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Oscillatory Behavior of Euler Type Delay Dynamic Equations with *p*-Laplacian Like Operators

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Abstract. We consider the Euler type delay dynamic equation with *p*-Laplacin like operators $(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2})^{\Delta}+a(v)x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}+r(v)x(\delta(v))|x(\delta(v))|^{p-2}=0,$ where $v\in[v_0,\infty)_T$. By using new inequality technique, we give some new criteria, which complement related contributions results.

1 Introduction

We consider the Euler type dynamic equation with a *p*-Laplacin like operator

$$(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2})^{\Delta}+a(v)x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}$$
$$+r(v)x(\delta(v))|x(\delta(v))|^{p-2}=0, \qquad (1)$$

where $v\in V, V=[v_0,\infty)_T$.

Assume that:

(H1) $p\geq 2$ is a real number, $a, r\in C_{rd}(V,\mathbb{R})$ satisfying $1-a(v)\mu(v)>0$, $r(v)>0$.

(H2) $\delta\in C_{rd}(V,T)$, $v\geq\delta(v)$, $\lim_{t\to\infty}\delta(v)=\infty$.

In (1), $a(v)x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}$ also called the damping term of this equation, oscillation of equations with damping term has attracted more and more attention, we refer the reader to [1-3].

Zhan [1] studied a dynamic equation

$$(a(v)\Phi(x^{\Delta}(v)))^{\Delta}+q(v)f(\Phi(x(\tau(v))))$$
$$+p(v)\Phi(x^{\Delta}(v))=0, \qquad (2)$$

where $\Phi(s)=|s|^{\gamma-2}s$, $\gamma>1$, and $a,p,q:T\to R$ satisfying $\frac{-p}{a}\in R^{+}$. The author consider the two cases

$$\int_{v_0}^{\infty}(a^{-1}(v)e_{-p/a}(v,v_0))^{1/(\gamma-1)}=\infty$$

and

$$\int_{v_0}^{\infty}(a^{-1}(v)e_{-p/a}(v,v_0))^{1/(\gamma-1)}<\infty,$$

by using new technique, some new properties of (2) are obtained.

Our objective of this paper are establishing some new oscillation criteria for equation (1) by using new transformation technique, which complement some known results.

We consider that all functional inequalities are eventually established when v is sufficiently large in this paper.

2 Main results

Lemma 1. ([4]) Assume $q\in C_{rd}(V,\mathbb{R})$, $q(v)\mu(v)+1>0$. Then $e_q(\cdot,v_0)>0$ is a solution of the initial value problem

$$\begin{cases} y^{\Delta}(v)=y(v)q(v),\\ y(v_0)=1.\end{cases}$$

Lemma 2. ([4]) If f is differentiable, then

$$(g^n)^{\Delta}=ng^{\Delta}\int_0^1[hg^{\sigma}+(1-h)g]^{n-1}dh,$$

where n is a constant.

Further, letting

$$C(v)=\frac{(e_{-a}(v,v_0))^{\frac{1}{p-1}}\left(\frac{\beta(\sigma(v))}{e_{-a}^{\sigma}(v,v_0)}\right)^{\frac{1}{p-1}}p\alpha(v)}{\alpha(\sigma(v))}+\frac{\alpha^{\Delta}(v)}{\alpha(\sigma(v))},$$

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$$\psi(v)=\alpha(v)\left(\left(\frac{\int_{v_0}^{\delta(v)}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s}{\int_{v_0}^{v}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s}\right)^{p-1}\frac{r(v)}{e_{-a}^{\sigma}(v,v_0)}\right.$$

$$\left.-\frac{\beta(v)}{e_{-a}^{\sigma}(v,v_0)}+\left(\frac{\beta(\sigma(v))}{e_{-a}^{\sigma}(v,v_0)}\right)^{\frac{p}{p-1}}(e_{-a}(v,v_0))^{\frac{1}{p-1}}\right),$$

where $\alpha,\beta\in C_{rd}^1(V,R)$ satisfying $\alpha(v)>0$, $\beta(v)\geq 0$.

Theorem 1. If (H1), (H2),

$$\int_{v_0}^{\infty}(e_{-a}(v,v_0))^{\frac{1}{p-1}}\Delta v=\infty \tag{3}$$

hold. If

$$\limsup_{v\to\infty}\int_{v_0}^{v}[\psi(s)+\frac{C^p(s)\alpha^p(\sigma(s))}{p^p\alpha^{p-1}(s)e_{-a}(s,v_0)}]\Delta s=\infty, \tag{4}$$

then (1) is oscillatory.

