is frequently used that equals the quotient of the Rayleigh number divided by the quadratic invariant of the deformation speed tensor [3, 24, 26, 28, 31, 32]

$$Ri = Ra / J^2 ,$$

where for the axisymmetric flow ($\partial/\partial\theta = 0$) it is possible to write

$$J^2 = 2 \left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{u_r}{r} \right)^2 + \left(\frac{\partial u_x}{\partial x} \right)^2 \right] + \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)^2 + \left(\frac{\partial u_r}{\partial x} + \frac{\partial u_x}{\partial r} \right)^2 + \left(\frac{\partial u_\theta}{\partial x} \right)^2 =$$

$$= \frac{1}{2} \left[\left(\frac{\sigma_{rr}}{\mu} \right)^2 + \left(\frac{\sigma_{\theta\theta}}{\mu} \right)^2 + \left(\frac{\sigma_{xx}}{\mu} \right)^2 \right] + \left(\frac{\tau_{r\theta}}{\mu} \right)^2 + \left(\frac{\tau_{\theta x}}{\mu} \right)^2 + \left(\frac{\tau_{xr}}{\mu} \right)^2 .$$

Fig. 5. Zones of local stability of flows with oppositely-swirled layers: **a** - two-layered, **b** - four-layered.

The Rayleigh number and the quadratic invariant of the deformation rate tensor have the same dimension. Consequently, the Richardson number obtained by dividing one by another characterizes the local ratio of the forces acting on a liquid particle during its accidental small displacements and contributing to either their suppression or increment, to forces (stresses) constantly acting at a given point of the liquid medium generated by the deformation tensor and independent of random causes. In turn, the strain tensor itself is the source that generates instability of the flow, and high deformations in the liquid medium stratum contribute to growth of random perturbations. Taking into account the foregoing,

the Richardson number, which is the ratio of random factors both destabilizing and stabilizing the current, to stationary factors, is, in this respect, an indicator of local conditions of flow stability. So, if the Richardson number is essentially positive ($Ri > 1$), and this, since $J^2 > 0$, is possible only under the conditions when $Ra > J^2$, then the effect of stabilizing forces will obviously exceed the effect of the forces determined by the strain tensor generating instability and in this case we will have a stable flow in the local area under study with a tendency of suppression of perturbation in the field of centrifugal forces. With Richardson number close to zero ($Ri \approx \pm 0$ with $Ra \approx \pm 0$ or $|Ra| \ll J^2$), the current destabilizing and stabilizing force are in relative local equilibrium with the forces generated by the strain tensor. In this case the flow in the area under study will be in an intermediate state. With high negative values of Richardson number ($Ri \ll 0$ for $Ra < 0$ and $|Ra| \gg J^2$) the flow in the local area under study will lose its stability, since in this case random perturbations will significantly exceed the deformations of the natural level characteristic of or corresponding to stationary factors determining the state of the flow. The latter can cause destabilization of the flow as a whole, if in the surrounding areas of the flow negative values of the Richardson criterion are observed, that is, prevailing are the forces favoring growth of perturbations.

Fig. 5 shows the maps of isoclines of Richardson numbers obtained for computational two- (Fig. 5,a) and four- (Fig. 5,b) layer flows. Shown is a section of up to $80R$ length that exceeds the length of the active zone. In view of symmetry of the flow relative to the pipe axis, the radial range in the figures is taken from 0 to $r/R = 1$. The boundaries of transition of Richardson and Rayleigh numbers through zero ($Ri = Ra = 0$) are shown by solid white lines. The figures inside the area indicate the local values of the Richardson numbers.

In Fig. 5, one can see the presence of areas with unstable flow indicated by letter "A", in which random perturbations tend to increase and areas with stable flow indicated by letter "B", where the perturbations are suppressed by the field of centrifugal forces.

