Some Oscillation Criteria for Higher-Order Neutral Emden-Fowler Delay Dynamic Equations on Time Scales

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Abstract. New oscillation criteria are established for higher-order Emden-Fowler delay dynamic equation $q(v)x^\beta(\delta(v)) + (r(v)(z^{\Delta^{n-1}}(v))^\alpha)^\Delta = 0$ on time scales, $z(v) := p(v)x(\tau(v)) + x(v)$. Our results extend and supplement those reported in literatures in the sense that we study a more generalized neutral delay equation and do not require $r^\Delta(v) \geq 0$ and the commutativity of the jump and delay operators.

1 Introduction

We will consider the even order dynamic equation

$$q(v)x^\beta(\delta(v)) + ((z^{\Delta^{n-1}}(v))^\alpha r(v))^\Delta = 0, \quad t \in V, \quad (1)$$

where $z(v) := p(v)x(\tau(v)) + x(v)$, $V = [t_0, \infty)_T$, and n is an even integer, $n \geq 2$, $\alpha > 0$, $\beta > 0$ are quotient of odd integers.

Assume that:

(A1) $p(v), r(v), q(v) \in C_{rd}(V, (0, \infty))$, where

$$p(v) < 1 \text{ and } \int_{v_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \Delta s = \infty.$$

(A2) $\tau(v), \delta(v) \in C_{rd}(V, T)$, such that $v \geq \tau(v)$,

$$v \geq \delta(v), \text{ and } \lim_{v \rightarrow \infty} \tau(v) = \lim_{v \rightarrow \infty} \delta(v) = \infty.$$

Grace [1] studied an even order equation

$$q(v)(x(v))^\alpha + ((x^{\Delta^{n-1}}(v))^\alpha a(v))^\Delta = 0, \quad (2)$$

where $n \geq 2$, $\alpha \geq 1$ is the ratio of positive odd integers.

Under the conditions $a^\Delta(t) \geq 0$ and $\int_{a^\Delta(s)}^{\infty} \frac{1}{a^\Delta(s)} \Delta s = \infty$,

some oscillation criteria of (2) were established.

Tunç [2] considered the oscillatory of even order equation

$$\begin{aligned} &F(v, x(v), x(\tau(v)), x^\Delta(v), x^\Delta(\tau(v))) \\ &+ (x^{\Delta^{n-1}}(v) | x^{\Delta^{n-1}}(v) |^{\alpha-1} r(v))^\Delta = 0, \end{aligned} \quad (3)$$

where $n \geq 2$, $\alpha > 0$ is a constant and exists a rd-continuous function $q(v) > 0$ such that

$$\frac{F(v, x, u, i, w)}{|u|^{\alpha-1} u} \geq q(v).$$

Furthermore, assume $\tau(v)$ satisfy $\tau^\Delta(v) > 0$, $\tau(\sigma(v)) = \sigma(\tau(v))$. By using Taylor monomials, some sufficient conditions for ensuring (3) oscillatory are established.

On the basis research of Grace [1] and Tunç [2], we study a neutral delay dynamic equation (1), by Riccati transformation with inequality technique and using comparison principles, sufficient conditions for oscillation of (1) extend and supplement some known results, and allow to relax some restrictive conditions imposed on the equation, like $r^\Delta(v) \geq 0$.

2 Auxiliary lemmas

We use the polynomials $\{h_k(v, s)\}_{k=0}^\infty$ (see [2]), where

$$\eta_i(v, s) = \begin{cases} \int_s^v \eta_{i-1}(u, s) \Delta u, & i \in N \\ 1, & i = 0 \end{cases}$$

Lemma 1. (Taylor's formula [3]). If $f \in C_{rd}^n(T, R)$, $n \in N$. For $s \in T$

$$\begin{aligned} f(v) = & \int_s^{\rho^{n-1}(v)} f^{\Delta^n}(\eta) h_{n-1}(v, \sigma(\eta)) \Delta \eta \\ & + \sum_{k=0}^{n-1} f^{\Delta^k}(s) h_k(v, s), \quad v \in T \end{aligned}$$

Lemma 2. [4]. If $f \in C_{rd}(T, R)$, $n \in N$, for $s, v \in T$,

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$$\begin{aligned} & \int_s^v f(\eta) h_n(s, \sigma(\eta)) \Delta \eta (-1)^n \\ &= \int_s^v \int_{\eta_{n+1}}^v \cdots \int_{\eta_2}^t f(\eta_1) \Delta \eta_1 \Delta \eta_2 \cdots \Delta \eta_{n+1} \end{aligned}$$