Proof. Let x is a non-oscillatory solution of (1). For $v\in[v_1,\infty)_T$, $v_1\geq v_0$, assume that $x(v)>0$, $x(\delta(v))>0$. From (1),

$$a(v)x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}+(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2})^{\Delta}$$
$$=-r(v)x(\delta(v))|x(\delta(v))|^{p-2}<0, \tag{5}$$

From Lemma 1 we know that $e_{-a}(\cdot,v_0)$ is a positive solution of the problem

$$\begin{cases}y^{\Delta}(v)=-a(v)y(v),\\ y(v_0)=1.\end{cases}$$

Then from (5),

$$\begin{aligned}&\left(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}e_{-a}^{-1}(v,v_0)\right)\\ &=\frac{e_{-a}(v,v_0)(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2})^{\Delta}}{e_{-a}^{\sigma}(v,v_0)e_{-a}(v,v_0)}\\ &-\frac{x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}(e_{-a}(v,v_0))^{\Delta}}{e_{-a}^{\sigma}(v,v_0)e_{-a}(v,v_0)}\\ &=\frac{(x^{\Delta}(v)|x^{\Delta}(v)|^{p-2})^{\Delta}+a(v)x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}}{e_{-a}^{\sigma}(v,v_0)}\\ &=-\frac{r(v)x(\delta(v))|x(\delta(v))|^{p-2}}{e_{-a}^{\sigma}(v,v_0)}<0\end{aligned} \tag{6}$$

Hence, $x^{\Delta}(v)>0$ or $x^{\Delta}(v)<0$, $v\in[v_2,\infty)_T$, $v_2\in[v_1,\infty)_T$.

We claim that $x^{\Delta}(v)>0,\ v\in[v_2,\infty)_T$. If not, then from (6) there exists $v_3\in[v_2,\infty)_T$ and $M>0$ such that, for $v\in[v_3,\infty)_T$,

$$\begin{aligned}&x^{\Delta}(v)|x^{\Delta}(v)|^{p-2}e_{-a}^{-1}(v,v_0)\\ &\leq x^{\Delta}(v_3)|x^{\Delta}(v_3)|^{p-2}e_{-a}^{-1}(v_3,v_0)=-M<0,\end{aligned}$$

which yields

$$x^{\Delta}(v)\leq-(e_{-a}(v,v_0))^{\frac{1}{p-1}}M^{\frac{1}{p-1}},$$

then

$$x(v)\leq x(v_3)-M^{\frac{1}{p-1}}\int_{v_3}^{v}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s.$$

From (3), $\lim_{t\to\infty}x(v)=-\infty$. Therefore, (1) can be rewrite as

$$^{\Delta}(x^{\Delta}(v))^{p-1}a(v)+((x^{\Delta}(v))^{p-1})+(x(\delta(v)))^{p-1}r(v)=0, \tag{7}$$

In addition, from (6), for $s\leq v$, we have

$$(x^{\Delta}(s))^{p-1}e_{-a}^{-1}(s,v_0)\geq(x^{\Delta}(v))^{p-1}e_{-a}^{-1}(v,v_0),$$

then

$$x(v)\geq(e_{-a}(v,v_0))^{-\frac{1}{p-1}}x^{\Delta}(v)\int_{v_2}^{v}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s.$$

Therefore,

$$\left(x(v)\left(\int_{v_2}^{v}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s\right)^{-1}\right)^{\Delta}\leq 0,$$

then

$$\begin{aligned}&x(\delta(v))x^{-1}(v)\geq\\ &\int_{v_2}^{\delta(v)}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s\left(\int_{v_2}^{v}(e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s\right)\end{aligned} \tag{8}$$

Define a Ricati transformation

$$\begin{aligned}\omega(v)=&(x^{\Delta}(v))^{p-1}\alpha(v)e_{-a}^{-1}(v,v_0)x^{1-p}(v)\\ &+\beta(v)\alpha(v)e_{-a}^{-1}(v,v_0).\end{aligned} \tag{9}$$

