MATEC Web of Conferences **252**, 02006 (2019) https://doi.org/10.1051/matecconf/201925202006
*CMES'18*

# The influence of the Lorenz system fractionality on its recurrensivity

*Magdalena* Gregorczyk[1,*], and *Andrzej* Rysak[1]

[1]Lublin University of Technology, Faculty of Mechanical Engineering, Nadbystrzycka 36, 20-618 Lublin, Poland

**Abstract.** In this work, we investigate the recurrensivity of the Lorenz system with fractional order of derivatives occurring in its all three differential equations. Several solutions of the system for varying fractional orders of individual derivatives were calculated, which was followed by an analysis of changes in the selected recurrence quantifiers occurring due to modifications of the fractional orders {q1, q2, q3}. The results of the recurrence analysis were referred to the time series plots, phase diagrams and FFT spectra. The obtained results were finally used to examine the influence of fractional derivatives on the chaos - periodicity transition of the system dynamics.

## 1 Introduction

The first mention of the fractional derivative appeared at the turn of the 17th and 18th centuries, whereas initial attempts to define it were made by Laplace in 1812. To this day, the interest in this subject is constantly growing. Fractional derivative is used in many fields of knowledge, including non-linear dynamic systems, through which we can express many natural phenomena.

The Recurrence plot (RP) method, for a given dynamic system, allows visualising the return of the system to its previous states on the recursive diagram. It was introduced by Eckerman in 1987 [1]. Techniques and methods of investigating recurrent diagrams were intensively developed at the turn of the 20th and 21st centuries by Zbilut and Webber [2], Marwan [3, 4] and others [5] up to the present day. Very often the RP method is used to study non-linear systems.

In this work, we study the Lorenz system, one of the most popular non-linear dynamic systems. It was introduced by Lorenz in 1963 [6] as a system of three non-linear differential equations. We assume that the order of each system's differential equations can be fractional.

In [7], the dynamics of this system was examined, following the change of q1 - the order of the derivative in the first equation, assuming a constant value of the remaining orders (q2 = q3 = 1). A transition from chaos to periodicity was detected occurring for q1 = 1.2.

In this article, we examine the recursiveness of the Lorenz system with a fractional order of the derivative occurring in all three differential equations. The research is divided into two research tasks, done for two values of q1, for which the Lorenz system is in various dynamic states - chaos and periodicity.

In both cases, we find the system solutions for the changing values of the orders of individual derivatives {q2, q3} and define their basic types. In the narrow range of q2 values, we determine changes in the dynamics of the system based on phase diagrams and changes in the values of recursive quantifiers (RQA).

The layout of the work is as follows. In Section 2, we present the Lorenz system with a fractional derivative. The RP method is described in Section 3. The results of tests and analyses are presented in Section 4, and Section 5 contains a short summary and final conclusions.

## 2 Lorenz system

The Lorenz system is a system of three non-linear differential equations modelling the phenomenon of thermal convection in the atmosphere.

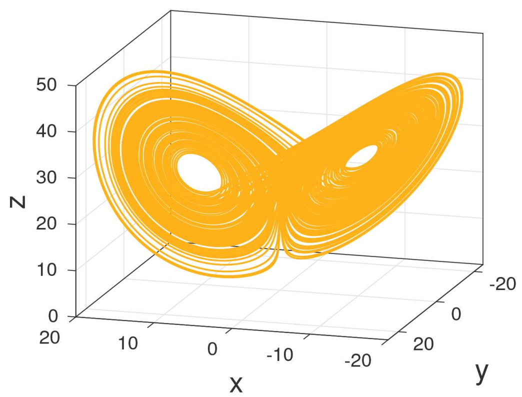


**Fig. 1.** A solution in the Lorenz attractor, where $\sigma = 10$, $r = 28$, and $b = 8/3$.

In its basic form, it is described by the following equations

* Corresponding author: m.gregorczyk@pollub.pl

$$\dot{x} = -\sigma(x - y)$$
$$\dot{y} = -xz + rx - y \qquad (1)$$
$$\dot{z} = xy - b$$

where we take σ = 10, r = 28, and b = 8/3 according to [8]. A typical solution for this system with given parameters is shown in Figure 1.