Lemma 3. [5]. If $\sup T = \infty$, $f \in C_{rd}^n(V, R)$, $n \in \mathbb{N}$, then

$$(i) \liminf_{v \rightarrow \infty} f^{\Delta^n}(v) > 0$$

$$\text{Then } \lim_{v \rightarrow \infty} f^{\Delta^i}(v) = \infty, \quad i \in [0, n]_Z.$$

$$(ii) \limsup_{v \rightarrow \infty} f^{\Delta^n}(v) < 0$$

$$\text{Then } \lim_{v \rightarrow \infty} f^{\Delta^i}(v) = -\infty, \quad i \in [0, n]_Z.$$

Lemma 4. (Kiguradz's lemma [5]). Let $\sup T = \infty$, $f \in C_{rd}^n(V, R^+)$, $n \in \mathbb{N}$. Suppose that either $f^{\Delta^n} \leq 0 (\neq 0)$ or $f^{\Delta^n} \geq 0 (\neq 0)$, $[v_0, \infty)_T$. Then, $f^{\Delta^n}(v)(-1)^{n-m} \geq 0$, $m \in [0, n]_Z$, $v \in [v_1, \infty)_T$, $v_1 \in [v_0, \infty)_T$,

$$(i) f^{\Delta^i}(v) > 0, \quad i \in [0, m]_Z, v \in [v_1, \infty)_T.$$

$$(ii) f^{\Delta^i}(v)(-1)^{a+i} > 0, \quad i \in [a, n]_Z, v \in [v_1, \infty)_T.$$

Lemma 5. [3] x is differentiable,

$$(g^n)^\Delta(v) = ng^\Delta(v) \int_0^1 [hg^\sigma(v) + (1-h)g(v)]^{n-1} dh,$$

n is constant.

Lemma 6. Assume that (A1)–(A2), and $x(v) > 0$ is an solution of (1). If

$$\int_{v_0}^\infty r^{-\frac{1}{\alpha}}(v) h_1(\sigma(w), v_0) \left(\int_w^\infty Q(s) \Delta s \right)^{\frac{1}{\alpha}} \Delta w = \infty, \quad (4)$$

where $(-p(\delta(v)) + 1)^\beta q(v) = Q(v)$.

For $v_1 \in [v_0, \infty)_T$

$$z^{\Delta^k}(v) > 0, \quad (k = 0, 1, \dots, n-1),$$

$$((z^{\Delta^{n-1}}(v))^\alpha r(v))^\Delta < 0. \quad (5)$$

Proof. If $x(t) > 0$ is solution of (1), then $x(\tau(v)) > 0$ and $x(\delta(v)) > 0$ for $v \in [v_1, \infty)_T$, $v_1 \in [v_0, \infty)_T$ then $z(v) > 0$. From (1) and (A1),

$$-x^\beta(\delta(v))q(v) = ((z^{\Delta^{n-1}}(v))^\alpha r(v))^\Delta < 0, \quad (6)$$

thus $z^{\Delta^{n-1}}(v)$ is eventually of one sign.

We claim that

$$z^{\Delta^{n-1}}(v) > 0, \quad v \in [v_1, \infty)_T. \quad (7)$$

Not, $z^{\Delta^{n-1}}(v) < 0$ on $[v_2, \infty)_T$, $v_2 \geq v_1$. In view of (6), for $c > 0$, $v \in [v_2, \infty)_T$

$$(z^{\Delta^{n-1}}(v))^\alpha r(v) \leq (z^{\Delta^{n-1}}(v_2))^\alpha r(v_2) = -c.$$

Thus,

$$z^{\Delta^{n-1}}(v) \leq -\left(\frac{c}{r(v)}\right)^{\frac{1}{\alpha}}, \quad v \in [v_2, \infty)_T.$$

Then

$$z^{\Delta^{n-2}}(v) \leq -\int_{v_2}^v \left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}} c^{\frac{1}{\alpha}} \Delta s + z^{\Delta^{n-2}}(v_2), \quad v \in [v_2, \infty)_T.$$

Letting $v \rightarrow \infty$ and from (A1), thus $\lim_{v \rightarrow \infty} z^{\Delta^{n-2}}(v) = -\infty$. From Lemma 3, $\lim_{v \rightarrow \infty} z(v) = -\infty$. This contradicts $z(v) > 0$.

