

Dissemination of shock waves in thin isotropic plates

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Abstract. The article deals with the propagation of shock waves in thin isotropic plates. The solution is done analytically. The geometric model of Kirchhoff and Rayleigh was used for the solution. Material model was used Hook. The shock wave is induced by the impact of the punch into the centre of the plate. The longitudinal waves propagate in the x and y axis, the transverse wave propagates in the axis. Comparison of the longitudinal and transverse wave propagation of both models at selected points is made. Comparisons are displacements, velocities and tensions in x , y and z axis, depending on time. The rectangular plate is made of unidirectional rolled aluminium sheet of high purity. The plate is around the perimeter fixed. The MATLAB program was used to solve this problem.

Keywords: plate, wave, MATLAB

1 Introduction

The issue of vibrations of thin plates is solved for more than 200 years. Euler made the first mathematical expression on the plates in 1776. He analysed the free oscillation of the plates. After that, many scientists, such as Chladni, Bernoulli and others, worked on the issue, all of which came from the work of Euler. Significant progress in solving these problems occurred in 1850, when Kirchhoff published his theory of thin plates [1, 2]. Other authors followed this basic theory. Rayleigh extended Kirchhoff's geometric model by the effect of cross-sectional rotation and corresponding inertial effects. Flügge introduced the effect of shear on the resulting vertical displacement, but did not consider turning the cross-sections. Timoshenko-Mindlin's model considers both cross-section and shear. In addition to the above-mentioned core model correction models, some additional corrections were proposed to improve the approximation [2].

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In addition to these geometric models, different models of materials. The basic material model is Hook's, which was further developed by Vight-Kelvin, Maxwell, and Zener.

The article deals with the examination of the transient oscillation of the thin isotropic rectangular plate that is around the perimeter fixed. The solution of this plate is done analytically. Kirchhoff and Rayleigh's geometric model and Hook's material model were selected. The thin plate of dimensions $a \times b \times h$ is loaded in its geometric center by the impact force $F(t)$ induced by the impact of the punch of the radius c on the upper face of the plate (Fig. 1). When the deformation plate from the propagating waves are assumed small deformation (Fig. 2).

The aim of the solution is to compare the deformations (displacement), the propagation velocity of the shockwave and the tensions of both plate models.

2 Solution

Based on the geometry of the plate, the basic relations for the displacement u, v, w in the direction of the axes x, y and z were derived from the propagating shock wave [2, 3].

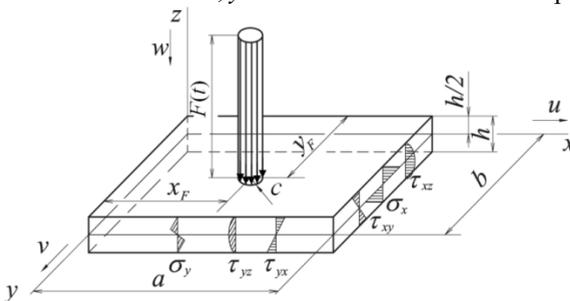


Fig. 1. Scheme of load plate [2]

Legend: a, b, h – dimensions of the plate, c – radius of impact, $F(t)$ – load force, x_F, y_F – force coordinates, σ_x, σ_y – bending stresses, τ_{yx}, τ_{xy} – torsional stresses, $\tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$ – shear force

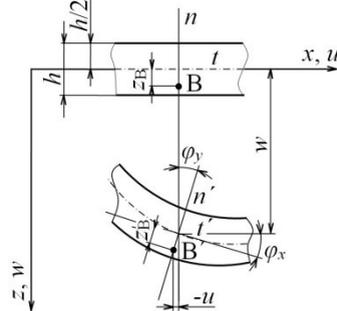


Fig. 2. Geometry of plate deformation [2]

Legend: n, n' – normals, t, t' – tangents, u, v, w – displacement in x and y direction, ϕ_x, ϕ_y – angles of the cut in the direction of the x and y axes, B, B' – watch point, $z_B, z_{B'}$ – distance of point B, h – thickness of the plate