The fractional derivative operator takes the following form:

$$D_t^q x = \frac{d^q}{dt^q} \qquad (2)$$

where $L$ denotes the memory length applied in the short memory principle [9, 10]. The Lorenz system modified by using of fractional derivatives in all three equations has the following form

$$_L D_t^{q1} x = -\sigma(x - y)$$
$$_L D_t^{q2} y = -xz + rx - y \qquad (3)$$
$$_L D_t^{q3} z = xy - b.$$

## 3 Recurrence analysis

The recurrence plots method for testing the nonlinear systems, require moving a considered time series to an m-dimensional embedding space. However, in our case, for the considered system the complete set of differential equations is known and the recursive analysis is carried out in a space with a dimension determined by the number of equations. In this paper, it is assumed that changes of the system's dimension due to the fractional derivatives are small and all interesting phenomena are preserved when a trajectory of the system is projected onto a space with an integer number of dimensions. Next, for the vector time series, being the solution of a nonlinear system, a distance matrix R is defined and whose elements determine neighbours for each vector of the series. The elements of the matrix R are defined using the Heaviside function

$$R_{i,j}^{\varepsilon} = \Theta\left(\varepsilon \|x_i - x_j\|\right) \qquad (4)$$

where the Euclidean $L_2$-norm was used, $N$ - the number of considered states $x_i$□ℝ $^m$ for $i,j=1,2,\ldots,N$. Moreover, $R_{i,j}=1$ for every $i,j=1,2,\ldots,N,$ which is called the *line of identity (LOI)*. The threshold parameter □ determines a proximity condition.

In this research, some recurrence measures of RQA were calculated. The best known of them is *recurrence rate* (RR)

$$RR(\varepsilon, N) = \frac{1}{N^2 - N} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} R_{i,j}^{\varepsilon} \qquad (5)$$

which counts the black points of the RP chart excluding the *LOI*. It is a measure that analyses the density of recurrence points.

The next one – *ratio,* given by the formula

$$RATIO = (N^2 - N) \frac{\sum_{l=2}^{N-1} l H_D(l)}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{N} R_{i,j}^{\varepsilon}\right)^2} \qquad (6)$$

is calculated using the length of all diagonal lines. This variable is very useful in detecting dynamic transitions in systems.

*Laminarity* (LAM) is defined as follows

$$LAM = \frac{\sum_{v=2}^{N} v H_V(v)}{\sum_{i=j=1}^{N} R_{i,j}^{\varepsilon}} \qquad (7)$$

where the length of vertical lines is used. LAM indicates the frequency of occurrence of laminar states in the system.

## 4 Results

### 4.1 Modelling of the Lorenz system

The numerical tests were performed for two values of the parameter q1, q1 = 1.15 and q1 = 1.25, which in the further part of the work are described as a case A and B, respectively. Both values of q1 were adopted on the basis of work [7], because for the starting values q2 = 1, q3 = 1, they correspond to the chaotic (A) and periodic (B) states near the transition of chaos – periodicity which occurs for q1 = 1.2.

On the basis of time and phase diagrams, the obtained solutions were classified into the basic types as follows: P – periodic, CH – chaotic, NS – unstable and DS – damped. The results for both q1 values are presented in diagrams, respectively in Figs 2 (A) and 3 (B).

The following values of the q2 and q3 orders were adopted in the calculations: q2 = 0.90, 0.95, 1.00, 1.05, 1.10, 1.15, 1.20, 1.25, and q3 = 0.95, 1.00, 1.05, 1.10.

| $q_3$ \ $q_2$ | 0.90 | 0.95 | 1.00 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 |
|---|---|---|---|---|---|---|---|---|
| 0.95 | DS | DS | CH | CH | P | CH | P | P |
| 1.00 | CH | CH | CH | CH | P | CH | P | P |
| 1.05 | CH | CH | CH | CH | P | CH | CH | NS |
| 1.10 | CH | CH | CH | CH | P | CH | NS | NS |

**Fig. 2.** Basic types of solutions of the fractional Lorenz system, obtained for q1 = 1.15 and various values of q2 and q3 (case A).

