An inverse geometric problem in steady state heat conduction - the solution and stability analysis

Leszek Hożejowski *

Faculty of Management and Computer Modelling, Kielce University of Technology, al.Tysiąclecia P. P. 7, 25-314 Kielce, Poland

Abstract. The paper addresses a numerical method for boundary identification in a problem governed by Laplace's equation. The proposed numerical procedure for discrete reconstruction of the unknown boundary from the given temperature data is based on the Trefftz method. In contrast to the procedures described in the reference papers, the present approach requires significantly less and easier computation. The paper undertakes analysis of the resistance of the solution to small perturbations of the prescribed temperature condition at the unknown part of the boundary. We define and then estimate a sensitivity factor which allows quantitative assessment of the relationship between temperature measurement errors and boundary identification errors, even if the exact solution is not known. The included numerical examples demonstrate the effectiveness of the proposed method for boundary reconstruction and present the analysis of numerical stability using a sensitivity factor.

1 Introduction

There are different kinds of inverse geometry problems like shape and design optimization or identification of defects in materials. In this paper, however, we address a problem which consists in identification of the unknown part of a domain boundary. More precisely, given a problem described by a differential equation, a numerical reconstruction of the unknown boundary has to be achieved from the available boundary condition. There have been different propositions of the efficient solution algorithms to this problem. One method, however, seems very suitable for approaching the problem, namely the Trefftz method and the method of fundamental solutions (MFS) which are mathematically equivalent though differ in formulation [1]. In recent years some researchers proposed their solution procedures based on the Trefftz method (or collocation Trefftz method) and concerning the problems governed by the Laplace equation [2-6], biharmonic equation [7] or the modified Helmholtz equation [8]. Also, there appeared some successful applications of MSF to boundary identification [9-11].

The present study employs the Trefftz method for discrete reconstruction of the unknown boundary. Unlike in [2-4, 7, 8], we recover the boundary points one after the other. In terms of computation this means solving one equation at a time (in contrast to solving a large system of equations). A new approach to stability analysis proposed in the paper allows to predict boundary identification errors, calculated on the basis of the known measurement errors.

2 Two dimensional boundary identification problem

In this section a certain inverse geometry problem will be set in terms of a mathematical model. Besides, a solution algorithm and a new approach to stability analysis will be proposed.

2.1 Problem formulation

Consider a problem in a bounded two-dimensional region Ω . Assume that its boundary $\partial\Omega$ is composed of two disjoint curves γ_0 and γ_1 :

$$\partial\Omega = \gamma_0 \cup \gamma_1 \quad (1)$$

such that γ_0 is known and γ_1 unknown. The governing equation has a form

$$\Delta T = 0 \quad \text{in } \Omega \quad (2)$$

where Δ denotes the Laplace operator so the problem can be interpreted as describing steady state heat conduction. We solve (2) subject to

$$T = T_0 \quad \text{on } \gamma_0 \quad (3)$$

$$\frac{\partial T}{\partial \mathbf{n}} = q_0 \quad \text{on } \gamma_0 \quad (4)$$

$$T = T_1 \quad \text{on } \gamma_1 \quad (5)$$

* Corresponding author: hozej@tu.kielce.pl

where $\partial/\partial \mathbf{n}$ denotes differentiation along the unit outward normal $\mathbf{n}$. Conditions (3) and (4) mean that both the temperature and its gradient along γ_0 are known while equation (5) indicates that the unknown part of the boundary, γ_1 , has to be inferred from the given temperature data. Assuming that T_1 is known from measurement as a set of discrete data (which better corresponds to the real situation), condition (5) should be rewritten as

$$T = T_1^m \text{ at } P_m \in \gamma_1 \quad (m = 1, 2, \dots, M) \quad (6)$$

2.2 Solution method

The problem described by (2) – (4) and (6) will be solved using the Trefftz method which consists in approximating the solution by a linear combination of the basis functions (named T-complete functions) which satisfy the differential equation (2). Then we have

$$T \approx \sum_{n=1}^N c_n V_n \quad (7)$$

where V_n denote the T-complete functions and c_n – the scalar coefficients which we specify to minimize the functional

$$\Phi = \left\| \sum_{n=1}^N c_n V_n \Big|_{\gamma_0} - T_0 \right\|^2 + \left\| \sum_{n=1}^N c_n \frac{\partial V_n}{\partial \mathbf{n}} \Big|_{\gamma_0} - q_0 \right\|^2 \quad (8)$$

In the above, $\|\cdot\|$ denotes L^2 norm. Note that finding c_n from the minimum of (8) requires solving a system of linear algebraic equations.