The displacement u, v and w in the direction of the x, y and z axis from the propagating shock wave are

$$u = u(x, y, z, t) \quad v = v(x, y, z, t) \quad w = w(x, y, t) \quad (1)$$

After deformation of the plate, the normal, resp. the tangent at point B is rotated by the angle ϕ – with respect to the x axis by the angle ϕ_x , with respect to the y axis of the angle ϕ_y . Then, the displacement along the x axis and the y axis direction for both models can be expressed as

$$u = -z \phi_x \quad v = -z \phi_y \quad (2)$$

The velocities of wave propagation $\dot{u}, \dot{v}, \dot{w}$ are for both models the same

$$\dot{u} = \frac{\partial u}{\partial t} \quad \dot{v} = \frac{\partial v}{\partial t} \quad \dot{w} = \frac{\partial w}{\partial t} \quad (3)$$

Vertical displacement of the isotropic thin plate $w(x, y, t)$ [2, 3]

for the Kirchhoff's model

$$w(x, y, t) = \frac{16}{abc} \frac{F_0}{\rho h} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_1(\gamma_{mn}c)}{\gamma_{mn}} \frac{\sin \alpha_n x_F \cos \beta_m y_F}{\omega_{mn}^2} \sin \alpha_n x \sin \beta_m y \sin^2 \frac{\omega_{mn}}{2} t \quad (4)$$

$$\omega_{mn} = (\alpha_n^2 + \beta_m^2) \sqrt{\frac{D}{\rho h}} = (\alpha_n^2 + \beta_m^2) \frac{ch}{\sqrt{12}}$$

for the Rayleigh's model

$$w(x, y, t) = \frac{16}{abc} \frac{F_0}{\rho h} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_1(\gamma_{mn}c)}{\gamma_{mn}} \frac{\sin \alpha_n x_F \cos \beta_m y_F}{\left[\frac{h^2}{12} (\alpha_n^2 + \beta_m^2) + 1 \right] \omega_{mn}^2} \sin \alpha_n x \sin \beta_m y \sin^2 \frac{\omega_{mn}}{2} t \quad (5)$$

$$\omega_{mn} = \frac{\alpha^2 + \beta^2}{\sqrt{\frac{h^2}{12} (\alpha_n^2 + \beta_m^2) + 1}} \sqrt{\frac{D}{\rho h}} = \frac{\alpha^2 + \beta^2}{\sqrt{\frac{h^2}{12} (\alpha_n^2 + \beta_m^2) + 1}} \frac{ch}{\sqrt{12}}$$

The coefficient γ_{mn} , the plate stiffness D and the constants α, β are the same for both models

$$\gamma_{mn} = \sqrt{\alpha_n^2 + \beta_m^2} \quad D = \frac{Eh^3}{12(1-\mu^2)} \quad \alpha_n = n \frac{\pi}{a} \quad \beta_m = m \frac{\pi}{b} \quad (6)$$

Normal and tangential stresses σ_x, σ_y are for both model the same

$$\sigma_x(x, y, z, t) = -\frac{z Ex}{1 - \mu_{xy}\mu_{yx}} \left(\frac{\partial^2 w}{\partial x^2} + \mu_{xy} \frac{\partial^2 w}{\partial y^2} \right) \quad (7)$$

$$\sigma_y(x, y, z, t) = -\frac{z Ey}{1 - \mu_{xy}\mu_{yx}} \left(\frac{\partial^2 w}{\partial y^2} + \mu_{xy} \frac{\partial^2 w}{\partial x^2} \right)$$

where a, b, h – dimensions of the plate, c – radius of punch, ρ – density of the plate material, F_0 – load force, $J_1(\gamma_{mn}c)$ – Bessel's first-order function, the first order for the $\gamma_{mn}c$ argument, α, β – constants, m, n – variables, x_F, y_F – force coordinates, x, y, z – the coordinate of the examined point, ω_{mn} – own frequency, t – time, D – plate stiffness, E, G – tensile and shear modulus, μ – Poisson's number.

3 Results

Wave propagation in a thin plate was analytically solved in the MATLAB program for the Al 99.9 isotropic aluminium plate [5, 6, 7, 8]. The dimensions of the plate were $0.2 \times 0.1 \times 0.002$ m. The material properties are shown in Tab. 1.

