A penny-shaped crack in a functionally graded layer between two half-spaces with different elastic properties

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Abstract. The axisymmetric static problem is considered on a penny-shaped mode I crack in an elastic inhomogeneous isotropic space. Young's modulus of the elastic space material is non-symmetrical with respect to the crack. The procedure is proposed for approximate problem solution and determination of the stress intensity factor.

Introduction

Prediction of inhomogeneous materials behavior subjected to different mechanical actions is constantly receiving attention from researchers. A special place in such studies is given to strength calculations, in particular, to the study of the development of penny-shaped cracks in inhomogeneous media. The classic problem for a penny-shaped crack in a homogeneous isotropic elastic media was first investigated by Sneddon [1] (1946) by dual integral equation method. In 1960-70 this problem for two-layered materials and graded interlayers was considered in works of Sneddon, Erdogan and others [2–5]. The behavior of disk cracks in complex media is remaining in research focus at the present time [6-8].

Earlier [9,10] we investigated behavior of a penny-shaped crack in case of symmetric distribution of material properties in the area adjacent to the crack. This distribution can be observed in real-world applications, but more often one may need to predict behavior of a crack for variation of mechanical characteristics which is arbitrary with respect to the crack plane. The present paper is devoted to the solution of this problem.

1 Problem statement

An axisymmetric static problem for a mode I penny-shaped crack in an elastic inhomogeneous isotropic space is considered. The cylindrical coordinate system (r, φ, z) is connected with the elastic space. The crack having radius R is placed in the $z = 0$ plane and its center coincides with the coordinates origin.

It is assumed that Young's modulus of the half-space material is an arbitrary continuous (or piecewise-continuous) non-symmetrical function of distance from the crack plane:

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$$E(z) = \begin{cases} E_1, & H_1 \leq z < \infty; E_1 f_1(z), & 0 \leq z \leq H_1; \\ E_2 f_2(z), & -H_2 \leq z \leq 0; E_2, & -\infty < z \leq -H_2. \end{cases} \quad (1.1)$$

The conditions are fulfilled

$$\begin{aligned} \min_{z \in (0; \infty)} \Theta_j(z) \geq c > 0, \quad \max_{z \in (0; \infty)} \Theta_j(z) \leq c < \infty, \quad \lim_{z \rightarrow \infty} \Theta_j(z) = \text{const} \\ \Theta_j(z) = \frac{G_j(z)}{1 - \nu_j(z)} = \frac{0.5 E_j(z)}{1 - \nu_j(z)^2} = 2M_j(z) \frac{\Lambda_j(z) + M_j(z)}{\Lambda_j(z) + 2M_j(z)}, j = 1, 2 \end{aligned} \quad (1.2)$$

The crack faces are subjected to distributed pressure $p(r)$ from the inside of crack. Due to symmetry of the material elastic properties in the crack plane it is considered that stresses, strains and deformations are not dependent on an angular coordinate φ . This leads to the following system of mixed boundary value problems for two half-spaces. The boundary conditions are:

$$\begin{aligned} \tau_{j,rz}(r, 0) = 0, \quad 0 < r < \infty; \sigma_{j,z}(r, 0) = (-1)^j p(r), \quad 0 \leq r \leq R; \\ \sigma_{1,z}(r, 0) = \sigma_{2,z}(r, 0), \quad r > R, w_1(r, 0) = w_2(r, 0), \quad r > R, j = 1, 2 \end{aligned} \quad (1.3)$$

At $z = (-1)^{j-1} H_j, j = 1, 2$ the displacements and stresses are matched. Further index $j=1, 2$ is used to specify two adjacent half-spaces.