Obviously, $\omega(v)>0$. Then

$$
\begin{aligned}
&\omega^{\Delta}(v) \\
&= x^{1-p}(v)\alpha(v)(e_{-a}^{-1}(v,v_0) \\
&\quad (x^{\Delta}(v))^{p-1})^{\Delta} + (x^{1-p}(v)\alpha(v))^{\Delta} \\
&\quad (e_{-a}^{-1}(v,v_0)(x^{\Delta}(v))^{p-1})^{\sigma} \\
&\quad + (\frac{\beta(v)}{e_{-a}(v,v_0)})^{\Delta}\alpha(v) \\
&\quad + (\frac{\beta(v)}{e_{-a}(v,v_0)})^{\sigma}\alpha^{\Delta}(v) \\
&= x^{1-p}(\sigma(v))((x^{\Delta}(v))^{p-1}\alpha^{\Delta}(v) \\
&\quad e_{-a}^{-1}(v,v_0))^{\sigma} \\
&\quad + (\beta(v)e_{-a}^{-1}(v,v_0))^{\sigma}\alpha^{\Delta}(v) \\
&\quad + x^{1-p}(v)((x^{\Delta}(v))^{p-1}\alpha(v) \\
&\quad e_{-a}^{-1}(v,v_0))^{\Delta} \\
&\quad - (x^{p-1})^{\Delta}x^{1-p}(\sigma(v))x^{1-p}(v) \\
&\quad \alpha(v)((x^{\Delta}(v))^{p-1}e_{-a}^{-1}(v,v_0))^{\sigma} \\
&\quad + (\beta(v)e_{-a}^{-1}(v,v_0))^{\Delta}\alpha(v)
\end{aligned} \tag{10}
$$

From Lemma 2, we have

$$
\begin{aligned}
&(x^{p-1}(v))^{\Delta} \\
&= \int_0^1 [x(v)(1-h)+x^{\sigma}(v)h]^{p-2}dh \times x^{\Delta}(v)(p-1) \\
&\geq x^{p-2}(v)(p-1)x^{\Delta}(v)
\end{aligned} \tag{11}
$$

Then from (6) and (9)–(11),

$$
\begin{aligned}
&\omega^{\Delta}(v) \\
&\leq \omega^{\sigma}(v)\frac{\alpha^{\Delta}(v)}{\alpha^{\sigma}(v)} - \frac{r(v)}{e_{-a}^{\sigma}(v,v_0)}\frac{x^{p-1}(\delta(v))}{x^{p-1}(v)}\alpha(v) \\
&\quad - x(\sigma(v))(p-1)x^{-1}(v)\left((\tfrac{(x^{\Delta}(v))^{p-1}}{e_{-a}(v,v_0)})^{\sigma} \right)^{1+\frac{1}{p-1}} \\
&\quad x^{\Delta}(v)\alpha(v)\left((\tfrac{(x^{\Delta}(v))^{p-1}}{e_{-a}(v,v_0)})^{\sigma} \right)^{-\frac{1}{p-1}} x^{-p}(\sigma(v)) \\
&\quad + (e_{-a}^{-1}(v,v_0)\beta(v))^{\Delta}\alpha(t)
\end{aligned} \tag{12}
$$

From (6)

$$
x^{\Delta}(\sigma(v))\left(\frac{(e_{-a}(v,v_0))^{\frac{1}{p-1}}}{(e_{-a}^{\sigma}(v,v_0))^{\frac{1}{p-1}}} \right) \leq x^{\Delta}(v),
$$

$$
x(v) \leq x(\sigma(v)).
$$

Then, from (8), (9) and (12),

$$
\begin{aligned}
&\omega^{\Delta}(v) \\
&\leq \frac{\alpha^{\Delta}(v)}{\alpha^{\sigma}(v)}\omega^{\sigma}(v) - \alpha(v)\frac{r(v)}{e_{-a}^{\sigma}(v,v_0)} \\
&\quad \left(\int_{v_2}^{\delta(v)} (e_{-a}(s,v_0))^{\frac{1}{1-p}}\Delta s (\int_{v_2}^{v} (e_{-a}(s,v_0))^{\frac{1}{p-1}}\Delta s)^{-1} \right)^{p-1} \\
&\quad - \alpha(v)(p-1)(e_{-a}(v,v_0))^{\frac{1}{p-1}} \\
&\quad \left((\omega(v)\alpha^{-1}(v) - e_{-a}^{-1}(v,v_0)\beta(v))^{\sigma} \right)^{1+\frac{1}{p-1}} \\
&\quad + \alpha(v)e_{-a}^{-1}(v,v_0)\beta(v)
\end{aligned} \tag{13}
$$