Table 1. Material properties of Al plate

Název	Označení	Jednotka	Hodnota
Modul pružnosti v tahu	E	Pa	7.1×10^{10}
Poissonovo číslo	μ	-	0.34
Hustota	ρ	$\text{kg} \cdot \text{m}^{-3}$	2 699

The plate was about perimeter fixed and in the center of impact ($x_F = 0.1 \text{ m}$, $y_F = 0.05 \text{ m}$) was loaded with a force of $F(t) = 1 \text{ N}$ from a punch with a radius $c = 0.025 \text{ m}$ (Fig. 3). Size of the displacements, velocity and deformations of the wave propagating in a thin plate for the Kirchhoff and Rayleigh models were determined at the points $K_1 [0.02; 0.00; 0.00]$, $K_2 [0.00; 0.02; 0.00]$.

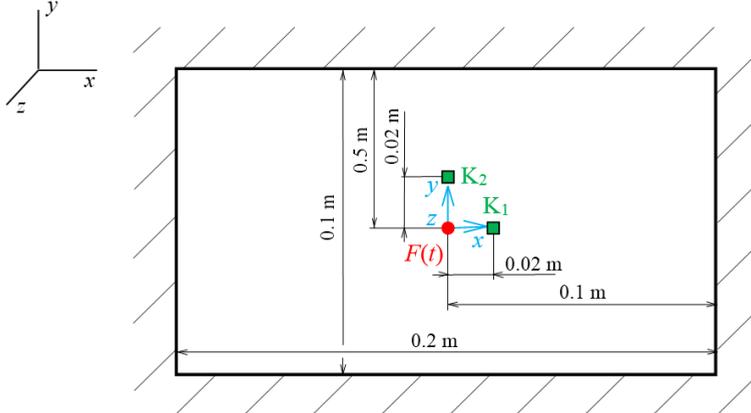


Fig. 3. Plate dimensions and centre of impact, Legend: $F(t)$ – load force in centre of plate, K_1, K_2 – points

On Fig. 4 and Fig. 5 are shown the waveforms w (in the z axis) at points K_1, K_2 for the Kirchhoff and Rayleigh models. The corresponding numerical values are shown in Tab. 2.

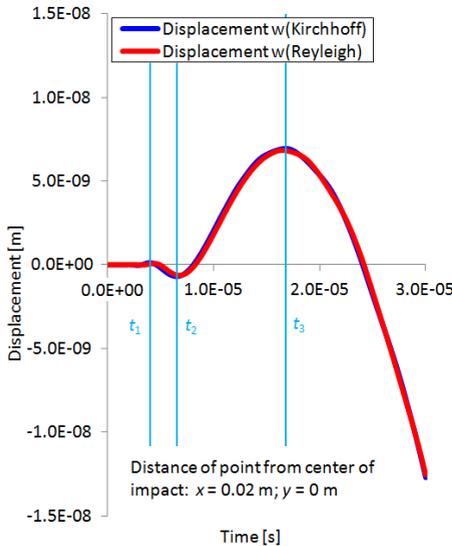


Fig. 4. Wave displacement after impact w in point K_1 for Kirchhoff's and Reyleigh's model

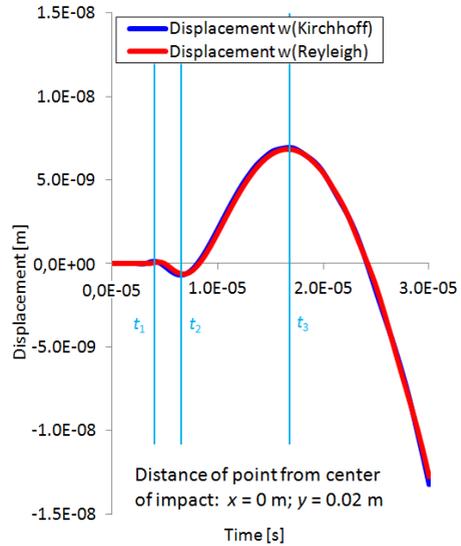


Fig. 5. Wave displacement after impact w in point K_2 for Kirchhoff's and Reyleigh's model

Table 2. Wave displacement after impact (w) in point K_1 and K_2

Time [s]	Point	K_1		K_2	
		Displacement w [m]			
		<i>Kirchhoff</i>	<i>Reyleigh</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>
t_1	4.000×10^{-6}	1.284×10^{-10}	5.259×10^{-11}	1.284×10^{-10}	5.437×10^{-11}
t_2	6.500×10^{-6}	-6.864×10^{-10}	-6.132×10^{-10}	-6.859×10^{-10}	-6.133×10^{-10}
t_3	1.680×10^{-5}	6.966×10^{-9}	6.825×10^{-9}	6.939×10^{-9}	6.830×10^{-9}

From the above it follows that the profile of the transverse displacement is the same at both points of observation in both models. Both points are equally distant from the shock center, the point K_1 lies 0.02 m on the x axis, point K_2 at the same distance on the y axis (see Fig. 3). In transverse displacement at Rayleigh's model does not have the effect of cross-sectional rotation, this effect is only apparent in longitudinal waves. (see below).