2 Solution of non-symmetrical crack problem

For graded medium there are following connection between normal stresses $\sigma_{j,z}(r, 0)$ and normal displacements $w_i(r, 0)$ of half-space surfaces:

$$w_j(r, 0) = \Theta_{j,0}^{-1} = \Theta_{j,0}^{-1} \int_0^\infty Q(\alpha) L_j(\alpha) J_0(\alpha r) d\alpha, \quad 0 \leq r < \infty \quad (2.1)$$

where

$$\begin{aligned} q_j(r) = \sigma_{j,z}(r, 0) = \int_0^\infty Q(\alpha) J_0(\alpha r) \alpha d\alpha, \quad Q(\alpha) = \int_0^\infty q(\rho) J_0(\alpha \rho) \rho d\rho, \\ w_j(r) = \int_0^\infty W_j(\alpha) J_0(\alpha r) \alpha d\alpha, \quad W_j(\alpha) = \int_0^\infty w_j(\rho) J_0(\alpha \rho) \rho d\rho, \quad \Theta_{j,0} = \Theta_j(0). \end{aligned}$$

The function $L(\alpha) \equiv 1$ is determined numerically by the modulating functions method [11]. When conditions (1.1), (1.2) are fulfilled it have the following properties:

$$L(\alpha) = A + B|\alpha| + o(\alpha^2), \quad \alpha \rightarrow 0; \quad L(\alpha) = 1 + D|\alpha|^{-1} + o(\alpha^{-2}), \quad \alpha \rightarrow \infty \quad (2.2)$$

For a homogeneous half-space $L(\alpha) \equiv 1$.

It can be shown that taking into account (1.2), (1.3) stresses can be expressed in terms of displacements as

$$\Theta_{j,0} \int_0^\infty W_j(\alpha) \frac{\alpha^2}{L_j(\alpha)} J_0(r\alpha) d\alpha = q_j(r), \quad 0 \leq r < \infty \quad (2.3)$$

Let's consider the case of uniform pressure applied to the crack faces and reduce the problem solution to the system of dual integral equations:

$$\begin{cases} \Theta_{j,0} \int_0^\infty W_j(\alpha) \frac{\alpha^2}{L_j(\alpha)} J_0(r\alpha) d\alpha = p, & 0 \leq r < R, \\ \int_0^\infty W_j(\alpha) J_0(r\alpha) \alpha d\alpha = (-1)^j w(r), & R \leq r < \infty \end{cases} \quad (2.4)$$

Introducing the following dimensionless quantities

$$\alpha H_j = u_j; \quad \lambda_j = \frac{H_j}{R}; \quad r' = \frac{r}{R}; \quad \rho' = \frac{\rho}{R}; \quad \frac{u_j}{\lambda_j} = \alpha; \quad 0 \leq r' \leq 1, j=1,2 \quad (2.5)$$

the system (2.4) can be rewritten in the following way (primes are omitted henceforward):

$$\begin{cases} \Theta_{j,0} \int_0^\infty \tilde{\Delta}_j(\alpha) \frac{\alpha^2}{L_j(\lambda_j \alpha)} J_0(r\alpha) d\alpha = p - \tilde{p}_j(r), & 0 \leq r < 1, \\ \int_0^\infty \tilde{\Delta}_j(\alpha) J_0(r\alpha) \alpha d\alpha = 0, & 1 < r < \infty \end{cases} \quad (2.6)$$

$$\tilde{\Delta}_j(\alpha) = (W_j(\alpha) - (-1)^j \tilde{W}_0(\alpha)), \quad W_0(\alpha) = \int_0^\infty w(r) J_0(r\alpha) r dr,$$

$$\tilde{p}_j(r) = (-1)^j \Theta_{j,0} \int_0^\infty \frac{\alpha^2 \tilde{W}_0(\alpha)}{L_j(\alpha)} J_0(r\alpha) d\alpha$$

Solution of the system (2.6) will be found by iteration procedure. It is assumed that $\tilde{p}_{j,0}(r) = 0$.

