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Buckling of a composite bar made of a functional-gradient material with an inhomogeneous pre-stress field

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Abstract. Within the 3D nonlinear elasticity we discuss the linear instability of a composite bar made of a functially graded material and having initial stresses. The bar consists of two layers which are inflated for a annular wedge of a circular cylinder. We present the linearized boundary0value problem and obtain its non-trivial solutions. The influence of the material inhomogeneity and the initial stresses are discussed.

1 Introduction

The inhomogeneous materials are very often used in the engineering. A very promising class of nonhomogeneous materials is functionally graded solids and structures, see [1-3] and the reference therein. Another type of inhomogeneity is the distribution of the initial or residual stresses or the presence of prestressed inclusions in the body, see, e.g., [4].

Such inclusions can be formed as a result of various artificial or natural processes in separate parts of the body, examples of which are phase transformations, growth processes, chemical reactions, plastic deformations, etc. Among examples of such structures are thin films deposited on rubber-like substrate [5-7], which are applied for devices of stretchable electronics [8, 9]. For these structures the methods of nonlinear elasticity are very useful [10, 11]. A characteristic peculiarity of nonlinear elastic bodies with prestressed inclusions is that they do not have an unique natural (non-stressed) reference configuration that is the same for the whole body. In a number of cases, a single reference configuration can be chosen in such a way such that it is prestressed for some parts of the body and stess-free for the rest.

The analysis of the stability of a two-layer composite bar made of an incompressible functionally gradient neo-Hookean material is carried out. The bar is symmetrical in thickness. Each layer of the composite strip was obtained as a result of rectification of a sector of a ring with some initial angle. The shear modulus μ is a function of the layer thickness coordinate. After straightening, the sectors are rigidly glued together and the resulting composite ифк is deformed as a single object. Thus, in the composite bar a field of non-uniform preliminary deformations is formed. In this paper, the basics of the static linear stability analysis are presented. The corresponding linearized problem is formulated for a two-layered bar with of inhomogeneous prestresses in each layer.

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Linearized equilibrium equations and boundary conditions, supplemented by corresponding linearized incompressibility condition are solved using semi-analytical technique. Various laws of variation of shear modulus μ are considered. Analysis of the dependence of critical loads on the parameters of the initial angleis carried out. Studies have shown that the initial stress fields may change both the values of the critical loads as well as the shape of the buckling modes.

2 Statement of the problem

2.1 Governing equations

We consider a two-layered elastic bar of thickness $2h$ with other dimensios a and b, all in a reference placement. The layers were manufactured with in inflation of annular wedge of a circular cylinder, see the scheme shown in Fig. 1.

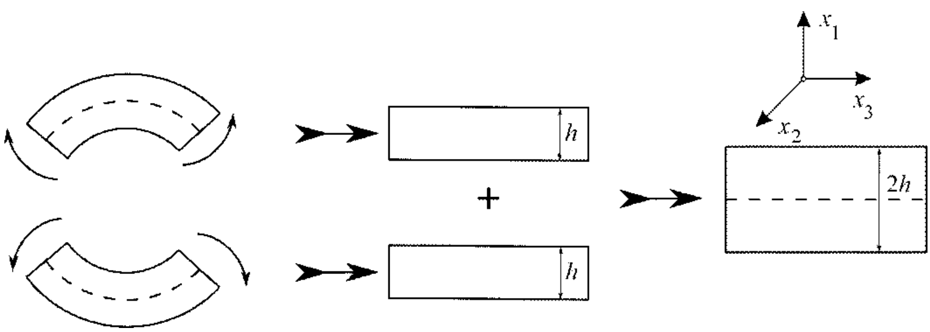

Fig. 1. Rectangular two-layered plate made of two layers after their inflation.

This deformation belongs to the class of so-called universal deformation and is given by [10, 11]

$$X=X(r), \quad Y=\chi\varphi, \quad Z=z, \tag{1}$$

where $\chi>0$ is deformation parameter, X, Y, Z and r, φ, z are referential Cartesian and actual polar coordinates, respectively.