Areas with unstable flow. The first such area (A_1) lies along the walls of the pipe ($r \approx 1$). The instability of the flow in this peripheral area is caused by viscous liquid diffusion at the solid flow boundaries. The second area (A_2) embraces the zone located between the co-axial oppositely swirled layers. The physical reason for instability of the flow is obvious here, because it is determined by the mutually - opposite direction of azimuthal velocities at the boundary of the layers. In a multilayer oppositely-swirled flow (with 4 layers) additional unstable flow areas indicated in Fig. 5,b as A_3 and A_4 , are repeated between the remaining coaxial oppositely- swirled layers. The last unstable flow area is the central vortex core. In a two-layer flow (Fig. 5,a), this area is indicated as A_3 , that in the multilayered area as A_5 . Instability of this area is determined by physical factors characteristic of the zone of the vortex cord of a damping circulation-longitudinal flow with uniform swirling [28]. In the vortex core, three zones can be distinguished: a zone of weak instability with local Richardson numbers up to $Ri = -1$, passing into the zone of flow destabilization with high Richardson numbers ($Ri = -10 \dots -100$), in turn, transforming to a zone with rapidly increasing instability with the values greater than $Ri \geq -1000$. The latter can be defined as a zone of stability loss with a "current crisis", which is completed by a "collapse or decay of the flow", or "vortex breakdown". It can be seen that in the flow with two oppositely-swirled layered (Fig. 5,a), the point of "decay" lies in the area of site $32R$ and in the four-oppositely swirled layered one - at the $16R$ site. The "decay" of the flow with oppositely-swirled layers determines, on the whole, the extent of the active zone. It is interesting to note that in publication [24] devoted to study of the circulation-longitudinal flow with homogeneous swirl at the point of "vortex breakdown" in the area of $74R$ site also corresponds to minimum value of Richardson number ($Ri \geq -1000$) the calculation was made with $Re = 500$). The areas with unstable flow of A_3 , A_4 and A_5 in the flow with four-layer swirling join in one common area of A_{345} . So, the cardinal difference between the flow

with oppositely-swirled layers and the circulation-longitudinal flow consists in the presence of unstable areas between oppositely-swirled layers. These areas and the area of vortex cord are the generators of processes initiating the flow laminar-turbulent transition, mass change between the layers, and finally the “vortex breakdown” of the initial flow with oppositely-swirled layers and its transition to longitudinal-axial flow.

The areas with stable current $B_1 - B_4$ are localized inside the layers with opposite swirl far from the boundaries of their interaction. These areas tend to decay, since the areas with unstable flow destroy them. At the exit from the active zone in the four-layered flow there remains just one of stable areas B_2 , which is characteristic of degenerating circulation-longitudinal homogeneous flow [24].

In general, the obtained the isoline maps of Richardson criterion correspond to physical idea of local stability zones in the flow with oppositely-swirled layers.

4 Conclusions

1. The article deals with a theoretical study of the hydrodynamics of a flow with coaxial layers rotating in opposite directions moving along a cylindrical tube. Such flows in the turbulent range have a great potential for application in technologies of mixing inhomogeneous media, as well as in order to quench the excess mechanical energy of the flows of liquids or gases. The effective development of these technologies is impossible without knowledge of the hydrodynamics of such flows. In addition to the above, such flows arise in draft tubes behind Francis hydraulic turbines in non-optimal operation modes. This, with insufficient knowledge of the physics of such flows, has repeatedly been the cause of accidents at large hydroelectric power stations.
2. The article is a continuation of work [6], devoted to the description of the kinematic structure of the flow under study. The theoretical model of the flow under study is based on the Navier – Stokes's equations and the Fourier's method of decomposition of systems of partial differential equations. In this paper, the calculated formulas for radial-axial distributions of tangential, axial and radial velocities in these flows, as well as current functions and components of vortices, which are series or products of Fourier – Bessel's series, are obtained.
3. A brief analysis of previously obtained results (for more details, see [6]) shows that both of the investigated laminar flows within the active zone are transformed into longitudinal flows without swirling. The length of the active zone, within which there is intense viscous dissipation of the swirling of the interacting oppositely rotating coaxial layers, in a two-layer flow with a Reynolds number equal to $Re = 500$, corresponds to 40 radii of pipe; in a four-layer flow in the same conditions, it's reduced by 2 times – to twenty radii. In general, the kinematic structure of flows with interacting layers is characterized by high radial and longitudinal gradients of tangential and axial velocities. This is reflected on the acting internal force fields determined by viscous tangential and normal stresses.
4. In accordance with the distributions of tangential, axial and radial velocities, in this work we obtained radial-axial distributions of viscous tangential ($\tau_{r\theta}$, $\tau_{\theta x}$, τ_{xr}) and normal (σ_{xx} , $\sigma_{\theta\theta}$, σ_{rr}) stresses. The most significant voltages are observed at the beginning of the active zone. Here, the shear stresses τ_{xr} reach values in excess of hundreds of normalized units. This is caused by the intense interaction of the oppositely rotating layers. As a result, at the beginning of the active zone in the studied flows, due to high internal viscous stresses, the flow loses the most significant part of the mechanical energy. This fundamentally distinguishes such flows from circulation-longitudinal flows, where significant stresses are concentrated in the vortex core, and from longitudinal-axial flows, where shear stresses τ_{xr} have maximum values at the pipe walls equal to 4 normalized units.