Method for solution of systems of kind (2.6) is given in [9, 10]. For kernel transforms from (2.6) satisfying (2.2) conditions this method uses approximation by an expression

$$L_j^{-1}(\lambda_j \alpha) \cong \prod_{i=1}^N \frac{(\alpha^2 + B_{j,i}^2 \lambda_j^{-2})}{(\alpha^2 + A_{j,i}^2 \lambda_j^{-2})} = \prod_{i=1}^N \frac{(\alpha^2 + \tilde{B}_{j,i}^2)}{(\alpha^2 + \tilde{A}_{j,i}^2)} = L_N^{-1}(\lambda_j \alpha) = 1 + \sum_{i=1}^N \frac{G_{j,i} \tilde{A}_{j,i}^2}{\alpha^2 + \tilde{A}_{j,i}^2}$$

Value of $\tilde{W}_0(\alpha)$ can be determined using expression for stresses along the crack line

$$\begin{aligned} p_{1,0}(r) - p_{2,0}(r) &= \int_0^\infty \alpha^2 \left(\frac{\Theta_{1,0} \tilde{\Delta}_1(\alpha)}{L_{1,N}(\alpha)} - \frac{\Theta_{2,0} \tilde{\Delta}_2(\alpha)}{L_{2,N}(\alpha)} \right) J_0(ar) d\alpha = \\ &= \int_0^\infty \alpha^2 \left(\frac{\Theta_{1,0} W_1(\alpha)}{L_{1,N}(\alpha)} - \frac{\Theta_{2,0} W_2(\alpha)}{L_{2,N}(\alpha)} \right) J_0(ar) d\alpha + \\ &+ \int_0^\infty \alpha^2 \left(\frac{\Theta_{1,0}}{L_{1,N}(\alpha)} + \frac{\Theta_{2,0}}{L_{2,N}(\alpha)} \right) \tilde{W}_0(\alpha) J_0(ar) d\alpha, \quad r > 1 \end{aligned} \quad (2.7)$$

According to the problem boundary conditions (1.3) stresses of adjacent half-spaces match along the crack line. Because of that, first integral in the last expression of (2.7), always turn to zero. Thus, applying the Hankel's inverse transform,

$$\begin{aligned}\tilde{W}_0(\alpha) &= \frac{L_1(\alpha\lambda_1)L_2(\alpha\lambda_2)}{\Theta_{1,0}L_2(\alpha\lambda) + \Theta_{2,0}L_1(\alpha\lambda)} \left(\frac{\Theta_{1,0}\tilde{\Delta}_1(\alpha)}{L_{1,N}(\alpha)} - \frac{\Theta_{2,0}\tilde{\Delta}_2(\alpha)}{L_{2,N}(\alpha)} \right) = \\ &= \tilde{\Theta}_{1,0}F_{1,K}(\alpha, \lambda)\tilde{\Delta}_1(\alpha) - \tilde{\Theta}_{2,0}F_{2,K}(\alpha, \lambda)\tilde{\Delta}_2(\alpha)\end{aligned}\quad (2.8)$$

Now value of $\tilde{p}_{j,1}(r)$ can be determined:

$$\begin{aligned}\tilde{p}_{s,1}(r) &= (-1)^s \Theta_{s,0} \int_0^\infty \tilde{W}_0(\alpha) \frac{\alpha^2}{L_s(\alpha)} J_0(r\alpha) d\alpha = \\ &= (-1)^s \sum_{j=1}^2 \frac{L_{j,N}(0)}{\Theta_{j,0}} \tilde{\Theta}_0 \left\{ \sum_{i=1}^N C_{j,i} \tilde{B}_{j,i}^2 I_0(\tilde{B}_{j,i}r) \left[\sum_{k=1}^K \frac{E_k \tilde{H}_k^2}{\tilde{B}_{j,i}^2 - \tilde{H}_k^2} - 1 \right] - \right. \\ &\quad \left. - \sum_{k=1}^K \frac{E_k \tilde{H}_k^2}{e^{\tilde{H}_k}} I_0(\tilde{H}_k r) \left[\frac{1 + \tilde{H}_k}{\tilde{H}_k^2} + \sum_{i=1}^N C_{j,i} \frac{\text{sh}(\tilde{B}_{j,i}) \tilde{H}_k \tilde{B}_{j,i} + \tilde{B}_{j,i}^2 \text{ch}(\tilde{B}_{j,i})}{\tilde{B}_{j,i}^2 - \tilde{H}_k^2} \right] \right\}, r < 1\end{aligned}\quad (2.9)$$