The equilibrium equations without mass forces and constitutive relations of hyperelastic solids are [10]

$$\nabla\cdot\mathbf{P}=\mathbf{0}, \quad \mathbf{P}=\mathbf{P}(\mathbf{F}), \tag{2}$$

where **P** and **F** are the Piola stress tensor and deformation gradient, respectively. Note that here **F** is defined as in [10, 12]. Obviously, **P** and **F** depend on a reference placement. Usually one uses so-called natural that is stress-free reference placement. For materials with prestressed inclusions or layers it is impossible to choose the same natural reference placement, in general, so one needs to consider explicitly few initial placements.

The deformation gradients transform as follows [1-3, 7]

$$\mathbf{F}=\mathbf{A}'\cdot\mathbf{F}', \quad \mathbf{F}':\kappa'\to X, \quad \mathbf{F}'':\kappa''\to X, \quad \mathbf{A}':\kappa'\to K', \quad \mathbf{A}'':\kappa''\to K'', \tag{3}$$

where κ' and κ'' are two initial placements, K' and K'' are two respective intermediate placements, and X is an actual placement. Tensors $\mathbf{A}',\mathbf{A}''$ are deformation gradients from

κ' into K' and κ'' into K'', respectively. With the incompressibility condition these tensors take the form

$$\mathbf{A}' = \frac{dX(r)}{dr}\mathbf{e}_r\mathbf{i}_1 + \frac{\chi}{r}\mathbf{e}_r\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3,$$
$$\mathbf{A}'' = -\frac{dX(r)}{dr}\mathbf{e}_r\mathbf{i}_1 + \frac{\chi}{r}\mathbf{e}_r\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3, \tag{4}$$

where $\mathbf{e}_j$ and $\mathbf{i}_k$ are the base vectors and in polar coordinates in the reference placement and are Cartesian base vectors in the actual placement, respectively, $j = r, \phi, z$, $k = 1,2,3$.

The plate is loaded by normal forces applied to lateral surface $x_3 = 0, a, x_2 = 0, b$. The faces $x_1 = -h$ and $x_1 = h$ are loads-free. Thus, the initial deformed state of the plate is affine and is given by

$$\mathbf{F}_0' = (\lambda_3\lambda_2)^{-1}\mathbf{i}_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3\mathbf{i}_3, \qquad 0 \le x_1 \le h,$$
$$\mathbf{F}_0'' = (\lambda_3\lambda_2)^{-1}\lambda_3\mathbf{i}_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3\mathbf{i}_3, -h \le x_1 \le 0, \tag{5}$$
$$\lambda_3 = \text{const} > 0, \lambda_2 = \text{const} > 0.$$

Here λ_3, λ_2 are stretch parameters.

The linearized boundary conditions take the form corresponding to simple clamping

$$x_3 = 0, a: \quad w_2(x_1, x_2, x_3) = 0, \quad w_1(x_1, x_2, x_3) = 0, \quad \dot{D}_{33} = 0,$$
$$x_2 = 0, b: \quad w_1(x_1, x_2, x_3) = 0, \quad w_3(x_1, x_2, x_3) = 0, \quad \dot{D}_{22} = 0. \tag{6}$$

For the interface between layers we assume the perfect contact.

In what follows we use the constitutive relations of the neo-Hookean incompressible material [1, 8]. The strain energy density and the Piola stress tensor are given by

$$W = \frac{\mu}{2}\left[\text{tr}(\mathbf{F}\cdot\mathbf{F}^T) - 3\right], \quad \mathbf{P} = \mu\mathbf{F} - p\mathbf{F}^{-T}, \tag{7}$$

where $\mu = \mu(x_1)$ is the shear modulus, and p is the pressure, $\mu(x_1) = \mu_0 e^{\gamma|x_1|}$, $\mu_0 = const > 0, \gamma = const > 0$.