5. In the thickness of the studied flows, there is a frequent change of sign for all components of the viscous stress tensor except $\sigma_{\theta\theta}$. This intensifies the suppression of the swirling of the interacting layers and contributes to the tendency to the loss of stability of the laminar flow and its transition to turbulent one with a high intensity of molar and energy exchanges between the interacting layers.

6. Hydrodynamic stability is considered in the work as the ability of a flow in a local region to maintain stability, suppressing the occurring random perturbations, or losing it, with an increase in random perturbations. To determine the regions and zones of local stability, Rayleigh (Ra) and Richardson (Ri) criteria were used. It is established that the criterion of local Rayleigh stability allows us to distinguish the generation regions of random perturbations and the regions of their suppression, but does not allow us to estimate the degree of flow stability to these random perturbations.

7. A more detailed estimate of the local stability of the flow can be obtained on the basis of the Richardson criterion, equal to the ratio of the Rayleigh criterion to the quadratic invariant of the deformation speed tensor J^2 . In this formulation, the number of Richardson is the ratio of random destabilizing or stabilizing factors to the stationary stresses generated in liquid environment by the strain tensor and independent of random causes. If the Richardson number is greater than unity ($Ri > 1$), which is possible when $Ra > J^2$, then, obviously, the influence of stabilizing forces will exceed the influence of the forces determined by the strain tensor generating instability. In this case, a steady flow with a tendency to suppress perturbations by the field of centrifugal forces will be observed in the local area under study. When the values of the Richardson number close to zero ($Ri \approx \pm 0$ at $Ra \approx \pm 0$ or $|Ra| \ll J^2$), the forces destabilizing and stabilizing the flow are in relative local equilibrium with the forces generated by the strain tensor. At high negative values of the Richardson number ($Ri \ll 0$ at $Ra < 0$ and $|Ra| \gg J^2$), the flow in the local area under study will lose stability, since random perturbations will significantly exceed the natural level of deformations. This can cause destabilization of the flow as a whole, if negative values of the Richardson criterion are also observed in the surrounding areas of the flow.

8. It was established in the work that regions with unstable flow are localized along the pipe walls, at the contact boundaries between interacting oppositely rotating layers, and on the pipe axis in the vortex core of the flow. Three zones can be distinguished in the vortex core: a zone of weak instability with local Richardson numbers up to $Ri = -1$, passing into a flow destabilization zone with high negative Richardson numbers ($Ri = -10$ to -100), in turn, passing into the zone with rapidly growing instability up to values of $Ri = -1000$. This is a zone of loss of stability, culminating in "crisis and decay of the flow" or "vortex breakdown". Thus, the fundamental difference between the studied flows and the circulation-longitudinal ones is the presence of unstable regions between oppositely rotation layers. These areas and the zone of the vortex cord are the generators of processes that promote the laminar-turbulent transition, the mass transfer between the layers, and ultimately the collapse of the initial flow with the swirling layers and its transition into an axial flow.

9. Regions with a steady flow are localized inside the swirling layers far from the boundaries of their interaction. These areas tend to decay, because areas with an unstable flow destroy them. On the whole, the obtained maps of isolines of the Richardson criterion correspond to physical ideas about the regions and zones of local stability in the studied complicated spatial flows.