On Fig. 6 and Fig. 7 are shows displacements u, v (in x and y axis) at points K_1, K_2 for Kirchhoff and Rayleigh models. The corresponding numeric values are shown in Tab. 3. At point K_1 is the displacement in the y axis direction of zero (displacement v). At point K_2 is the displacement in the x axis direction of zero (displacement u).

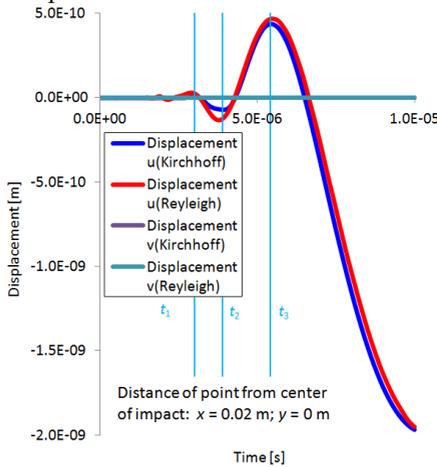


Fig. 6. Wave displacement after impact u, v in point K_1 for Kirchhoff's and Reyleigh's model

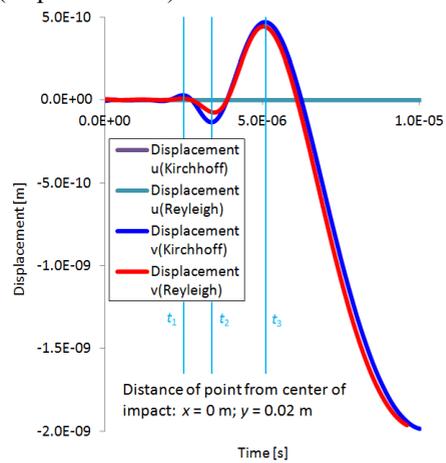


Fig. 7. Wave displacement after impact u, v in point K_2 for Kirchhoff's and Reyleigh's model

Table 3. Wave displacement after impact (u, v) in point K_1 and K_2

Time [s]	Point	K_1		K_2	
		Displacement u [m]		Displacement v [m]	
		<i>Kirchhoff</i>	<i>Reyleigh</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>
t_1	3.000×10^{-6}	1.254×10^{-11}	2.383×10^{-11}	3.362×10^{-11}	2.391×10^{-12}
t_2	3.900×10^{-6}	-7.081×10^{-11}	-1.246×10^{-10}	-1.294×10^{-10}	9.947×10^{-12}
t_3	5.400×10^{-6}	4.373×10^{-10}	4.647×10^{-10}	4.716×10^{-10}	2.118×10^{-10}

The distance of the points under consideration is from the center of impact in both directions the same, so the wave will arrive at both points at the same time. Differences during displacement between the models are due to the fact that Rayleigh's computational model involves the effect of cross-section rotation, whereas the Kirchhoff model does not.

On Fig. 8 and Fig. 9 are show velocities w (in z axis) at points K_1, K_2 for Kirchhoff and Rayleigh models. The corresponding numeric values are shown in Tab. 4.

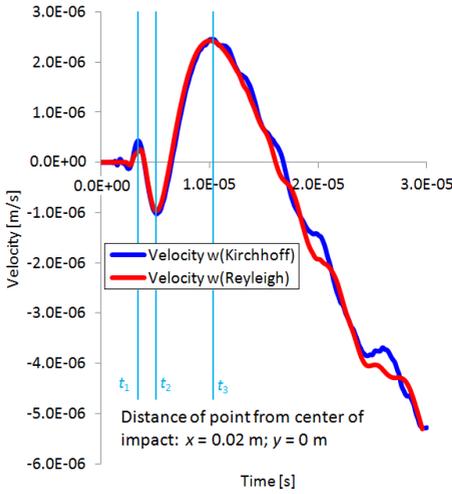


Fig. 8. Wave velocity after impact \dot{w} in point K_1 for Kirchhoff's and Reyleigh's model