Here the following notation is used:

$$\begin{aligned}F_j(\alpha\lambda) &= \tilde{\Theta}_{j,0}^{-1} \frac{L_1(\alpha\lambda_1)L_2(\alpha\lambda_2)\Theta_{j,0}}{\Theta_{1,0}L_2(\alpha\lambda_2) + \Theta_{2,0}L_1(\alpha\lambda_1)} \frac{1}{L_j(\alpha\lambda_j)} \cong F_{j,K}(\alpha\lambda) = \\ &= \frac{\prod_{k=1}^K \alpha^2 + G_{j,k}^2 \lambda_j^{-2}}{\prod_{k=1}^K \alpha^2 + H_{j,k}^2 \lambda_j^{-2}} = \frac{\prod_{k=1}^K \alpha^2 + \tilde{G}_{j,k}^2}{\prod_{k=1}^K \alpha^2 + \tilde{H}_{j,k}^2} = 1 + \sum_{k=1}^K \frac{E_{j,k} \tilde{H}_{j,k}^2}{\alpha^2 + \tilde{H}_{j,k}^2}, \tilde{\Theta}_{j,0} = \frac{\Theta_{j,0}}{\Theta_{1,0} + \Theta_{2,0}} \\ F(\alpha\lambda) &= \tilde{\Theta}_0^{-1} \frac{L_1(\alpha\lambda_1)L_2(\alpha\lambda_2)\Theta_{1,0}\Theta_{2,0}}{\Theta_{1,0}L_2(\alpha\lambda_2) + \Theta_{2,0}L_1(\alpha\lambda_1)} \cong F_{j,K}(\alpha\lambda), \tilde{\Theta}_0 = \frac{\Theta_{1,0}\Theta_{2,0}}{\Theta_{1,0} + \Theta_{2,0}}\end{aligned}\quad (2.10)$$

The coefficients $C_{j,i}$ are to be determined from the system of equations

$$\sum_{i=1}^N C_{j,i} \frac{\tilde{B}_{j,i}^2 \text{ch}(\tilde{B}_{j,i}) + \tilde{A}_{j,m} \tilde{B}_{j,i} \text{sh}(\tilde{B}_{j,i})}{\tilde{B}_{j,i}^2 - \tilde{A}_{j,m}^2} = -\frac{1 + \tilde{A}_{j,m}}{\tilde{A}_{j,m}^2} m = 1, 2, \dots, N, \quad j = 1, 2 \quad (2.11)$$

The expression $\tilde{p}_{j,1}(r)$ can be then represented in the form of the Fourier – Bessel series :

$$\tilde{p}_{j,1}(r) = p \sum_{k=1}^\infty s_{j,k} J_0(\mu_k r), \quad s_{j,k} = \frac{2}{J_1^2(\mu_k)} \int_0^1 x \tilde{p}_{j,1}(x) J_0(\mu_k x) dx. \quad (2.12)$$

Here $\mu_1, \mu_2, \dots, \mu_n, \dots$ denote positive zeros of the Bessel function $J_0(x)$ in the ascending order.

Finally, the relations for stress intensity factor are obtained:

$$\begin{aligned}
K_I &= \lim_{r \rightarrow 1+0} \sqrt{r-1} p(r) = \\
&= -\frac{\sqrt{2}}{\pi} p \sum_{j=1}^2 \tilde{\Theta}_0 \frac{L_{j,N}(0)}{\Theta_{j,0}} \left(1 - \sum_{i=1}^N C_{j,i} \tilde{B}_{j,i} \operatorname{sh}(\tilde{B}_{j,i}) + \sum_{j=1}^K s_k \frac{L_N(\mu_k)}{L_{j,N}(0)} \right) \quad (2.13)
\end{aligned}$$

It follows that for a mode I crack on the boundary of two different inhomogeneous half-spaces the stress intensity factor is approximately equal to average of stress intensity factors for each of two half-spaces which are determined from the condition that the each half-space joins the same.

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