Oscillation Criteria of Second-Order Mixed Neutral Delay Dynamic Equations on Time Scales

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Abstract. In this artical, we consider a second-order neutral dynamic equation on a time scales. A number of oscillation theorems are shown that supplement and extend some known results in the eassay.

1 Introduction

In this essay, we will consider the oscillation of the second-order mixed neutral delay dynamic equation

$$(r(n)x^{\Delta}(n))^{\Delta}+q(n)y(\delta(n))=0,\ n\in[n_0,\infty)_T, \quad (1)$$

where $x(n):=y(n)+a(n)y(n-\tau_1)+b(n)y(n+\tau_2)$, τ_1 and τ_2 are nonnegative constants, $n_0\in T$ and $[n_0,\infty)_T$ $:=[n_0,\infty)\cap T$ denotes a time scale interval with $\sup T=\infty$. We shall assume the following conditions:

(C1) $a,b,q\in C_{rd}([n_0,\infty)_T,R)$, where

$0\le a(n)\le a_0<\infty$, $0\le b(n)\le b_0<\infty$, and $q(n)>0$.

(C2) $r\in C_{rd}([n_0,\infty)_T,R)$, $r(n)>0$, $\int_{n_0}^{\infty}\frac{1}{r(n)}\Delta n<\infty$.

(C3) $\delta\in C_{rd}([n_0,\infty)_T,T)$, $\delta^{\Delta}(n)>0$, $\delta(n)\le n$, $\lim_{t\to\infty}\delta(n)=\infty$.

Neutral functional difference and differential equations have many applications in many fields, like electric networks. As an important part, difference and differential equations with mixed type neutral term have attracted many researcher's attention, see [1-4].

Han et al. [1] investigated the equations of mixed type

$$\begin{aligned}&[(y(n)+p_1y(n-\tau_1)+p_2y(n+\tau_2))^{\gamma}]''\\&=q_1(n)y^{\gamma}(n-\sigma_1)+q_2(n)y^{\gamma}(n+\sigma_2),\end{aligned}$$

where $\gamma\ge 1$ is the ratio of positive odd numbers. Some oscillation theorems are established and the previous results are generalized.

The theory of time scale, which has received numerous attention, was introduced by Hilger's landmark contribution article [5], in order to unify discrete and continuous calculus theory, and it also extends these classical cases to cases 'in between'. A time scale T is an arbitrary nonempty closed subset of the real numbers field R.

The oscillation of solutions of various dynamic equations on time scales has attracted wide attention in the last several years. For the oscillation research on time scales, readers can refer to the [6-11], and the references cited therein. There are a few results regarding the oscillation of second order neutral dynamic equations with mixed neutral term.

Our objective of this paper is establish some new criteria for the more generalized second order neutral dynamic equation (1), which also allow to relax some restrictive conditions imposed on the equation.

If there are infinitely large generalized zeros on $[T_y,\infty)_T$, then the solution y of equation (1) is oscillatory; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2 Main results

Theorem 1. Assume that *δ(n ± α) = δ(n) ± α(α*>o). If

$$\int_{n_0}^{\infty}Q(s)\Delta s=\infty, \quad (2)$$

and

$$\limsup_{n\to\infty}\int_{n_0}^{n}[\phi(s)Q(s)-\frac{1+a_0+b_0}{4r(s)\phi(s)}]\Delta s=\infty, \quad (3)$$

where

$$Q(n)=\min\{q(n),q(n-\tau_1),q(n+\tau_2)\},\ \phi(n)=\int_n^{\infty}\frac{1}{r(s)}\Delta s.$$

Then equation (1) oscillation.

Proof. Suppose that x is a eventually positive

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solution of (1). Yet the general, we may assume that $n_1 \ge n_0$ exists to make $y(n)$, $y(n-\tau_1)$, $y(n+\tau_2)$ and $y(\delta(n))$ are all positive for $n \in [n_1, \infty)_T$.