From the inequality (see [5])

$$
-[G(\frac{1}{n}+1) - \frac{E}{n}]E^{\frac{1}{n}} + G^{1+\frac{1}{n}} \leq (G-E)^{1+\frac{1}{n}},
$$
$$
G \geq 0,\ \ E \geq 0, n \geq 1,
$$

define

$$
G := (\alpha^{-1}(v)\omega(v))^{\sigma}, \quad E := (e_{-a}^{-1}(v,v_0)\beta(v))^{\sigma},
$$

we have

$$
\begin{aligned}
&\left((\alpha^{-1}(v)\omega(v) - e_{-a}^{-1}(v,v_0)\beta(v))^{\sigma} \right)^{\frac{p}{p-1}} \\
&\geq \omega^{1+\frac{1}{p-1}}(\sigma(v))\alpha^{-1-\frac{1}{p-1}}(\sigma(v)) \\
&\quad + \beta^{1+\frac{1}{p-1}}(\sigma(v))(e_{-a}^{\sigma}(v,v_0))^{-1-\frac{1}{p-1}}(p-1)^{-1} \\
&\quad - (\tfrac{\beta(\sigma(v))}{e_{-a}^{\sigma}(v,v_0)})^{\frac{1}{p-1}}(p-1)^{-1}\alpha^{-1}(\sigma(v))p
\end{aligned} \tag{14}
$$

From (13), (14),

$$
\begin{aligned}
&\omega^{\Delta}(v) \\
&\leq \omega(\sigma(v))C(v) - \psi(v) - (e_{-a}(v,v_0))^{\frac{1}{p-1}} \\
&\quad \alpha^{-1-\frac{1}{p-1}}(\sigma(v))\alpha(v)(p-1)\omega^{\frac{p}{p-1}}(\sigma(v))
\end{aligned} \tag{15}
$$

Define

$$
\begin{aligned}
G := {}& \alpha^{\frac{1}{p-1}}(v)(p-1)^{\frac{1}{p-1}}\alpha^{-1}(\sigma(v)) \\
& (e_{-a}(v,v_0))^{\frac{1}{p}}\omega(\sigma(v))
\end{aligned}
$$

and

$$
\begin{aligned}
E := {}& C^{p-1}(v)\alpha(\sigma(v))(\alpha^{\frac{p-1}{p}}(v) \\
& p(e_{-a}(v,v_0))^{\frac{1}{p}}(p-1)^{\frac{p-1}{p}})^{-1}.
\end{aligned}
$$

Applying variation of the Young inequality

$$E^{\frac{p}{p-1}}(p-1)^{-1} \geq -G^{\frac{p}{p-1}} + G(p-1)^{-1} p E^{\frac{1}{p-1}},$$

we have

$$\begin{aligned} &-(e_{-a}(v,v_0))^{\frac{1}{p-1}} \alpha^{-1-\frac{1}{p-1}}(\sigma(v))(p-1) \\ &\omega^{1+\frac{1}{p-1}}(\sigma(v))\alpha(v) + \omega(\sigma(v))C(v) \\ &\leq \alpha^p(\sigma(v))(p^p e_{-a}(v,v_0)\alpha^{p-1}(v))^{-1} C^p(v). \end{aligned} \tag{16}$$

Using (16) in (15), we conclude that

$$\int_{t_2}^{t} [\psi(s) + \frac{C^p(v)\alpha^p(\sigma(s))}{p^p \alpha^{p-1}(s) e_{-a}(s,t_0)}]\Delta s \leq \omega(v_2).$$

This contradiction completes the proof.

Further, a function $H \in C_{rd}(D,[0,\infty))$ (where $D \equiv \{(v,s): v_0 \leq s \leq v,\ v,s \in V\}$) belongs to a class H if

(i) $H(v,v) = 0$, $v \geq v_0$, $H(v,s) > 0$, $v \geq s \geq v_0$;

(ii) H has a rd-continuous Δ -partial derivative $H^{\Delta_s}(v,s) \geq 0$ on D_0 with the second variable and satisfying

$$\begin{aligned} &\alpha^{-1}(\sigma(s))h(v,s)H^{\frac{p-1}{p}}(v,s) \\ &= C(s)H(v,s) + H^{\Delta_s}(v,s). \end{aligned} \tag{17}$$

Theorem 2. If (H1), (H2), and (3) hold, and for $H \in H$, $v_2 \in V$,

$$\begin{aligned} &\limsup_{v\to\infty} \frac{1}{H(v,v_2)} \int_{t_0}^{t} [\psi(s)H(v,s) \\ &-(p^p \alpha^{p-1}(s) e_{-a}(s,v))^{-1} h_+^p(v,s)]\Delta s = \infty, \end{aligned} \tag{18}$$

here $h_+(v,s) := \max\{h(v,s),0\}$. Then (1) is oscillatory.