The Piola stress tensors with respect to κ' and κ'' are given by

$$\mathbf{P}' = \mu\mathbf{P}'^T \cdot \mathbf{P}' \cdot \mathbf{P}' - p'\mathbf{P}'^{-T},$$
$$\mathbf{P}'' = \mu\mathbf{P}''^T \cdot \mathbf{P}'' \cdot \mathbf{P}'' - p''\mathbf{P}''^{-T}. \tag{8}$$

The linearized equilibrium equations and the linearized incompressibility condition are given by [4, 13]

$$\nabla \cdot \dot{\mathbf{P}} = \mathbf{0}, \tag{9}$$

$$\dot{\mathbf{P}}' = \mu\mathbf{B}' \cdot \nabla\mathbf{w}' - \dot{p}'\mathbf{F}_0'^{-T} + p_0'\mathbf{F}_0'^{-T} \cdot \nabla\mathbf{w}' \cdot \mathbf{F}_0'^{-T}, \quad \mathbf{B}' = \mathbf{A}'^T \cdot \mathbf{A}', \tag{10}$$

$$\dot{\mathbf{P}}'' = \mu\mathbf{B}'' \cdot \nabla\mathbf{w}'' - \dot{p}''\mathbf{F}_0''^{-T} + p_0''\mathbf{F}_0''^{-T} \cdot \nabla\mathbf{w}'' \cdot \mathbf{F}_0''^{-T}, \quad \mathbf{B}'' = \mathbf{A}''^T \cdot \mathbf{A}''. \tag{11}$$

$$\text{tr}(\mathbf{C}_0^{-1} \cdot \nabla\mathbf{w}) = 0. \tag{12}$$

Initial stresses are given by

$$P_{11}' = \mu\lambda_3\lambda_2^{-1}\frac{r^2}{\chi^2} - p_0'\lambda_3\lambda_2, \quad P_{22}' = \mu\lambda_2\frac{\chi^2}{r^2} - p_0'\lambda_2^{-1}, \qquad P_{33}' = \mu\lambda_3 - p_0'\lambda_3^{-1}, \tag{13}$$

$$P_{11}'' = \mu\lambda_3\lambda_2^{-1}\frac{r^2}{\chi^2} - p_0''\lambda_3\lambda_2, \quad P_{22}'' = \mu\lambda_2\frac{\chi^2}{r^2} - p_0''\lambda_2^{-1}, \quad P_{33}'' = \mu\lambda_3 - p_0''\lambda_3^{-1}. \tag{14}$$

The values of p_0' and p_0'' , $\mathrm{X} = \mathrm{X}(r)$ and χ are given

$$p_0' = \mu\lambda_3^{-2}\lambda_2^{-2}\frac{r^2}{\chi^2}, p_0'' = \mu\lambda_3^{-2}\lambda_2^{-2}\frac{r^2}{\chi^2}. \ \mathrm{X}(r) = \frac{r^2}{2\chi} + C. \qquad \chi = \left[\frac{\mu}{3}(r_2^3 - r_1^3)(r_1^{-1} - r_2^{-1})^{-1}\right]^{\frac{1}{4}}.$$

2.2 Infinitesimal superimposed displacements

The vector **w** has the general form

$$\mathbf{w} = w_1(x_1,x_2,x_3)\mathbf{i_1} + w_2(x_1,x_2,x_3)\mathbf{i_2} + w_3(x_1,x_2,x_3)\mathbf{i_3}, \tag{15}$$

where the components of **w** and linearized pressure can be represented as follows [7]

$$\begin{gathered} w_1 = W_1(x_1)\sin\frac{\pi n x_3}{a}\sin\frac{\pi m x_2}{b},\ w_2 = W_2(x_1)\sin\frac{\pi n x_3}{a}\cos\frac{\pi m x_2}{b}, \\ w_3 = W_3(x_1)\cos\frac{\pi n x_3}{a}\sin\frac{\pi m x_2}{b},\ \dot{p} = P(x_1)\sin\frac{\pi n x_3}{a}\sin\frac{\pi m x_2}{b}. \end{gathered} \tag{16}$$