Then we have $x(n) > 0$. We get that $r(n)x^\Delta(n) < 0$ from (1), then $x^\Delta(n)$ one sign at an end. Hence, there exist two cases for $n \in [n_1, \infty)_T$, that is,

(i) $x(n) > 0, x^\Delta(n) > 0, (r(n)x^\Delta(n))^\Delta \le 0$;

(ii) $x(n) > 0, x^\Delta(n) < 0, (r(n)x^\Delta(n))^\Delta \le 0$.

Firstly, assume that case (i) hold. Applying (1), we have

$$(r(n)x^\Delta(n)(n))^\Delta + q(n)y(\delta(n)) + a_0(r(n-\tau_1)x^\Delta(n-\tau_1))^\Delta$$
$$+a_0q(n-\tau_1)y(\delta(n-\tau_1)) + b_0(r(n+\tau_2)x^\Delta(n+\tau_2))^\Delta$$
$$+b_0q(n+\tau_2)y(\delta(n+\tau_2)) = 0.$$

From $y(n)$ and $Q(n)$, we obtain

$$(r(n)x^\Delta(n))^\Delta + a_0(r(n-\tau_1)x^\Delta(n-\tau_1))^\Delta$$
$$+b_0(r(n+\tau_2)x^\Delta(n+\tau_2))^\Delta + Q(n)y(\delta(n)) \le 0. \quad (4)$$

Integrating (4) from n_2 to n, and from $x(n) \ge c > 0$ (c is a constant) for $n \in [n_2, \infty)_T$, we obtain

$$c\int_{n_2}^{n} Q(s)\Delta s \le r(n_2)x^\Delta(n_2) + a_0 r(n_2-\tau_1)x^\Delta(n_2-\tau_1)$$
$$+b_0 r(n_2+\tau_2)x^\Delta(n_2+\tau_2),$$

which conflict with (2).

Next, suppose that case (ii) hold. Define a function ω by

$$\omega(n) = \frac{r(n)x^\Delta(n)}{x(n)}, \ n \in [n_2, \infty)_T. \quad (5)$$

Obviously, $\omega(n) < 0$. Since $r(n)x^\Delta(n)$ is decreasing, we have

$$x(l) \le x(n) + r(n)x^\Delta(n)\int_n^l \frac{1}{r(s)}\Delta s, \ l \ge n \ge n_2.$$

When $l \to \infty$, we conclude

$$x(n) + r(n)x^\Delta(n)\phi(n) \ge 0, \ n \in [n_2, \infty)_T.$$

Therefore, we can obtain

$$\frac{r(n)x^\Delta(n)}{x(n)}\phi(n) \ge -1, \ n \in [n_2, \infty)_T,$$

and

$$-1 \le \omega(n)\phi(n) \le 0, \ n \in [n_2, \infty)_T. \quad (6)$$

Define u as

$$u(n) = \frac{r(n-\tau_1)x^\Delta(n-\tau_1)}{x(n)}, \ n \in [n_2, \infty)_T. \quad (7)$$

Similarly, we know that $u(n)<0$, and from $r(n)x^\Delta(n)<0$, we have $r(n-\tau_1)x^\Delta(n-\tau_1) \ge r(n)x^\Delta(n)$. Then, $u(n) \ge \omega(n)$, because of (6), we get

$$-1 \le u(n)\phi(n) \le 0, \ n \in [n_2, \infty)_T. \quad (8)$$

Next, define function v by

$$v(n) = \frac{r(n+\tau_2)x^\Delta(n+\tau_2)}{x(n)}, \ n \in [n_2, \infty)_T. \quad (9)$$

We know that $v(n) < 0$, and since $r(n)x^\Delta(n)$ is decreasing, we have

$$r(s)x^\Delta(s) \le r(n+\tau_2)x^\Delta(n+\tau_2), \ s \ge n+\tau_2 \ge n_2.$$

Through calculation, we have

$$x(l) \le x(n) + r(n+\tau_2)x^\Delta(n+\tau_2)\int_n^l \frac{1}{r(s)}\Delta s,$$
$$l \ge n+\tau_2 \ge n_2.$$