From Lemma 4, there exists $m \in [0, n-1]_Z$ such that $0 \leq z^{\Delta^{n-1}}(v)(-1)^{n-1-m}$, so $m \in \{1, 3, \dots, n-1\}$ such that (i) and (ii) of Lemma 4 hold, $v \in [v_1, \infty)_T$, $z^\Delta(v) > 0$ on $v \in [v_1, \infty)_T$. Then, from (A1) and $z(v)$, we have

$$\begin{aligned} x(v) &= -p(v)x(\tau(v)) + z(v) \\ &\geq -p(v)z(\tau(v)) + z(v) \\ &\geq (-p(v) + 1)z(v). \end{aligned}$$

From (1),

$$0 \geq z^\beta(\delta(v))Q(v) + ((z^{\Delta^{n-1}}(v))^\alpha r(v))^\Delta. \quad (8)$$

Now, we claim that $m = n-1$. If not, we have

$$0 < z^{\Delta^{n-3}}(v), 0 > z^{\Delta^{n-2}}(v), \quad v \in [v_1, \infty)_T.$$

Since $\lim_{v \rightarrow \infty} \delta(v) = \infty$ and $z^\Delta(v) < 0$, $[v_1, \infty)_T$, we can choose $v_2 \in [v_1, \infty)_T$, $v_1 \leq \delta(v)$, $v \in [v_2, \infty)_T$, and so

$$0 < c = z(v_1) \leq z(\delta(v)), \quad v \in [v_2, \infty)_T. \quad (9)$$

Integrating (8) v to $u \geq v$, $u \rightarrow \infty$, by (9),

$$\left(\int_t^\infty Q(s) \Delta s \frac{1}{r(v)} \right)^{\frac{1}{\alpha}} c^{\frac{\beta}{\alpha}} \leq z^{\Delta^{n-1}}(v), \quad v \in [v_2, \infty)_T. \quad (10)$$

Integrating (10) v to $w \geq t$, $w \rightarrow \infty$, then

$$-\int_v^\infty \left(\int_u^\infty Q(s) \Delta s \frac{1}{r(u)} \right)^{\frac{1}{\alpha}} \Delta u \times c^{\frac{\beta}{\alpha}} \geq z^{\Delta^{n-2}}(v).$$

Integrating v_2 to v , and by Lemma 1,

$$\begin{aligned} z^{\Delta^{n-3}}(t_2) &\geq c^{\frac{\beta}{\alpha}} \int_{v_2}^v \int_w^\infty \left(\int_u^\infty Q(s) \frac{1}{r(u)} \Delta s \right)^{\frac{1}{\alpha}} \Delta u \Delta v \\ &= -c^{\frac{\beta}{\alpha}} \int_{v_2}^v h_1(v_2, \sigma(w)) \left(\int_w^\infty Q(s) \Delta s \frac{1}{r(w)} \right)^{\frac{1}{\alpha}} \Delta w \\ &= c^{\frac{\beta}{\alpha}} \int_{v_2}^v h_1(\sigma(w), v_2) \left(\int_w^\infty Q(s) \Delta s \frac{1}{r(w)} \right)^{\frac{1}{\alpha}} \Delta w, \end{aligned}$$

Which contradicts (4). Therefore, $m = n - 1$, and from Lemma 4, we have

$$\begin{aligned} z(v) &> 0, \\ z^\Delta(v) &> 0, \\ \dots, \\ z^{\Delta^{n-1}}(v) &> 0, \quad v \in [v_1, \infty)_T. \end{aligned}$$

This completes the proof.

Lemma 7. Let $n > 2$. Assume that $x(v)$ is an eventually positive solution of (1), and satisfy the conditions of Lemma 6. Then $z^{\Delta^{n-2}}(v) \left(\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s \right)^{-1}$ is non-increasing on $[v_1, \infty)_T$, and

$$\begin{aligned} z(v) &\geq \frac{z^{\Delta^{n-2}}(v)}{\int_{v_1}^{\rho^{n-3}(v)} \frac{1}{r^{1/\alpha}(s)} \Delta s} \int_{v_1}^v \left(h_{n-3}(v, \sigma(\eta)) \int_{v_1}^\eta \frac{1}{r^{1/\alpha}(s)} \Delta s \right) \Delta \eta. \end{aligned}$$