Discrete reconstruction of γ_1 should be interpreted as: given the x -coordinate of the point P_m , find its y -coordinate. However, depending on a particular problem, we would sometimes prefer a formulation in polar coordinates.

Since one has to calculate y_m from (6), where y_m denotes the y -coordinate of P_m , we propose to solve M nonlinear equations (one after the other), each with one unknown:

$$T(x_m, y_m) = T_1^m, \quad m = 1, 2, \dots, M \quad (9)$$

This is in contrast to other solution algorithms based on the Trefftz method [2-4, 7,8] which include both the coefficients c_n of T-complete functions and the sought coordinates y_m of P_m 's in a system of simultaneous nonlinear equations whose solution might be problematic, especially if M is very large.

Numerical methods for solving nonlinear equations usually require an initial guess. In our problem, the y -coordinate of the point joining γ_0 and γ_1 could make the initial guess for the first equation (corresponding to $m=1$) while in subsequent steps the previously

obtained y_m seems a reasonable initial point for the $(m+1)$ -th equation.

2.3 Stability analysis

Since our task is to recover γ_1 from the temperature data T_1 , the problem of solution sensitivity comes down to the question: If T_1 is perturbed with a small error into $T_1 + \Delta T_1$, how will the position of the recovered boundary points P_m change in comparison with that obtained from the original data?

In the reference papers addressing a boundary identification problem, both computation and further analysis is based on synthetic data coming from numerical simulations. Then, stability analysis consists in disturbing the inputs with simulated random errors and observing if the boundary identification errors have more or less the same magnitude as input errors. Such an approach seems insufficient. In fact, the crucial problem lies in comparison of the two ratios: $\Delta T_1/T_1$ and $\Delta y/y$ which represent relative errors of temperature measurement and boundary identification, respectively. We can write

$$\frac{\frac{\Delta T}{T}}{\frac{\Delta y}{y}} = \frac{y}{T} \frac{\Delta T}{\Delta y} \approx \frac{y}{T} \frac{\partial T}{\partial y} \quad (10)$$

where the latter comes from replacing the differential quotient by the partial derivative. For simplicity, we will write “=” instead of “ $\approx$ ”. Let $\varepsilon_{T,y}$ denote the right-hand side of (10):

$$\varepsilon_{T,y} = \frac{y}{T} \frac{\partial T}{\partial y} \quad (11)$$

$\varepsilon_{T,y}$ is known as a partial elasticity of the function T with respect to y and is defined locally (at a point). Equation (10) referred to γ_1 gives

$$\left| \frac{\Delta y}{y} \right| = \left| \varepsilon_{T,y} \right|^{-1} \cdot \left| \frac{\Delta T_1}{T_1} \right| \text{ on } \gamma_1 \quad (12)$$

Formula (12) is crucial to further analysis. It says that a 1% perturbation of the temperature will result in $\left| \varepsilon_{T,y} \right|^{-1}$ times larger percent error of the boundary identification. In view of this interpretation, $\left| \varepsilon_{T,y} \right|^{-1}$ could be called a sensitivity factor. Evaluating $\varepsilon_{T,y}$ is not problematic because the Trefftz method allows for obtaining the derivative $\partial T/\partial y$ by analytical differentiation of (7). On the other hand, $\Delta T_1/T_1$ can be treated as given because it denotes either a simulated error or a measurement error, usually known by the experimenter. As a consequence, we are able to

effectively estimate $|\Delta y/y|$. Besides, one should note that the evaluation of errors using a sensitive factor does not need the exact analytical solution of Laplace's equation which serves only for simulating temperature measurements.

3 Numerical examples

For two-dimensional Laplace's equation the T-complete functions in (x,y) -coordinates are

$$\{1, \operatorname{Re}(x+iy)^n, \operatorname{Im}(x+iy)^n : n=1,2,\dots\} \quad (13)$$

where Re and Im denote real and imaginary part of a complex number, respectively, and i stands for the imaginary unit. We recognize the functions (13) as harmonic polynomials.

In further considerations, the choice of an analytical solution to Laplace's equation will allow for numerical simulation of the required temperature T_0 and T_1 and the temperature gradient q_0 .