Letting $l \to \infty$, and from (9), then

$$-1 \le v(n)\phi(n) \le 0, \ n+\tau_2 \ge n_2. \quad (10)$$

Differentiating (5), we obtain

$$\omega^\Delta(n) = \frac{(r(n)x^\Delta(n))^\Delta}{x(\sigma(n))} - \frac{r(n)(x^\Delta(n))^2}{z(n)x(\sigma(n))}$$
$$\le \frac{(r(n)x^\Delta(n))^\Delta}{x(\sigma(n))} - \frac{r(n)(x^\Delta(n))^2}{x^2(n)}$$
$$= \frac{(r(n)x^\Delta(n))^\Delta}{x(\sigma(n))} - \frac{\omega^2(n)}{r(n)} < 0, \ n \ge n_3 \ge n+\tau_2. \quad (11)$$

Differentiating (7) and (9), as the similar process of (11), we have

$$u^\Delta(n) \le \frac{(r(n-\tau_1)x^\Delta(n-\tau_1))^\Delta}{x(\sigma(n))} - \frac{u^2(n)}{r(n)} < 0, \ n \in [n_3, \infty)_T. \quad (12)$$

And

$$v^\Delta(n) \le \frac{(r(n+\tau_2)x^\Delta(n+\tau_2))^\Delta}{x(\sigma(n))} - \frac{v^2(n)}{r(n)} < 0, \ n \in [n_3, \infty)_T. \quad (13)$$

Combining with (11), (12) and (13), we get

$$\omega^\Delta(n) + a_0u^\Delta(n) + b_0v^\Delta(n)$$
$$\le \frac{(r(n)x^\Delta(n))^\Delta}{x(\sigma(n))} + \frac{a_0(r(n-\tau_1)x^\Delta(n-\tau_1))^\Delta}{x(\sigma(n))} + \frac{b_0(r(n+\tau_2)x^\Delta(n+\tau_2))^\Delta}{x(\sigma(n))}$$

$$-\frac{\omega^2(n)}{r(n)}-a_0\frac{u^2(n)}{r(n)}-b_0\frac{v^2(n)}{r(n)},\ n\in[n_3,\infty)_T. \quad (14)$$

Since, $x(n)$ is decreasing, from (4) and (13), we can obtain

$$\omega^\Delta(n)+a_0u^\Delta(n)+b_0v^\Delta(n)$$
$$\le -Q(n)-\frac{\omega^2(n)}{r(n)}-a_0\frac{u^2(n)}{r(n)}-b_0\frac{v^2(n)}{r(n)},\ n\in[n_3,\infty)_T. \quad (15)$$

Multiplying both side of (15) by $\phi(n)$ and integrating from n_3 to n, using the integration by parts on time scales, we have

$$\phi(n)\omega(n)-\phi(n_3)\omega(n_3)-\int_{n_3}^n\phi^\Delta(s)\omega(\sigma(s))\Delta s+\int_{n_3}^n\frac{\phi(s)\omega^2(s)}{r(s)}\Delta s$$
$$+a_0\phi(n)u(n)-a_0\phi(n_3)u(n_3)-a_0\int_{n_3}^n\phi^\Delta(s)u(\sigma(s))\Delta s+a_0\int_{n_3}^n\frac{\phi(s)u^2(s)}{r(s)}\Delta s$$
$$+b_0\phi(n)v(n)-b_0\phi(n_3)v(n_3)-b_0\int_{n_3}^n\phi^\Delta(s)v(\sigma(s))\Delta s+b_0\int_{n_3}^n\frac{\phi(s)v^2(s)}{r(s)}\Delta s$$
$$+\int_{n_3}^n\phi(s)Q(s)\Delta s\le 0. \quad (16)$$