Proof. By Lemma 6 we get (5), then

$$\left[(z^{\Delta^{n-1}}(v))^\alpha r(v) \right]^\Delta < 0, \quad [v_1, \infty)_T,$$

and

$$\begin{aligned} -z^{\Delta^{n-2}}(v_1) + z^{\Delta^{n-2}}(v) &= \int_{v_1}^v \frac{(r(s)(z^{\Delta^{n-1}}(s))^\alpha)^{\frac{1}{\alpha}}}{r^{1/\alpha}(s)} \Delta s \\ &\geq \int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s \times ((z^{\Delta^{n-1}}(v))^\alpha r(v))^{\frac{1}{\alpha}} \end{aligned}$$

then

$$\int_{v_1}^v r^{-\frac{1}{\alpha}}(s) \Delta s \times ((z^{\Delta^{n-1}}(v))^\alpha r(v))^{\frac{1}{\alpha}} \leq z^{\Delta^{n-2}}(v). \quad (11)$$

Thus, we can obtain

$$\begin{aligned} &\left(\left(\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s \right)^{-1} z^{\Delta^{n-2}}(v) \right)^\Delta \\ &= \int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s \times z^{\Delta^{n-1}}(v) - z^{\Delta^{n-2}}(v) r^{-1/\alpha}(v) \\ &\quad \left(\int_{v_1}^{\sigma(v)} r^{-1/\alpha}(s) \Delta s \int_{v_1}^v r^{-1/\alpha}(s) \Delta s \right)^{-1} \\ &\leq 0. \end{aligned}$$

So $z^{\Delta^{n-2}}(v) \left(\int_{v_1}^v r^{-1/\alpha}(s) \Delta s \right)^{-1}$ is non-increasing on $[v_1, \infty)_T$. Next, using Lemma 1,

$$\begin{aligned} z(v) &= \int_{v_1}^{\rho^{n-3}(v)} h_{n-3}(v, \sigma(\eta)) z^{\Delta^{n-2}}(\eta) \Delta \eta + \sum_{k=0}^{n-3} h_k(v, s) z^{\Delta^k}(s) \\ &\geq \int_{v_1}^{\rho^{n-3}(v)} z^{\Delta^{n-2}}(\eta) h_{n-3}(v, \sigma(\eta)) \Delta \eta \\ &= \int_{v_1}^{\rho^{n-3}(v)} \left(\int_{v_1}^\eta \frac{1}{r^{1/\alpha}(s)} \Delta s \frac{z^{\Delta^{n-2}}(\eta)}{\int_{v_1}^\eta \frac{1}{r^{1/\alpha}(s)} \Delta s} h_{n-3}(v, \sigma(\eta)) \right) \Delta \eta \\ &\geq \left(\int_{v_1}^v r^{-1/\alpha}(s) \Delta s \right)^{-1} z^{\Delta^{n-2}}(v) \\ &\quad \int_{v_1}^{\rho^{n-3}(v)} \left(h_{n-3}(v, \sigma(\eta)) \int_{v_1}^\eta \frac{1}{r^{1/\alpha}(s)} \Delta s \right) \Delta \eta \end{aligned}$$

This completes the proof.

3 Main results

Let

$$\begin{aligned} A(v) &= \int_{v_0}^v r^{-\frac{1}{\alpha}}(s) \Delta s, \quad I_1(v) = A(v) - A(v_1), \\ I_i(v) &= \int_{v_1}^v I_{i-1}(s) \Delta s. \end{aligned}$$

Theorem 1. Assume that (A1)–(A2) and (4). If the equation

$$I_{n-1}^\beta(\delta(v)) Q(v) y^{\frac{\beta}{\alpha}}(\delta(v)) + y^\Delta(v) = 0, \quad (12)$$

is oscillatory, where Q is defined as in Lemma 6. Then equation (1) is oscillatory.