References

1. G.I. Krivchenko, A.P. Mordasov, E.V. Kvyatkovskaya, V.V. Volshanik, A.L. Zuikov, *Hydrotechnical Construction* **15** (10), 604-608 (1981)

2. G.I. Krivchenko, A.P. Mordasov, E.V. Kvyatkovskaya, V.V. Volshanik, A.L. Zuikov, A.V. Levanov, XX Congress IAHR **VIII**, 464-467 (1983)
3. V.K. Akhmetov, V.V. Volshanik, A.L. Zuikov, G.V. Orekhov, *Simulation and Calculation of Counter-Rotating Flows* (Publ. MISI-MGSU, Moscow, 2012)
4. V.K. Akhmetov, V.V. Volshanik, A.L. Zuikov, G.V. Orekhov, *Physical Modeling of Contrvirevii Structures and Equipment* (Publ. MISI-MGSU, Moscow, 2018)
5. State Standard of Russian Federation CO 34.21.308-2005
6. A. Zuikov, G. Orekhov, T. Suehtina, MATEC Web of Conferences **193**, 02024 (2018) doi.org/10.1051/mateconf/201819302024
7. V.V. Volshanik, G.V. Orekhov, Vestnik MGSU **7**, 87-104 (2015)
8. V.Ya. Karelin, G.I. Krivchenko, et.al. *Physical and Math. Modeling of Hydraulic Processes in the Study of Large-Scale Multi-Purpose Hydro "MH-89"*, (Publ. VNIIG, Leningrad, 1989)
9. V.V. Volshanik, A.L. Zuikov, A.P. Mordasov, *Swirling flows in hydraulic structures* (Publ. Energoatomizdat, Moscow, 1990)
10. A. Javadi, H. Nilsson, Engineering Applications of Computational Fluid Mechanics **11**, 1, 30-41 (2017), DOI: 10.1080/19942060.2016.1235515
11. Zhao Bang-tao, Chinese J. Process Eng. **2**, 178-180 (2006)
12. N.O. Zaytsev, Science and technical sheets SPSPU **5-1**, 197-202 (2006)
13. K.I. Strelets, Science and technical sheets SPSPU **5(108)**, 156-160 (2010)
14. A.A. Girgidov, K.I. Strelets, N.I. Vatin, Mag. Civil Eng. **5(23)**, 5-9 (2011)
15. A.L. Zuikov, *Hydraulics* (Publ. MISI-MGSU, Moscow, 2014)
16. A.L. Zuikov, Vestnik MGSU **3**, 195-199 (2009)
17. A.L. Zuikov, Vestnik MGSU **3**, 200-204 (2009)
18. A.L. Zuikov, Int. J. Comput. Civil and Struct. Eng. **8-2**, 82-96 (2012)
19. A.L. Zuikov, Vestnik MGSU **1**, 51-56 (2012)
20. A.L. Zuikov, G.V. Orekhov, V.V. Volshanik, Vestnik MGSU **5**, 150-161 (2013)
21. G.A. Korn, T.M. Korn, *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review* (Dover Publ., 2000)
22. A.L. Zuikov, Vestnik MGSU, Special Issue **2**, 91-95 (2009)
23. A.L. Zuikov, Vestnik MGSU, Special Issue **2**, 95-99 (2009)
24. A.L. Zuikov, *Hydrodynamics of Circulating Currents* (Publ. MISI-MGSU, Moscow, 2010)
25. A.A. Halatov, Izvestija VUZ. Aviacionnaja tehnika **3**, 105-109 (1976)
26. V.K. Shhukin, A.A. Halatov, *Heat Transfer, Mass Transfer and Hydrodynamics of Swirled Flows in Axisymmetrical Channels* (Pub. Mech. Eng., Moscow, 1982)
27. S. Leibovich, K.A. Stewartson, J. Fluid Mech. **126**, 54-98 (1983)
28. A.L. Zuikov, Izvestija VUZ. Stroitel'stvo **11-12**, 77-86 (2009)
29. G.I. Taylor, *Proc. of the Royal Soc., Ser. A* **135**, 685-701 (1932)
30. S. Goldstein, *Proc. Cambridge Philosophical Soc.* **31-3**, 351-359 (1935)
31. A.K. Gupta, D. Lilley, N. Syred, *Swirl Flows* (Publ. Abacus Press, 1984)
32. J.M. Delery, Progress in Aerospace Sciences **30**, 1-59 (1994)