Noting $\omega(n)$, $u(n)$ and $v(n)$ are decreasing, from (16), we obtain

$$\phi(n)\omega(n)-\phi(n_3)\omega(n_3)-\int_{n_3}^n\frac{\omega(s)}{r(s)}\Delta s+\int_{n_3}^n\frac{\phi(s)}{r(s)}\omega^2(s)\Delta s$$
$$+a_0\phi(n)u(n)-a_0\phi(n_3)u(n_3)-a_0\int_{n_3}^n\frac{u(s)}{r(s)}\Delta s+a_0\int_{n_3}^n\frac{\phi(s)}{r(s)}u^2(s)\Delta s$$
$$+b_0\phi(n)v(n)-b\phi(n_3)v(n_3)-b_0\int_{n_3}^n\frac{v(s)}{r(s)}\Delta s+b_0\int_{n_3}^n\frac{\phi(s)}{r(s)}v^2(s)\Delta s$$
$$+\int_{n_3}^n\phi(s)Q(s)\Delta s\le 0. \quad (17)$$

Using the averaging technique for (17), we obtain

$$\phi(n)\omega(n)-\phi(n_3)\omega(n_3)+a_0\phi(n)u(n)-a_0\phi(n_3)u(n_3)$$
$$+b_0\phi(n)v(n)-b_0\phi(n_3)v(n_3)$$
$$-\frac{1+a_0+b_0}{4}\int_{n_3}^n\frac{1}{r(s)\phi(s)}\Delta s+\int_{n_3}^n\phi(s)Q(s)\Delta s\le 0. \quad (18)$$

Combining (6), (8) and (10) with (18), we have

$$\int_{n_3}^n[\phi(s)Q(s)-\frac{1+a_0+b_0}{4r(s)\phi(s)}]\Delta s$$
$$\le(1+a_0+b_0)+\phi(n_3)(\omega(n_3)+a_0u(n_3)+b_0v(n_3)),$$

which conflict with (3). This proof is completes.

In next theorem, we need the following definition:

We say that a function $\varphi=\varphi(n,s,l)\in Y$, if $\varphi\in C(E,R)$, where $E=\{(n,s,l):n_0\le l\le s\le n<\infty\}$, which satisfies $\varphi(n,n,l)=0$, $\varphi(n,l,l)=0$ and $\varphi(n,s,l)>0$ for $l<s<n$, and has a nonnegative rd-continuous Δ-partial derivative φ^{Δ_s} on E that it is locally integrable with respect to s in E. Define the operator T by

$$T[g;l,n]=\int_l^n\varphi(n,s,l)g(s)\Delta s, \quad (19)$$

for $n\ge s\ge l\ge t_0$ and $g\in C_{rd}^1[n_0,\infty)_T$. $\psi=\psi(n,s,l)$ is defined by

$$\varphi^{\Delta_s}(n,s,l)=\psi(n,s,l)\varphi(n,s,l). \quad (20)$$

Then, T is a linear operator obviously, and

$$T[g^\Delta;l,n]=-T[\psi g^\sigma;l,n]. \quad (21)$$

Theorem 2. Assume that $\delta(n\pm\alpha)=\delta(n)\pm\alpha(\alpha>0)$ and $\delta(n)\le n-\tau_1$. If there is positive function $\eta\in C_{rd}^1([n_0,\infty)_T,R)$ such that for $\varphi\in Y$ we have

$$\limsup_{n\to\infty}T[\eta(s)Q(s)-\frac{(1+a_0+b_0)}{4\eta(s)}(\psi(n,s,l)$$
$$+\frac{\eta^\Delta(s)}{\eta(\sigma(s))})^2r(\delta(s))\eta^2(\sigma(s));l,n]>0, \quad (22)$$

and

$$\limsup_{n\to\infty}T[Q(s)-\frac{(1+a_0+b_0)}{4}\psi^2(n,s,l)r(s);l,n]>0, \quad (23)$$

where $Q(n)$ is shown in Th 1, T defined as (19), and $\psi=\psi(n,s,l)$ is shown by (20). Then equation (1) oscillation.

Proof. Suppose that x is a nonoscillatory solution of (1). Yet the general, we may assume that $n_1\ge n_0$ exists to make $y(n)$, $y(n-\tau_1)$, $y(n+\tau_2)$ and $y(\delta(n))$ are positive for $n\in[n_1,\infty)_T$.