Based on the analysis of the layout of the basic types of solutions shown in the diagrams, for both cases, some interesting transitions of the dynamics of the system were chosen, which were examined in more detail further in the work.

For the A-case, we examine the passage from chaos to chaos through the periodic column (q2 = 1.1) by choosing a constant value of q3 = 1 and q2 = varying from 1.02 to 1.15 every 0.01.

For the B-case, we focus on the chaos - periodicity transition occurring for q3 = 1.0 and a range of q2 from 0.95 to 1.05.

| $q_3$ \ $q_2$ | 0.90 | 0.95 | 1.00 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 |
|---|---|---|---|---|---|---|---|---|
| 0.95 | DS | DS | CH | P | P | P | P | NS |
| 1.00 | CH | CH | P | P | P | P | NS | NS |
| 1.05 | CH | P | P | P | P | NS | NS | NS |
| 1.10 | P | P | P | P | NS | NS | NS | NS |

**Fig. 3**. Basic types of solutions of the fractional Lorenz system, obtained for for q1 = 1.25 and various values of q2 and q3 (case B).

## 4.2 Phase diagrams

Phase diagrams are a good tool to show the general features of the dynamics of the system.

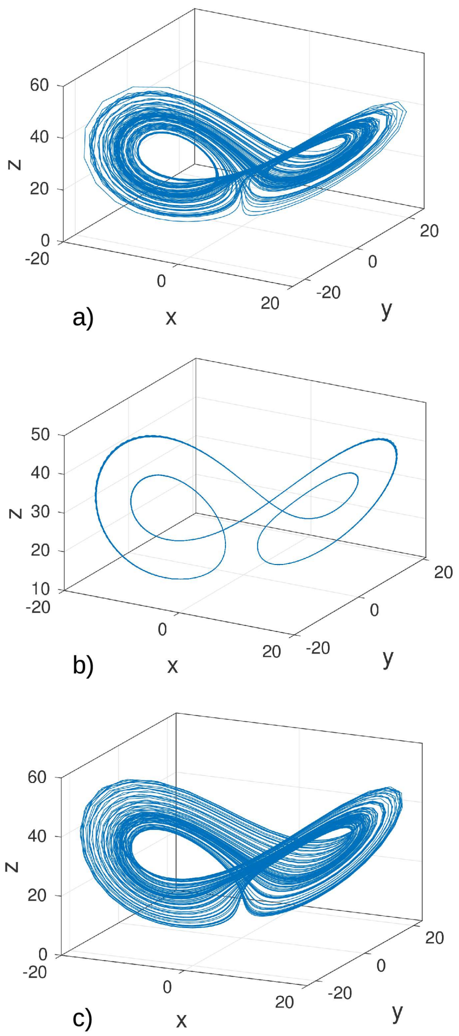


**Fig. 4**. Case A, q1 = 1.15, q3 = 1: a) q2 = 1.05, b) q2 = 1.10, c) q2 = 1.15.

To illustrate the evolution of the system following the change in the parameter q2 in ranges A and B, in both cases phase diagrams for three selected values of this parameter were plotted. The results are presented in Figs. 4 and 5, respectively.

According to the case diagrams A and B, Fig. 4 presents solutions: chaotic (a), periodic (b) and again chaotic (c), while Fig. 5 shows the chaotic solution (a) and two periodic solutions for higher values of q2 (b, c).