3.1 Example 1

Let the problem domain be

$$\Omega : 0 \leq x \leq 1, \quad f(x) \leq y \leq g(x) \quad (14)$$

Its upper boundary, $g(x)$, will be treated as unknown and has to be determined. In this example we choose $f(x) = (x-1)(x+1)^{-1}$, $g(x) = 2(2+x\sin 10x)^{-1}$ and

$T(x,y) = e^x \cos y$. Besides, we take $N = 17$ T-complete functions in expansion (7) and perform calculations with the simulated measurements at $M = 40$ points.

Using the unperturbed temperature data, the algorithm recovers the upper boundary with very high accuracy (errors less than 0.0016%). Figure 1 presents the results of boundary identification from the noisy data, with randomly generated errors ranging from -5% to 5% and normally distributed. For the purpose of stability analysis, we evaluated the sensitivity factor which turned out to vary (depending on x) from 0.2 to 1.6.

Fig. 1. Reconstructed part of the unknown boundary (γ_1) in Example 1.

Visual inspection of the graph in Fig. 1 shows barely noticeable influence of the introduced noise on the boundary reconstruction. This should not be a surprise since the sensitivity factor in this case is relatively small.

Figure 2 presents the true boundary identification errors and compares them to those estimated from equation (12).

Fig. 2. Boundary reconstruction errors in Example 1 (for noisy data).

The graph shows very good agreement between the true and predicted errors. The result also proves (though not explicitly) that we succeeded in evaluating the sensitivity factor with high accuracy.

3.2 Example 2

This example differs from the previous one only in a choice of an upper boundary to be determined. Here we take $g(x) = 1(2+x\sin 10x)^{-1}$, obtained by dividing the previous $g(x)$ by 2. All the other data, including the simulated errors, are remained the same.

The computation using errorless data yields the reconstruction having errors at most 0.0003%. In comparison to the previous example, the use of noisy data significantly perturbed the recovered boundary points locations. Such result was expected, since the magnitude of the sensitivity factor was considerably greater than in Example 1 (for details, see Tab.1).

Fig. 3. Reconstructed part of the unknown boundary (γ_1) in Example 2.

Similarly to the previous case, we present the percentage boundary reconstruction errors, both true and predicted (see: Fig. 4).

Fig. 4. Boundary reconstruction errors in Example 2 (for noisy data).

The results shown in Fig. 4 confirm what we could predict by analyzing the estimated sensitivity factor ($2.1 \div 7.3$), namely that the boundary reconstruction errors in this example must be a few times larger than in the previous example.

A tabular summary of the computation results concerning the stability analysis of both numerical examples is contained in Tab.1.

Table 1. Boundary reconstruction analysis - accuracy and stability.

	Sensitivity factor	Maximum boundary reconstruction error	
		exact input data	noisy input data
Example 1	$0.2 \div 1.6$	0.0016%	4.2%
Example 2	$2.1 \div 7.3$	0.0003%	18.7%

Generally, when a sensitivity factor grows, its estimation from noisy data becomes less accurate. This explains why the predicted errors shown in Fig. 4 are considerably underestimated or overestimated at some points. Nevertheless, in this and other similar cases, formula (12) still provides some useful information which can be indicative of the true accuracy of boundary reconstruction. Finally, we note that insufficient accuracy of the solution presented in Example 2 is not implied by imperfection of the used computational algorithm. A high sensitivity factor indicates problems where accurate boundary reconstructions from noisy data are very problematic or, in extreme cases, unavailable.

The stability analysis in problems described in polar coordinates goes similar to the analysis shown above. The only difference is that given such a problem, we seek for the radial coordinates of some unknown boundary points so the proper sensitivity coefficient contains the derivative of T with respect to r .

4 Conclusions

The paper presents an application of the Trefftz method for discrete reconstruction of an unknown part of the domain boundary in a heat conduction problem. In terms of computation, the algorithm outputs every recovered point by solving a single nonlinear equation with one unknown. Therefore, with a growing number of the unknown boundary points, the computation does not become more complicated or problematic, unlike in the related algorithms which employ large systems of nonlinear equations. Moreover, the presented method gives accurate boundary reconstructions, as shown in the included examples. The use of what was defined as a sensitivity factor, allows for credible estimation of the boundary identification errors without knowing the true boundary.

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