Proof steps similar to Theorem 1, for $n\in[n_1,\infty)_T$ we have case (i) and case (ii).

First assume that case (i) hold. Define

$$\omega(n)=\eta(n)\frac{r(n)x^\Delta(n)}{x(\delta(n))},\ n\in[n_1,\infty)_T. \quad (24)$$

Obviously $\omega(n)>0$, and we have

$$\omega^\Delta(n)$$
$$=\eta(n)\frac{(r(n)x^\Delta(n))^\Delta}{x(\delta(n))}+\frac{\eta^\Delta(n)}{\eta(\sigma(n))}\omega(\sigma(n))-\frac{\eta(n)x^\Delta(\delta(n))}{\eta(\sigma(n))x(\delta(n))}\omega(\sigma(n))$$
$$\le\eta(n)\frac{(r(n)x^\Delta(n))^\Delta}{x(\delta(n))}+\frac{\eta^\Delta(n)}{\eta(\sigma(n))}\omega(\sigma(n))-\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}\omega^2(\sigma(n)). \quad (25)$$

Define function

$$u(n)=\eta(n)\frac{r(n-\tau_1)x^{\Delta}(n-\tau_1)}{x(\delta(n))},\ n\in[n_1,\infty)_T. \quad (26)$$

Obviously $u(n)>0$, and we have

$$u^{\Delta}(n)$$
$$=\eta(n)\frac{(r(n-\tau_1)x^{\Delta}(n-\tau_1))^{\Delta}}{x(\delta(n))}+\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}u(\sigma(n))-\frac{\eta(n)x^{\Delta}(\delta(n))}{\eta(\sigma(n))x(\delta(n))}u(\sigma(n))$$
$$\le\eta(n)\frac{(r(n-\tau_1)x^{\Delta}(n-\tau_1))^{\Delta}}{x(\delta(n))}+\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}u(\sigma(n))-\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}u^2(\sigma(n)). \quad (27)$$

Define function

$$v(n)=\eta(n)\frac{r(n+\tau_2)x^{\Delta}(n+\tau_2)}{x(\delta(n))},\ n\in[n_1,\infty)_T. \quad (28)$$

Obviously $u(n)>0$, and we have

$$v^{\Delta}(n)$$
$$=\eta(n)\frac{(r(n+\tau_2)x^{\Delta}(n+\tau_2))^{\Delta}}{x(\delta(n))}+\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}v(\sigma(n))-\frac{\eta(n)x^{\Delta}(\delta(n))}{\eta(\sigma(n))x(\delta(n))}v(\sigma(n))$$
$$\le\eta(n)\frac{(r(n+\tau_2)x^{\Delta}(n+\tau_2))^{\Delta}}{x(\delta(n))}+\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}v(\sigma(n))-\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}v^2(\sigma(n)). \quad (29)$$

Combining (25), (27) and (29), and using (4), we obtain

$$\omega^{\Delta}(n)+a_0u^{\Delta}(n)+b_0v^{\Delta}(n)$$
$$\le-\eta(n)Q(n)+\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}\omega(\sigma(n))-\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}\omega^2(\sigma(n))$$
$$+a_0\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}u(\sigma(n))-a_0\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}u^2(\sigma(n))$$
$$+b_0\frac{\eta^{\Delta}(n)}{\eta(\sigma(n))}v(\sigma(n))-b_0\frac{\eta(n)}{r(\delta(n))\eta^2(\sigma(n))}v^2(\sigma(n)). \quad (30)$$

Applying operator T to (30), and from (21), we have

$$T[\eta(s)Q(s);l,n]$$
$$\le T[(\psi(n,s,l)+\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))})\omega(\sigma(s))-\frac{\eta(s)}{r(\delta(s))\eta^2(\sigma(s))}\omega^2(\sigma(s))$$
$$+a_0(\psi(n,s,l)+\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))})u(\sigma(s))-a_0\frac{\eta(s)}{r(\delta(s))\eta^2(\sigma(s))}u^2(\sigma(s))$$
$$+b_0(\psi(n,s,l)+\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))})v(\sigma(s))-b_0\frac{\eta(s)}{r(\delta(s))\eta^2(\sigma(s))}v^2(\sigma(s));l,n]. \quad (31)$$