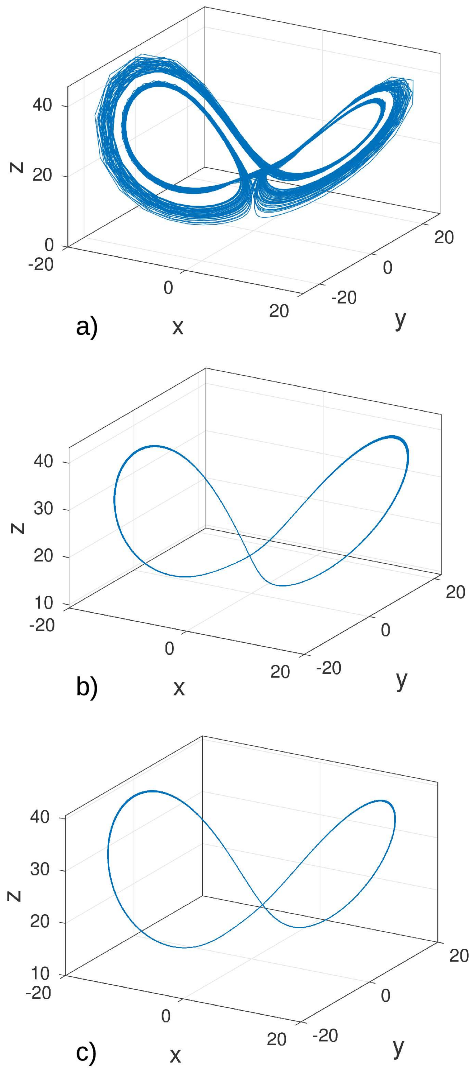


**Fig. 5.** Case B, q1 = 1.25, q3 = 1: a) q2 = 0.95, b) q2 = 1.00, c) q2 = 1.05.

For case A, it can be seen that on both sides of the periodic area the shape of the Lorenz attractor is similar, for that reason, it seems that after passing through the periodicity the dynamics of the system returns to the previous mode.

Loops representing two periodic solutions in case B only slightly differ in shape, therefore, in this range, the dynamics of the system does not change much when the parameter q2 is modified.

## 4.3 Recursive measures

For both numerical tests' series, A and B, the recurrence analyses were performed. As a result, the measures RR, LAM, and RATIO were calculated for respective values of q2 parameter. The results are shown in Fig. 6 for the A-case and in Fig. 7 for the B-case.

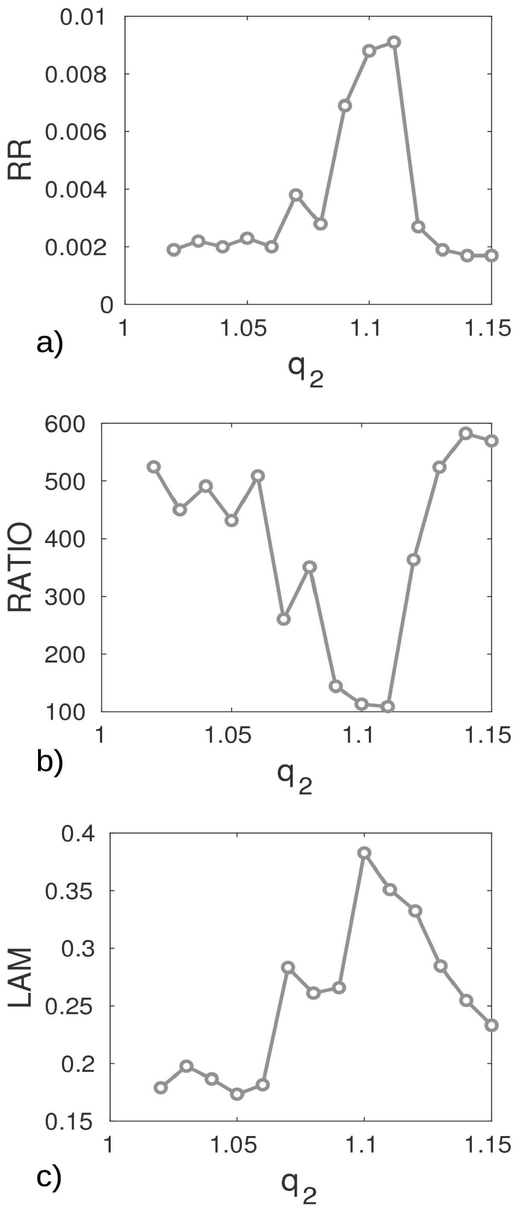


**Fig. 6.** RR, LAM, RATIO for q2 = 1.15.

The obtained results of the recurrence measures reflect changes in the dynamics of the Lorenz system resulting from changes in the fractionality of the system. When comparing the results of both cases, one can see the common features of the variability of the recurrence measures at the transition from chaos to periodicity.