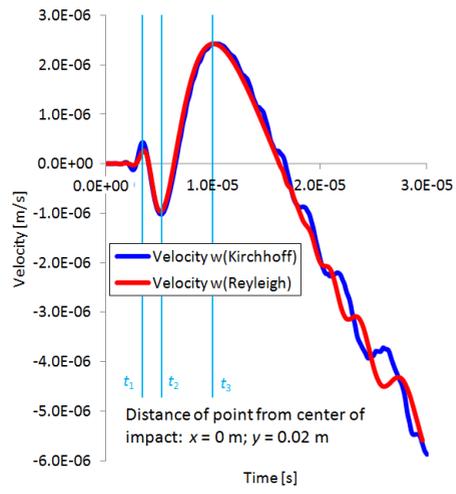


Fig. 9. Wave velocity after impact \dot{w} in point K_2 for Kirchhoff's and Reyleigh's model

Table 4. Wave velocity after impact (\dot{w}) in point K_1 and K_2

Point	K_1		K_2		
	Velocity \dot{w} [$m \cdot s^{-1}$]				
Time [s]	<i>Kirchhoff</i>	<i>Reyleigh</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>	
t_1	3.400×10^{-6}	4.292×10^{-7}	2.081×10^{-7}	4.316×10^{-7}	2.411×10^{-7}
t_2	5.100×10^{-6}	-1.027×10^{-6}	-9.666×10^{-7}	-1.022×10^{-6}	-9.569×10^{-7}
t_3	1.030×10^{-5}	2.452×10^{-6}	2.409×10^{-6}	2.425×10^{-6}	2.410×10^{-6}

At longer times is noticeable smoother velocity changes in the Rayleigh model. It's probably due to the fact that the equation is enhanced by the influence of the rotation section. Equations for speed are given by the derivation of the feed. At Kirchhoff, the effect of not turning the cross-section will be reflected after derivation of greater deviation in velocity ($\dot{w}(t)$). The longer time during the velocity on both models, show the wave reflections from the bottom edge of the plate.

On Fig. 10 and Fig. 11 are shows velocities \dot{u} , \dot{v} (in x and y axis) at points K_1 , K_2 for Kirchhoff and Rayleigh models. The corresponding numeric values are shown in Tab. 5. At point K_1 is the velocity in the y axis direction of zero (velocity \dot{v}). At point K_2 is the velocity in the x axis direction of zero (velocity \dot{u}).

Table 5. Wave velocity after impact (\dot{u} , \dot{v}) in point K_1 and K_2

Point	K_1		K_2		
	Velocity \dot{u} [$m \cdot s^{-1}$]		Velocity \dot{v} [$m \cdot s^{-1}$]		
Time [s]	<i>Kirchhoff</i>	<i>Kirchhoff</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>	
t_1	2.300×10^{-6}	8.711×10^{-5}	8.711×10^{-5}	8.716×10^{-5}	1.602×10^{-5}
t_2	3.000×10^{-6}	-2.895×10^{-4}	-2.895×10^{-4}	-2.895×10^{-4}	-1.432×10^{-4}
t_3	4.100×10^{-6}	5.579×10^{-4}	5.579×10^{-4}	5.579×10^{-4}	5.387×10^{-4}

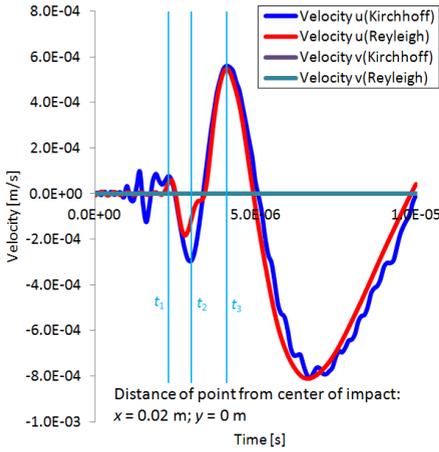


Fig. 10. Wave velocity after impact u , v in point K_1 for Kirchhoff's and Reyleigh's model