By the averaging technique, from (31), we obtain

$$T[\eta(s)Q(s);l,n]\le T[\frac{(1+a_0+b_0)}{4\eta(s)}(\psi(n,s,l)$$
$$+\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))})^2r(\delta(s))\eta^2(\sigma(s));l,n],$$

that is

$$T[\eta(s)Q(s)$$
$$-\frac{(1+a_0+b_0)}{4\eta(s)}(\psi(n,s,l)+\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))})^2r(\delta(s))\eta^2(\sigma(s));l,n]\le0,$$

which a contradiction to (22).

Next, assume case (ii) hold. Define ω, u and v as (5), (7) and (9), proofing as in Theorem 1, we obtain inequality (15). Applying operator T to (15), and from (21), we have

$$T[Q(s);l,t]\le$$
$$T[\psi(n,s,l)\omega(\sigma(s))+a_0\psi(n,s,l)u(\sigma(s))+b_0\psi(n,s,l)v(\sigma(s));l,n]$$
$$-T[\frac{\omega^2(s)}{r(s)}+a_0\frac{u^2(s)}{r(s)}+b_0\frac{v^2(s)}{r(s)};l,n]. \quad (32)$$

Since the decreasing property of $\omega(n)$, $u(n)$ and $v(n)$, from (32) we have

$$T[Q(s);l,n]\le T[\psi(n,s,l)\omega(s)+a_0\psi(n,s,l)u(s)$$
$$+b_0\psi(n,s,l)v(s)-\frac{\omega^2(s)}{r(s)}-a_0\frac{u^2(s)}{r(s)}-b_0\frac{v^2(s)}{r(s)};l,n].$$

By the averaging technique, we obtain

$$T[Q(s)-\frac{(1+a_0+b_0)}{4}\psi^2(n,s,l)r(s);l,n]\le0,$$

which a contradiction to (23). So the theorem is established.

Remark 2.1. We can obtain kinds of theorems of equation (1) by from Th 2 different choices of η and φ.

Theorem 3. Assume that $B(\delta(n))=(1-a_0-b_0\frac{R(\delta(n)+\tau_2)}{R(\delta(n))})>0$, and positive function $\xi\in C_{rd}^1([n_0,\infty)_T,R)$ exist to make $\frac{\xi(n)}{\phi(n)r(n)}+\xi^{\Delta}(n)\le0$ and $C(\delta(n))=(1-a_0\frac{\xi(\delta(n)-\tau_1)}{\xi(\delta(n))}-b_0)>0$, where $R(n)=\int_{n_0}^{n}\frac{1}{r(s)}\Delta s$ and $\phi(n)$ is defined as in Theorem 1. If there exist positive function $\theta\in C_{rd}^1([n_0,\infty)_T,R)$ then

$$\limsup_{n\to\infty}\int_{n_2}^{n}[\theta(s)q(s)B(\delta(s))-\frac{(\theta^{\Delta}(s))^2r(\delta(s))}{4\theta(s)}]\Delta s=\infty. \quad (33)$$

And

$$\limsup_{n\to\infty}\int_{n_2}^{n}[\phi(s)q(s)C(\delta(s))-\frac{1}{4\phi(s)r(s)}]\Delta s=\infty. \quad (34)$$

Then equation (1) oscillation.

Proof. Suppose that y is a eventually positive solution of (1). Yet the general, we may assume that $n_1\ge n_0$ exists to make $y(n)$, $y(n-\tau_1)$, $y(n+\tau_2)$ and $y(\delta(n))$ are positive for $n\in[n_1,\infty)_T$.

Proof steps similar to Theorem 1, for $n\in[n_2,\infty)_T$

we have case (i) and case (ii).