Proof. If x is a non-oscillatory solution of (1). $x(v) > 0$, $x(\delta(v)) > 0$, $x(\tau(v)) > 0$, $v \in [v_1, \infty)_T$, ($v_1 \geq v_0$). From Lemma 6, we have (5) and (8). Letting $(z^{\Delta^{n-1}}(v))^\alpha r(v) =: y(v)$, then

$$\left(\frac{y(v)}{r(v)} \right)^{\frac{1}{\alpha}} = z^{\Delta^{n-1}}(v). \quad (13)$$

We know that $y^\Delta(v) < 0$, $[v_1, \infty)_T$, then integrating (13), we can get

$$\int_{v_1}^v r^{-\frac{1}{\alpha}}(s) \Delta s \times y^{\frac{1}{\alpha}}(v) \leq z^{\Delta^{n-2}}(v),$$

i.e.

$$(A(v) - A(v_1)) y^{\frac{1}{\alpha}}(v) = I_1(v) y^{\frac{1}{\alpha}}(v) \leq z^{\Delta^{n-2}}(v). \quad (14)$$

Integrating (14) $(n-2)$ times from v_1 to v , we have

$$I_{n-1}(v)y^{\frac{1}{\alpha}}(v) \leq z(v),$$

then

$$I_{n-1}(\delta(v))y^{\frac{1}{\alpha}}(\delta(v)) \leq z(\delta(v)). \quad (15)$$

Combining (15) together with (8), and we get $y(v) > 0$ is a solution of

$$I_{n-1}^{\beta}(\delta(v))Q(v)y^{\frac{\beta}{\alpha}}(\delta(v)) + y^{\Delta}(v) \leq 0.$$

Thus, from [6], (12) admits a non-oscillatory solution, too. This contradiction completes the proof.

Theorem 2. Let $n > 2$, $\beta \geq \alpha$. Assume that (A1)–(A2) and (4). If there exist $\phi \in C_{rd}^1([v_0, \infty)_T, R)$ for $M > 0$, $v_1 \in [v_0, \infty)_T$ we have

$$\limsup_{v \rightarrow \infty} \int_{v_2}^v [Q(u)\phi(u)(\int_{v_1}^{\rho^{n-3}(\delta(u))} (h_{n-3}(\delta(u), \sigma(\eta))) \int_{v_1}^{\eta} r^{-1/\alpha}(s)\Delta s \Delta \eta (\int_{v_1}^u \frac{1}{r^{1/\alpha}(s)} \Delta s)^{-1})^{\beta} - r(u)\alpha^{\alpha}((\phi^{\Delta}(u))_+)^{\alpha+1}((\beta M)^{\alpha}(\alpha+1)^{\alpha+1}\phi^{\alpha}(u))^{-1}] \Delta u = \infty \quad (16)$$

where $v_2 \in [v_1, \infty)_T$, $\rho^{n-3}(\delta(v)) \geq v_1$ for $v \in [v_2, \infty)_T$, $Q(u)$ is defined as in Lemma 6, $(\phi^{\Delta}(u))_+ = \max\{\phi^{\Delta}(u), 0\}$. Then, (1) is oscillatory.

Proof. If x is a non-oscillatory solution of (1). $x(v) > 0$, $x(\tau(v)) > 0$, $x(\delta(v)) > 0$, $v \in [v_1, \infty)_T$, $(v_1 \geq v_0)$. By Lemma 6, we have (5) and (8). Define a generalized Riccati substitution

$$\omega(v) = \phi(v)(z^{\Delta^{n-1}}(v))^{\alpha}(z^{\Delta^{n-2}}(v))^{-\beta}r(v), \quad v \in [v_1, \infty)_T.$$

Obviously, $\omega(v) > 0$. Then

$$\begin{aligned} \omega^{\Delta}(v) &= (z^{\Delta^{n-2}}(v))^{-\beta} \phi(t)((z^{\Delta^{n-1}}(v))^{\alpha} r(v))^{\Delta} \\ &\quad + (z^{\Delta^{n-1}}(\sigma(v)))^{\alpha} r(\sigma(v))((z^{\Delta^{n-2}}(v))^{-\beta} \phi(t))^{\Delta} \\ &\leq -(z^{\Delta^{n-2}}(v))^{-\beta} Q(v)z^{\beta}(\delta(v))\phi(v) \\ &\quad + (z^{\Delta^{n-1}}(\sigma(v)))^{\alpha} r(\sigma(v))(z^{\Delta^{n-2}}(v))^{\beta}(z^{\Delta^{n-2}}(v))^{-\beta} \\ &\quad (z^{\Delta^{n-2}}(\sigma(v)))^{-\beta} \phi^{\Delta}(v) \\ &\quad - ((z^{\Delta^{n-2}}(v))^{\beta})^{\Delta} (z^{\Delta^{n-2}}(v))^{-\beta} \phi(v)(z^{\Delta^{n-2}}(\sigma(v)))^{-\beta} \\ &= -(z^{\Delta^{n-2}}(v))^{-\beta} Q(v)\phi(v)z^{\beta}(\delta(v)) \\ &\quad + \phi^{-1}(\sigma(v))\omega(\sigma(v))\phi^{\Delta}(v) \\ &\quad - \phi(v)(z^{\Delta^{n-1}}(\sigma(v)))^{\alpha}((z^{\Delta^{n-2}}(v))^{\beta})^{\Delta} r(\sigma(v)) \\ &\quad (z^{\Delta^{n-2}}(v))^{-\beta}(z^{\Delta^{n-2}}(\sigma(v)))^{-\beta}. \end{aligned} \quad (17)$$