Representing he linearized Piola stress tensors as follows

$$\dot{\mathbf{P}}' = \dot{P}_{ks}'\mathbf{i}_k\mathbf{i}_s, \quad k,s = 1,2,3; \ \dot{\mathbf{P}}'' = \dot{P}_{ks}''\mathbf{i}_k\mathbf{i}_s, \quad k,s = 1,2,3. \tag{17}$$

where

$$\dot{P}_{11}'' = 2\mu\frac{r^2}{\chi^2}\frac{\partial w_{1''}}{\partial x_1} - \dot{p}''\lambda_2\lambda_3, \quad \dot{P}_{12}'' = \mu\frac{r^2}{\chi^2}\left(\frac{\partial w_{2''}}{\partial x_1} + \lambda_2^{-2}\lambda_3^{-1}\frac{\partial w_{1''}}{\partial x_2}\right),$$

$$\dot{P}_{13}'' = \mu\frac{r^2}{\chi^2}\left(\frac{\partial w_{3''}}{\partial x_1} + \lambda_2^{-1}\lambda_3^{-2}\frac{\partial w_{1''}}{\partial x_3}\right), \quad \dot{P}_{21}'' = \mu\frac{\chi^2}{r^2}\frac{\partial w_{1''}}{\partial x_2} + \mu\frac{r^2}{\chi^2}\lambda_2^{-2}\lambda_3^{-1}\frac{\partial w_{2''}}{\partial x_1},$$

$$\dot{P}_{22}'' = \left(\mu\frac{\chi^2}{r^2} + \mu\frac{r^2}{\chi^2}\lambda_2^{-4}\lambda_3^{-2}\right)\frac{\partial w_{2''}}{\partial x_2} - \dot{p}''\lambda_2^{-1}, \quad \dot{P}_{23}'' = \mu\frac{\chi^2}{r^2}\frac{\partial w_{3''}}{\partial x_2} + \mu\frac{r^2}{\chi^2}\lambda_2^{-3}\lambda_3^{-3}\frac{\partial w_{2''}}{\partial x_3},$$

$$\dot{P}_{31}'' = \mu\frac{\partial w_{1''}}{\partial x_3} + \mu\frac{r^2}{\chi^2}\lambda_2^{-2}\lambda_3^{-1}\frac{\partial w_{3''}}{\partial x_1}, \quad \dot{P}_{32}'' = \mu\frac{\partial w_{2''}}{\partial x_3} + \mu\frac{r^2}{\chi^2}\lambda_2^{-3}\lambda_3^{-3}\frac{\partial w_{3''}}{\partial x_2},$$

$$\dot{P}_{33}'' = \left(\mu + mu\frac{r^2}{\chi^2}\lambda_2^{-2}\lambda_3^{-4}\right)\frac{\partial w_{3''}}{\partial x_3} - \lambda_3^{-1}\dot{p}'';$$

we obtain the linear system of ordinary differential equations of first and second-order, which is analysed by semi-analytical methods, such as in [7].

3 Results of instability analysis

First, let us consider the plane problem. The governing equations of the plane problem can be obtained using the following set of parameters: λ_3=1, n=0, and W_3=0. We introduce the stress resultant as a mean value of stresses

$$N_2 = \int_{-h}^{h} P_{22} dx_1.$$

The critical values of N_2 is given in Fig. 2. Here we are restricted ourselves by squared plate.

As a result, Eq. (9) take more simple form which is not presented here due to the lack of space. It is seen that the increase of γ leads to the increase of the absolute value of N_2.