Firstly, assume that case(i) hold. Since $r(n)x^{\Delta}(n)<0$, then

$$r(s)x^{\Delta}(s)\geq r(n)x^{\Delta}(n),\ \ n\geq s\geq n_1.$$

Through calculation, we have

$$x(n)\geq R(n)r(n)x^{\Delta}(n).$$

Then, we obtain

$$\begin{aligned}(\frac{x(n)}{R(n)})^{\Delta}&=\frac{x^{\Delta}(n)R(n)-x(n)R^{\Delta}(n)}{R(n)R(\sigma(n))}\\&=-\frac{1}{r(n)}[\frac{x(n)-r(n)x^{\Delta}(n)R(n)}{R(n)R(\sigma(n))}]\leq 0\cdot\end{aligned}$$

Since $\frac{x(n)}{R(n)}$ is nonincreasing, then

$$x(n+\tau_2)\leq\frac{R(n+\tau_2)}{R(n)}x(n). \tag{35}$$

From $x(n)$ and (35), we have

$$\begin{aligned}y(n)&\geq x(n)-a_0x(n-\tau_1)-b_0x(n+\tau_2)\\&\geq(1-a_0-b_0\frac{R(n+\tau_2)}{R(n)})x(n).\end{aligned} \tag{36}$$

Combining (1) with (35), then

$$(r(n)x^{\Delta}(n))^{\Delta}+q(n)B(\delta(n))x(\delta(n))\leq 0. \tag{37}$$

Define function

$$\omega(n)=\theta(n)\frac{r(n)x^{\Delta}(n)}{x(\delta(n))}\ \ n\in[n_2,\infty)_T. \tag{38}$$

Obviously, $\omega(n)>0$. Differentiating $\omega(n)$, and from (37), (38), we obtain

$$\begin{aligned}&\omega^{\Delta}(n)\\&=\frac{\theta(n)}{x(\delta(n))}(r(n)x^{\Delta}(n))^{\Delta}+\frac{\theta^{\Delta}(n)}{\theta(\sigma(n))}\omega(\sigma(n))-\frac{\theta(n)x^{\Delta}(\delta(n))}{\theta(\sigma(n))x(\delta(n))}\omega(\sigma(n))\\&\leq-\theta(n)q(n)B(\delta(n))+\frac{\theta^{\Delta}(n)}{\theta(\sigma(n))}\omega(\sigma(n))-\frac{\theta(n)}{r(\delta(n))\theta^2(\sigma(n))}\omega^2(\sigma(n))\cdot\end{aligned} \tag{39}$$

By the averaging technique, from (39), so

$$\omega^{\Delta}(n)\leq-\theta(n)q(n)B(\delta(n))+\frac{(\theta^{\Delta}(n))^2r(\delta(n))}{4\theta(n)}.$$

Integrating both sides of above form from n_2 to n, we obtain

$$\int_{n_2}^{n}[\theta(s)q(s)B(\delta(s))-\frac{(\theta^{\Delta}(s))^2r(\delta(s))}{4\theta(s)}]\Delta s\leq\omega(n_2),$$

which a contradiction to (33).

Next, assume case (ii) holds. Since $r(n)x^{\Delta}(n)$ is decreasing, we have

$$r(s)x^{\Delta}(s)\leq r(n)x^{\Delta}(n),\ \ s\geq n\geq n_2.$$

Through calculation, we have

$$x(l)\leq x(n)+r(n)x^{\Delta}(n)\int_n^l\frac{1}{r(s)}\Delta s.$$

When $l\to\infty$, we obtain

$$0\leq x(n)+\phi(n)r(n)x^{\Delta}(n). \tag{40}$$

Then, for $\xi(n)$, we have

$$\begin{aligned}(\frac{x(n)}{\xi(n)})^{\Delta}&=\frac{x^{\Delta}(n)\xi(n)-x(n)\xi^{\Delta}(n)}{\xi(n)\xi(\sigma(n))}\\&\geq-\frac{x(n)}{\xi(n)\xi(\sigma(n))}(\frac{\xi(n)}{\phi(n)r(n)}+\xi^{\Delta}(n))\geq 0\end{aligned}$$