RR increases its value when passing to periodicity. RATIO - a measure derived from RR - decreases its value in this pass. The measure LAM, for the B-case in the region of this transition, increases and decreases its value giving in result a wide peak.

The variability of this measure in case A is more complex the analysed area there is a double transition in the dynamics of the system of type: chaos - periodicity - chaos.

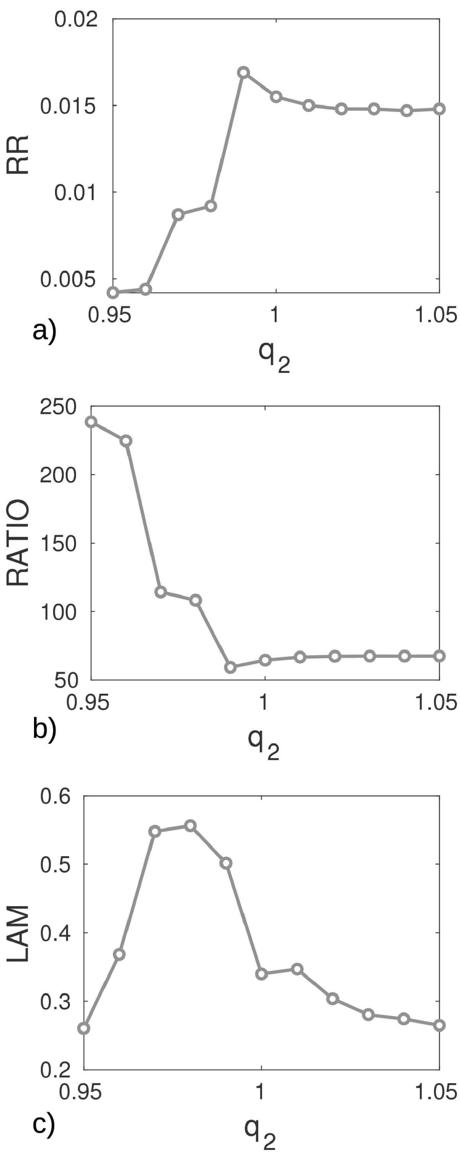


**Fig. 7.** RR, LAM, RATIO for q2 = 1.25.

## 5 Conclusions

This paper presents the results of recurrence studies of the Lorenz fractional system. System solutions for different fractional values of the orders of its differential equations (q1, q2, q3) were analysed, while a particularly close attention was given to the examination of the evolution of the system dynamics occurring at chaos-period transitions induced by the change of the q2 parameter.

The results of recurrence measures allow determining the transition location (on the scale of the q2 parameter) and its width. These may be employed to obtain more interesting details about transition if only the discretisation of q2 is high enough. Comparing the results of both cases leads to the conclusion that the recurrence measures show similar variability when the character of dynamics changes is of the same type.

## References

1. J.P. Eckmann, S.O. Kamphorst, D. Ruelle, Europhys. Lett. **5**, 973 (1987)
2. C.L. Webber, Jr., J.P. Zbilut, J. Appl. Physiology **76**, 965 (1994)
3. N. Marwan, Ph.D. thesis, Universitaet Potsdam, Potsdam (2003)
4. N. Marwan, M.C. Romario, M. Thiel, J. Kurths, Phys. Rep. **438**, 237 (2007)
5. M. Casdagli, Phys. D **108**, 12 (1997)
6. E.N. Lorenz, J. atmos. Sci. pp. 120–141 (1963)
7. A. Rysak, M. Gregorczyk, VETOMAC 2018, MATEC web conf. (to be published)
8. J.C. Sprott, *Chaos and Time-series Analysis* (Oxford University Press, Oxford, UK & New York, USA, 2003)
9. I. Petráš, *Fractional Order Nonlinear Systems* (Springer, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2011)
10. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999)