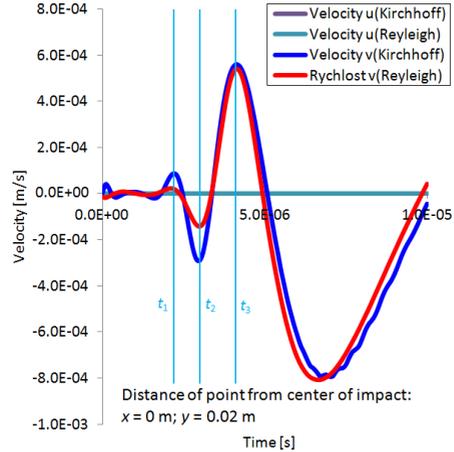


Fig. 11. Wave velocity after impact u , v in point K_2 for Kirchhoff's and Reyleigh's model

It can be seen from the course of velocity that at longitudinal velocities the influence of the cross-sectional rotation of the Rayleigh model is more pronounced. Essentially, this means that the effect of rotation is at the cross-section of the longitudinal wave undulations substantially greater than during the transverse velocity. It can be said that Rayleigh's model is closer to reality than the basic Kirchhoff model.

On Fig. 12 and Fig. 15 are shows the normal and tangential stresses σ_x , σ_y (in x and y axis) at points K_1 , K_2 for the Kirchhoff and Rayleigh models. We assume that the waves propagate from the center of the shock to the circle, the tangent stress at point K_1 is σ_y , at the point K_2 is σ_x and the normal stress at the point K_1 is σ_x , at the point K_2 is σ_y . The corresponding numerical values of the normal stresses are in Tab. 6. and tensile stresses are in Tab. 7.

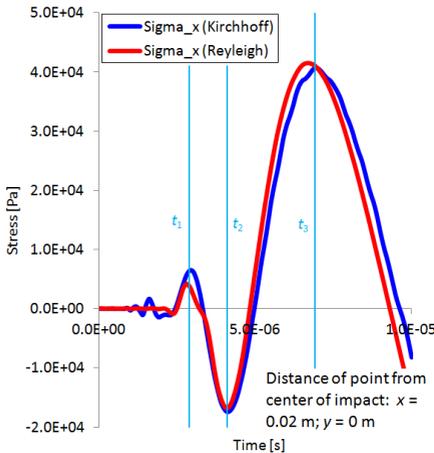


Fig. 12. Normal stresses σ_x in point K_1 for Kirchhoff's and Reyleigh's model

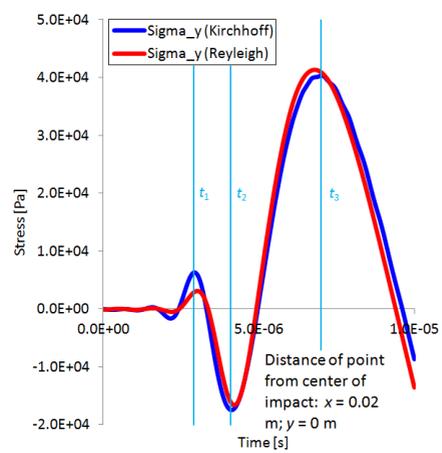


Fig. 13. Normal stresses σ_y in point K_2 for Kirchhoff's and Reyleigh's model

Table 6. Normal stresses (σ_x, σ_y) in point K₁ and K₂

Point	K ₁		K ₂		
	Normal stress σ_x [Pa]		Normal stress σ_y [Pa]		
	<i>Kirchhoff</i>	<i>Reyleigh</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>	
t_1	2.900×10^{-6}	6.307×10^3	3.650×10^3	6.307×10^3	2.815×10^3
t_2	4.100×10^{-6}	-1.745×10^4	-1.680×10^4	-1.745×10^4	-1.619×10^4
t_3	6.900×10^{-6}	4.037×10^4	4.101×10^4	4.037×10^4	4.045×10^4

For waveforms normal stress is evident that the magnitude of the stress at the Kirchhoff model is identical on both points. For Rayleigh model are noticeable differences, which range from 2% at time t_3 , up to 13% at time t_1 . These differences are minimal and can be attributed to inaccuracies. As for the magnitude of the stress, the differences between Rayleigh and Kirchhoff are quite significant in a short time, which can be attributed to the influence of the cross-sectional rotation and thus to the more accurate Rayleigh calculation.