Hence, we obtain

$$y(n)\geq(1-a_0\frac{\xi(n-\tau_1)}{\xi(n)}-b_0)x(n). \tag{41}$$

Combining (1) with (41), we conclude

$$(r(n)x^{\Delta}(n))^{\Delta}+q(n)C(\delta(n))x(\delta(n))\leq 0. \tag{42}$$

Define function

$$\nu(n)=\frac{r(n)x^{\Delta}(n)}{x(n)}\ \ n\in[n_2,\infty)_T. \tag{43}$$

Obviously, $\nu(n)<0$. From (40) and (43), we can obtain

$$-1\leq\phi(n)\nu(n)\leq 0. \tag{44}$$

Differentiating (43), and from (42), we conclude

$$\nu^{\Delta}(n)\leq-q(n)C(\delta(n))-\frac{\nu^2(n)}{r(n)}.$$

Then $\nu^{\Delta}(n)<0$. Multiplying both side of above inequality by $\phi(n)$ and integrating from n_2 to n, using the integration by parts on time scales, we have

$$\begin{aligned}&\phi(n)\nu(n)-\phi(n_2)\nu(n_2)-\int_{n_2}^{n}\frac{1}{r(s)}\nu(s)\Delta s\\&+\int_{n_2}^{n}\phi(s)q(s)C(\delta(s))\Delta s+\int_{n_2}^{n}\frac{\phi(s)}{r(s)}\nu^2(s)\Delta s\leq 0\cdot\end{aligned}$$

By the averaging technique, and from (44), we obtain

$$\int_{n_2}^{n}[\phi(s)q(s)C(\delta(s))-\frac{1}{4\phi(s)r(s)}]\Delta s\leq 1+\phi(n_2)\nu(n_2),$$

which a contradiction to (34). The proof is now completed.

3 Examples and summary

In this section, some examples are presented to explain the significance of our harvest.

Example 3.1. Consider

$$(t^2 x^{\Delta}(n))^{\Delta} + 3ny(n-c) = 0,\ n \in [1,\infty)_T, \quad (45)$$

where $x(n) = y(n) + y(n-c) + 2y(n+c)$, $c > 0$ is constant.

Here $r(n) = n^2$, $a(n) = 1$, $b(n) = 2$, $q(n) = 3n$, $\tau_1 = \tau_2 = c$, $\delta(n) = n - c$.

Then $\delta(n \pm \alpha) = \delta(n) \pm \alpha(\alpha > 0)$ holds, $Q(n) = 3(n-c)$, $\phi(n) = \int_n^{\infty} \frac{1}{s^2} \Delta s$, and it is not hard to verify that (2), (3) are satisfied. Therefore, equation (45) oscillation because of Theorem 1.

Example 3.2. Consider the following dynamic equation

$$(t^2 x^{\Delta}(n))^{\Delta} + \frac{1}{(n-c)^2} y(n-c) = 0,\ n \in [1,\infty)_T, \quad (46)$$

where $x(n)=y(n)+y(n-c)+2y(n+c)$, $c>0$ is constant. Here $r(n)=n^2$, $a(n)=1$, $b(n)=2$, $q(n)=3n$, $\tau_1=\tau_1=c$, $\delta(n)=n-c$. Then $\delta(n\pm\alpha)=\delta(n)\pm\alpha(\alpha>0)$ and $\delta(n)\leq n-\tau_1$ hold, $Q(n) = \frac{1}{n^2}$. Choosing $\varphi(n,s,l) = (n-s)(s-l)(n \geq s \geq l \geq 1)$, then $\varphi(n,s,l)$ belongs to function class Y , and $\varphi^{\Delta_s}(n,s,l) = n - 2s + l$, and from (20) we have $\psi(n,s,l) = \frac{n-2s+l}{(n-s)(s-l)}$. It is apt to educe that (22) and (23) established. Therefore, (46) is oscillatory due to Theorem 2.

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