By Lemma 7,

$$\begin{aligned} &(z^{\Delta^{n-2}}(v))^{-\beta} z^{\beta}(\delta(v)) \\ &= (z(\delta(v))(z^{\Delta^{n-2}}(\delta(v))z^{\Delta^{n-2}}(v))^{-1} z^{\Delta^{n-2}}(\delta(v)))^{\beta} \\ &\geq \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{v_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^{\beta} \end{aligned} \quad (18)$$

where $\rho^{n-3}(\delta(v)) \geq v_1$, $v \in [v_2, \infty)_T \subseteq [v_1, \infty)_T$.

Moreover, by Lemma 5, we obtain

$$\begin{aligned} &((\eta^{\Delta^{n-2}}(v))^{\beta})^{\Delta} \\ &= \int_0^1 [\eta^{\Delta^{n-2}}(\sigma(v))h + \eta^{\Delta^{n-2}}(v)(1-h)]^{\beta-1} dh \times \eta^{\Delta^{n-1}}(v)\beta \\ &\geq \begin{cases} z^{\Delta^{n-1}}(v)\beta(z^{\Delta^{n-2}}(v))^{\beta-1} & \text{if } \beta \geq 1, \\ z^{\Delta^{n-1}}(v)\beta(z^{\Delta^{n-2}}(\sigma(v)))^{\beta-1} & \text{if } 0 < \beta < 1. \end{cases} \end{aligned} \quad (19)$$

Using (18) and (19) in (17), we have: If $\beta \geq 1$, then

$$\begin{aligned} &\omega^{\Delta}(v) \\ &\leq -\phi(v) \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{v_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^{\beta} Q(v) \\ &\quad + \phi^{-1}(\sigma(v))\omega(\sigma(v))\phi^{\Delta}(v) \\ &\quad - \beta\phi(v)(z^{\Delta^{n-1}}(\sigma(v)))^{\alpha} z^{\Delta^{n-2}}(\sigma(v))r(\sigma(v)) \\ &\quad (z^{\Delta^{n-2}}(v))^{-1} z^{\Delta^{n-1}}(v)(z^{\Delta^{n-2}}(\sigma(v)))^{-\beta-1} \end{aligned} \quad (20)$$

If $0 < \beta < 1$, then

$$\begin{aligned} &\omega^{\Delta}(v) \\ &\leq -\phi(v) \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{v_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^{\beta} \\ &\quad Q(v) + \frac{\phi^{\Delta}(v)}{\phi(\sigma(v))} \omega(\sigma(v)) \\ &\quad - (z^{\Delta^{n-2}}(\sigma(v)))^{-\beta-1} \phi(v)(z^{\Delta^{n-1}}(\sigma(v)))^{\alpha} \beta z^{\Delta^{n-1}}(v) \\ &\quad (z^{\Delta^{n-2}}(v))^{-\beta} r(\sigma(v))(z^{\Delta^{n-2}}(\sigma(v)))^{\beta}. \end{aligned} \quad (21)$$

Since $z^{\Delta^{n-1}}(v) > 0$ and $((z^{\Delta^{n-1}}(v))^{\alpha} r(v))^{\Delta} < 0$ on $[v_1, \infty)_T$,

$$\begin{aligned} &z^{\Delta^{n-2}}(\sigma(v)) \geq z^{\Delta^{n-2}}(v), \\ &\left(\frac{r(\sigma(v))}{r(v)} \right) z^{\Delta^{n-1}}(\sigma(v)) \leq z^{\Delta^{n-1}}(v), \quad v \in [v_1, \infty)_T. \end{aligned}$$