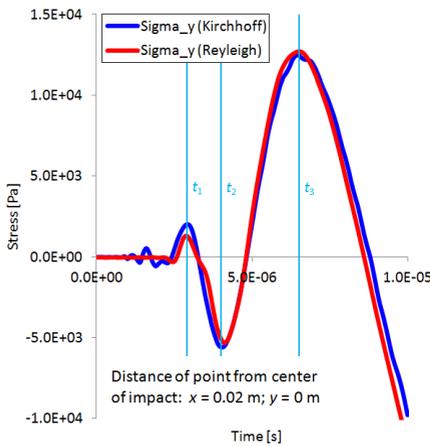


Fig. 14. Tangent stresses σ_y in point K₁ for Kirchhoff's and Reyleigh's model

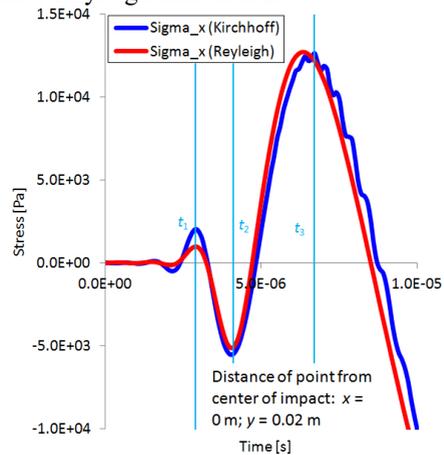


Fig. 15. Tangent stresses σ_x in point K₂ for Kirchhoff's and Reyleigh's model

Table 7. Tangent stresses (σ_x, σ_y) in point K₂ and K₁

Point	K ₂		K ₁		
	Tangent stress σ_x [Pa]		Tangent stress σ_y [Pa]		
	<i>Kirchhoff</i>	<i>Reyleigh</i>	<i>Kirchhoff</i>	<i>Reyleigh</i>	
t_1	2.900×10^{-6}	2.036×10^3	1.654×10^3	2.030×10^3	1.338×10^3
t_2	4.100×10^{-6}	-5.490×10^3	-5.107×10^3	-5.578×10^3	-5.060×10^3
t_3	6.700×10^{-6}	1.265×10^4	1.146×10^4	1.265×10^4	1.372×10^4

In the tangent stresses waveforms, the situation is similar. Rayleigh's model, which, unlike the Kirchhoff model, is considering cross-sectioning is more accurate. The differences between Kirchhoff and Rayleigh models are very small and can be attributed to inaccuracies when rounding off in the calculation program.

4 Conclusion

The article compares the course of displacements, velocities and stresses in impact loading of thin isotropic plates. From the above values, it can be deduced that the shock wave propagates evenly along the circles from the shock center, so it is possible to state that it is an isotropic material.

The analytical solution was performed in the MATLAB programming environment for Kirchhoff and Rayleigh's thin plate model. The impact force on the plate was realized by an impact that hit the geometric center of the plate. The impact was 5 mm in diameter, the impact force was 1 N. Subsequently, we compared the displacements, velocities and tensions of both models at points K_1 and K_2 . The points were distant from the center of impact (geometric center of the plate) at a distance of 0.02 m in the direction x (K_1) and direction y (K_2).

The solution was an isotropic thin plate measuring $0.2 \times 0.1 \times 0.002$ m. The plate was about perimeter fixed. The plate material was Al 99.9, the specified material values of $E_x = E_y = E_z = E$, $\mu_x = \mu_y = \mu_z = \mu$ are given in the Tab. 1. The time courses of the individual variables, i.e., the displacements, velocities and stresses at K_1 and K_2 , are almost staggered for both geometric models (Kirchhoff, Rayleigh). Differences between models do not exceed 15%. The tables show the values of the variables at times t_1 , t_2 and t_3 . At time t_1 , the individual values for the two models differ, as the Kirchhoff model only considers vertical displacements and their corresponding inertial effects, therefore has a different initial course than Rayleigh's model, which is expanded by the influence of the cross-sectional rotation.

It can be stated that the values given in the Rayleigh model are more accurate as it considers more parameters. It can be concluded from this that, when further expansion (i.e. the calculation according to Flügge and in particular according to Timoshenko-Mindlina), the values are even more accurate.

The extension of this theory on the composite plate (orthotropic) is important for the new design in many fields, it is assumed that the increase utilization of composite plate and shell elements.

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