Proof. Let x is a non-oscillatory solution of (1). $x(v) > 0$, $x(\delta(v)) > 0$, $v \in [v_1,\infty)_T$, $v_1 \geq v_0$. Define $\omega(v)$ as (9), then by Theorem 1, we obtain (15). Then

$$\begin{aligned} &\int_{v_2}^{v} \psi(s)H(v,s)\Delta s \leq \\ &-\int_{v_2}^{v} (e_{-a}(s,v_0))^{\frac{1}{p-1}} \alpha(s)(p-1)(\alpha^{1+\frac{1}{p-1}})^{-1}(\sigma(s)) \\ &\omega^{\frac{p}{p-1}}(\sigma(s))H(v,s)\Delta s \\ &+\int_{v_2}^{v} \omega(\sigma(s))H(v,s)C(s)\Delta s \\ &-\int_{v_2}^{v} \omega^{\Delta}(s)H(v,s)\Delta s \end{aligned} \quad . \tag{19}$$

Then

$$\begin{aligned} &\int_{v_2}^{v} \omega^{\Delta}(s)H(v,s)\Delta s = -\omega(v_2)H(v,v_2) \\ &-\int_{v_2}^{v} \omega(\sigma(s))\, H^{\Delta_s}(v,s)\Delta s. \end{aligned} \tag{20}$$

From (17), (19) and (20), we have

$$\begin{aligned} &\int_{t_2}^{t} \psi(s)H(v,s)\Delta s \\ &\leq \int_{v_2}^{v} C(s)H(v,s))\omega(\sigma(s)) \\ &\quad + H^{\Delta_s}(v,s)\Delta s + \omega(v_2)H(v,v_2) \\ &-\int_{v_2}^{v} (e_{-a}(s,v_0))^{\frac{1}{p-1}} \alpha(s)(p-1)\alpha^{-1-\frac{1}{p-1}}(\sigma(s)) \\ &\omega^{1+\frac{1}{p-1}}(\sigma(s))H(v,s)\Delta s \\ &\leq \int_{v_2}^{v} h_+(v,s)H^{\frac{p-1}{p}}(v,s)\alpha^{-1}(\sigma(s))\omega(\sigma(s))\Delta s \\ &+\omega(v_2)H(v,v_2) \\ &-\int_{v_2}^{v} (p-1)(e_{-a}(s,v_0))^{\frac{1}{p-1}} \alpha(s)\alpha^{-1-\frac{1}{p-1}}(\sigma(s)) \\ &\omega^{1+\frac{1}{p-1}}(\sigma(s))H(v,s)\Delta s \end{aligned} \tag{21}$$

Define

$$\begin{aligned} &G := \alpha^{-1-\frac{1}{p-1}}(\sigma(s))(p-1)(e_{-a}(s,v_0))^{\frac{1}{p-1}} \\ &\alpha(s)H(v,s), \\ &E := \frac{H^{\frac{p-1}{p}}(v,s)}{\alpha(\sigma(s))} h_+(v,s) \\ &u := \omega(\sigma(v)) \end{aligned} \quad .$$

Using the inequality

$$Ev - Gv^{\frac{\gamma+1}{\gamma}} \leq (\gamma+1)^{-\gamma-1}\gamma^{\gamma} \frac{E^{\gamma+1}}{G^{\gamma}},\ G > 0,$$

we have

$$\begin{aligned} &\int_{t_2}^{t} h_+(v,s)\alpha^{-1}(\sigma(s))\, H^{\frac{p-1}{p}}(v,s)\omega(\sigma(s))\Delta s \\ &-\int_{v_2}^{v} \alpha^{-1-\frac{1}{p-1}}(\sigma(s))(e_{-a}(s,t_0))^{\frac{1}{p-1}}(p-1)\alpha(s)H(v,s)\omega^{1+\frac{1}{p-1}}(\sigma(s))\Delta s \\ &\leq \int_{v_2}^{v} \frac{h_+^p(v,s)}{p^p \alpha^{p-1}(s) e_{-a}(s,v)} \Delta s \end{aligned} \tag{22}$$

By (22), (21),

$$\begin{aligned} &\omega(v_2)H(v,v_2) \geq \\ &\int_{v_2}^{v} \left[-e_{-a}^{-1}(s,v) \frac{h_+^p(v,s)}{p^p \alpha^{p-1}(s)} + \psi(s)H(v,s) \right] \Delta s, \end{aligned}$$

which contradicts (18). The proof is complete.

3 Conclusion

This paper is concerned with a delay dynamic equation of Euler type with *p*-Laplacin like operators, by using new transformation and inequality technique with specific analytical skills, new criteria for the oscillation of the equations are established. under condition (3) holds.

It is a interesting problem for future research of develop a different method to study equation (1) under the condition that integral formulae of (3) bounded.

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