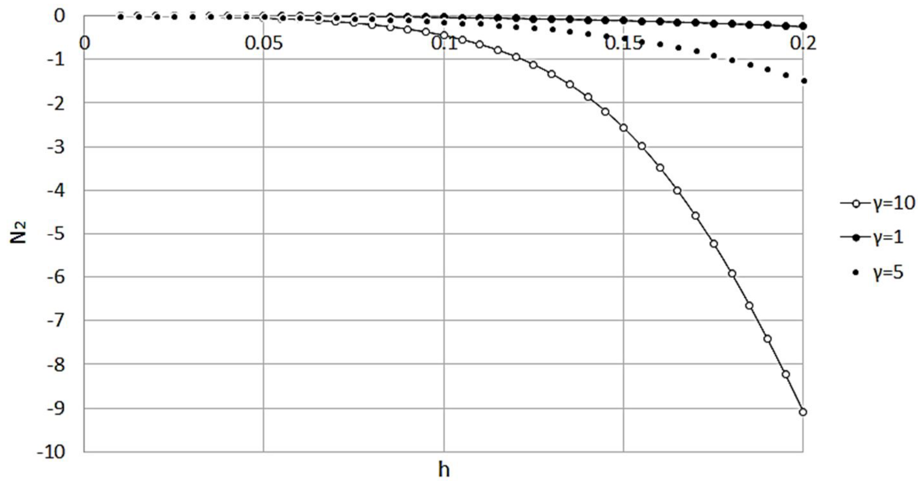

Fig. 2. Critical stress resultant vs. half-thickness *h*.

For spatial problem we consider two stress resultants as loading parameters. We introduce the stress resultants N_k, k=2,3, as integrated over the thickness stresses

$$N_k = \int_{-h}^{h} P_{kk} dx_3.$$

There is a dependence between stretch parameters λ_2 and λ_3 for which the linearized boundary-value problem has nontrivial solutions. This critical dependence between λ_2 and λ_3 are determined numerically. As N_k are functions λ_2 and λ_3 in the plane (N_2, N_3) there critical curves which separate area of stable and instable initial solutions. As stress-free state is stable, the stability area contains the point (N_2, N_3)=(0,0). The critical curves are given in Fig. 3 for m=1 and n=1 which determines the first buckling mode and for different values of γ. Is is shown that the increase of γ leads to stabilizing effect, thtat is to the increase of critical loads.

4 Conclusions

We discussed the instability of the composite elastic bar made of two layers of elastomeric functionally graded material. The layers are initially deformed due to inflation deformation of the same annular wedge but in different direction, so the initial stress field is self-equilibrated. Here use the Treloar model of hyperelasticity with the shear modulus exponentially dependent on the thickness coordinate. We apply here the technique of small deformation superposed on the initial finite deformation. The linearized boundary-value problem was derived and its non-trivial solutions were analysed in dependence on material properties and initial stresses.

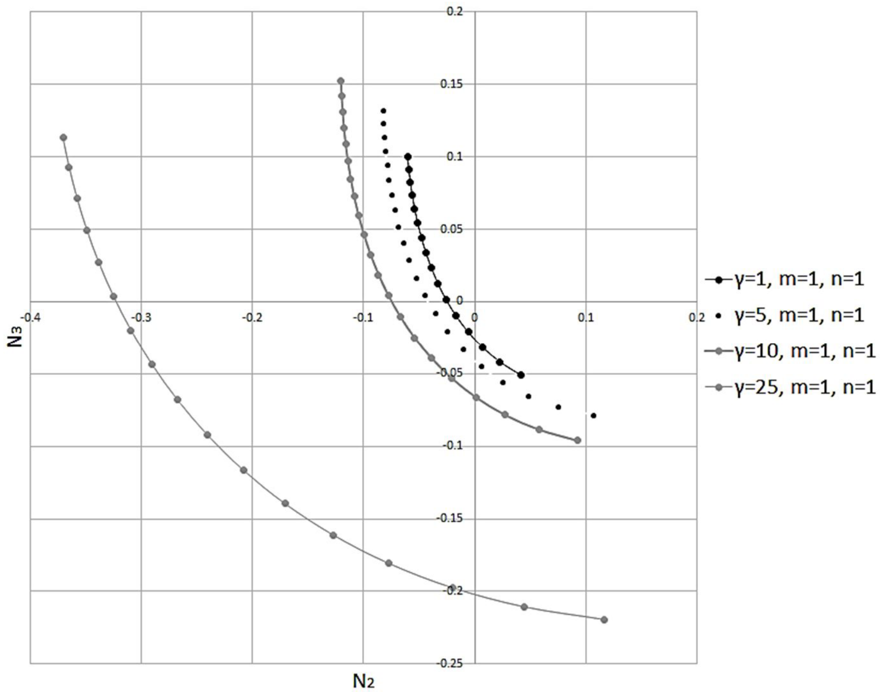

Fig. 3. Critical curves *vs*. γ.

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