So, (20), (21) yield

$$\begin{aligned} & \omega^\Delta(v) \\ & \leq -\phi(v) \left(\frac{\int_{t_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{t_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^\beta \\ & \quad Q(v) + \phi^{-1}(\sigma(v)) \omega(\sigma(v)) \phi^\Delta(v) \\ & \quad - (z^{\Delta^{n-1}}(\sigma(v)))^{\alpha+1} \beta (z^{\Delta^{n-2}}(\sigma(v)))^{-\beta-1} \phi(v) \\ & \quad r^{-1/\alpha}(v) r^{(\alpha+1)/\alpha}(\sigma(v)) \end{aligned} \quad (22)$$

Then

$$\begin{aligned} & -r(\sigma(v)) \phi(v) (z^{\Delta^{n-2}}(\sigma(v)))^{-\beta-1} \beta (z^{\Delta^{n-1}}(\sigma(v)))^{\alpha+1} = \\ & -\frac{\beta \phi(v)}{\phi^{(1+\alpha)/\alpha}(\sigma(v)) r^{1/\alpha}(v)} (z^{\Delta^{n-2}}(\sigma(v)))^{\frac{\beta-\alpha}{\alpha}} \omega^{\frac{1+\alpha}{\alpha}}(\sigma(v)). \end{aligned}$$

From (5), then $(z^{\Delta^{n-2}}(v))^\Delta > 0$, $[v_1, \infty)_T$, and $\beta \geq \alpha$, then exist a constant $M > 0$ such that $(z^{\Delta^{n-2}}(\sigma(v)))^{\frac{\beta-\alpha}{\alpha}} \geq M$ on $[v_1, \infty)_T$, hence

$$\begin{aligned} & -r(\sigma(v)) \phi(v) (z^{\Delta^{n-2}}(\sigma(v)))^{-\beta-1} \beta (z^{\Delta^{n-1}}(\sigma(v)))^{\alpha+1} \\ & \leq -\phi^{-(1+\alpha)/\alpha}(\sigma(v)) r^{-1/\alpha}(v) \beta M \phi(v) \omega^{\frac{1+\alpha}{\alpha}}(\sigma(v)). \end{aligned}$$

Using this in (22), we have

$$\begin{aligned} & \omega^\Delta(v) \\ & -\phi(v) \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{t_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^\beta \quad (23) \\ & Q(v) + \phi^{-1}(\sigma(v)) \omega(\sigma(v)) (\phi^\Delta(v))_+ \\ & -M \phi^{-(1+\alpha)/\alpha}(\sigma(v)) \beta r^{-1/\alpha}(v) \phi(v) \omega^{\frac{1+\alpha}{\alpha}}(\sigma(v)) \end{aligned}$$

Letting

$$E = \phi^{-1}(\sigma(v)) (\phi^\Delta(v))_+,$$

$$G = \frac{\beta M \phi(v)}{\phi^{(1+\alpha)/\alpha}(\sigma(v)) r^{1/\alpha}(\sigma(v))}$$

and $w = \omega(\sigma(v))$. Applying the inequality

$$Ew - Gw^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{E^{\alpha+1}}{G^\alpha}, \quad G > 0,$$

we conclude from (23) that

$$\omega^\Delta(v) \leq -Q(v) \phi(v) \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(v))} (h_{n-3}(\delta(v), \sigma(\eta))) \int_{t_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^v \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^\beta \quad (24)$$

Then

$$\begin{aligned} & \int_{v_2}^v [Q(u) \phi(u) \\ & \left(\frac{\int_{v_1}^{\rho^{n-3}(\delta(u))} (h_{n-3}(\delta(u), \sigma(\eta))) \int_{t_1}^{\eta} \frac{1}{r^{1/\alpha}(s)} \Delta s \Delta \eta}{\int_{v_1}^u \frac{1}{r^{1/\alpha}(s)} \Delta s} \right)^\beta \\ & - \frac{\alpha^\alpha r(u) ((\phi^\Delta(u))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1} (\beta M)^\alpha \phi^\alpha(u)}] \Delta u \leq \omega(v_2), \end{aligned}$$

which contradicts (16). This complete the proof.

4 Conclusion

In this paper, applying new technique, some criteria for (1), include comparison theorem and Philos-type oscillation criteria. Our result obtained are new even for the case $T = R$ and $T = Z$.

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