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Profiles of critical states in diagnostics of controlled processes

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Abstract. The forecasting problem of critical states of controlled processes is considered. The large deviations of controlled process from some regular state is the basis of forecasting. The main tool for the analysis of large deviations and prediction of critical states is asymptotic methods. The forecasting problem is reduced to the Lagrange-Pontryagin optimal control problem that in some conditions has a unique solution in the form of quasipotential. Based on quasipotential it is possible to build an effective prediction of critical states of controlled process.

1 Introduction

The forecasting problem has been and remains relevant in the analysis of various technological processes in industry and transport. Its significance is strengthened due to the increase of data amount, on the one hand, and the complexity of problems and the need to forecast and prevent all kinds of critical situations, emergencies and modes in controlled technological processes, on the other hand. It is quite natural, at the same time, to consider the problem from general statistical view, assuming as initial data the information about the object dynamics and the perturbations characteristic in the form of random processes.

As is known [1], a random process (or sequence) is a function of continuous time t (or discrete n). This view of the problem proves to be predominant, since it agrees well with Bellman's principle of optimality and is convenient for implementing recurrent calculations for Markov processes. But the requirements of computational convenience for control systems are not limited. The ability of the regulator to perform under conditions of various perturbations become increasingly relevant. In this case the ordinary reflex circuit [2] is not always able to cope. So in this situation, a non-reflex [2] knowledge-based system is needed [3]. For control systems such subject area is usually formed by situations when system trajectories escape to critical levels. This is another view of the solution of the stochastic system: to consider a random process as an event function, that takes values in the space of trajectories. In the general case, specifying the probability in such a space is a complex problem, but it is greatly facilitated by the use of asymptotic methods for analysing large deviations.

We begin consider a simple discrete case.

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2 Random walks and discrete systems

Random walks along the points of the set of all integers are described by a sequence

$$S_n = \xi_1 + \cdots + \xi_n = N_+ - N_-,\ n = 1, 2, \cdots, \tag{1}$$

or an equivalent recurrence equation:

$$S_k - S_{k-1} = \xi_k,\ k = 1, 2, \cdots,\ S_0 = 0, \tag{2}$$

where the random variable ξ_i takes on values $+1, -1$, with probabilities p_+, p_-, (respectively), and the values of the corresponding displacements (in the numbers of steps) are N_+, N_-, $N_+ + N_- = n$.

Like any random process, it can also be represented as a certain probability distribution on a set of paths [5], i.e. solutions φ of the equation

$$\varphi_i - \varphi_{i-1} = v_i, \tag{3}$$

in which the elements of the sequence $\{v_i\}$ belong to the pair $\{+1, -1\}$, moreover, each trajectory is uniquely determined by its sequence from $\{+1, -1\}$.

For (1) (or (2)) we consider the probability of a critical state (CS): $p_n(A,B) = P(A \le S_n \le B)$ for any interval $[A,B]$ that does not contain an average value S_n. For simplicity, we assume A, B, S_n of the same parity. In what follows we denote: $\nu_+ = N_+/n$, $\nu_- = N_-/n$, $a = A/n$, $b = B/n$, and then for the empirical mean value we have: $S_n/n = \nu_+ - \nu_- \to p_+ - p_-$, $n \to \infty$, (law of large numbers), and for the probability CS $P_n(a,b) = P(a \le S_n/n \le b)$:

$$P_n(a,b) = \sum_{a<\frac{N_+-N_-}{n}<b} \frac{n!}{N_+!\,N_-!} p_+^{N_+} p_-^{N_-} = \sum_{a<\frac{N_+-N_-}{n}<b} \frac{n!}{N_+!\,N_-!} e^{N_+ \ln p_+ + N_- \ln p_-}. \tag{4}$$

Hence, taking into account Stirling's formula and its various consequences [5]:

$$\ln n! = n(\ln n - 1) + o(n),\quad n! \approx n^n e^{-n}\sqrt{2\pi n},\quad n \to \infty,$$

$$\frac{n!}{N_+!\,N_-!} \approx e^{-(N_+ \ln \nu_+ + N_- \ln \nu_-)}$$

we have in (4):

$$P_n(a,b) = \sum_{a<\frac{N_+-N_-}{n}<b} e^{-(N_+ \ln \nu_+ + N_- \ln \nu_-) + N_+ \ln p_+ + N_- \ln p_- + o(1)},$$

where, introducing the notation for the relative (Kullback-Leibler-Sanova) entropy [4, 6]:

$$H(\nu, p) = \nu_+ \ln\frac{\nu_+}{p_+} + \nu_- \ln\frac{\nu_-}{p_-}, \tag{5}$$

we get:

$$P_n(a,b) = \sum_{a<\frac{N_+-N_-}{n}<b} e^{-nH(\nu,p)+o(1)} . \qquad (6)$$

For a symmetric walk on a line, when $p_+ = p_- = 1/2$, denote $\nu = \nu_+$, $p = p_+ = 1/2$. Then for entropy (5) we obtain:

$$H(\nu,p) = \nu \ln(2\nu) + (1-\nu)\ln[2(1-\nu)] . \qquad (7)$$

We use the Taylor expansion of the function $h(x) = H(1/2+x, 1/2)$ at zero point, taking into account that, by (7): $h(0)=0$, $h'(0)=0$, $h''(0)=4$. As a result, we get

$$h(x) = \frac{1}{2}(2x)^2 + o(x^2) . \qquad (8)$$

Moreover, in (8) we have: $2x = 2\nu - 1 = S_n / n$. Taking this into account, (8) is valid only for a sufficiently small $x = \nu - 1/2$, therefore we represent the walk as:

$$S_n = S_{n-k} + S_{n-k,n}, \qquad (9)$$

where $S_{n-k,n} = \xi_{n-k+1} + \cdots + \xi_n$. At some k: $2x \approx S_{n-k,n}/n$, $n \to \infty$, and under the additional condition $n-k \to \infty$:

$$(2x)^2 \approx \xi^2_{n-k+1} + \cdots + \xi^2_n . \qquad (10)$$

Returning to the original formula of the probability $p_n(A,B)$ and to the equation of the path (3), we obtain that, to within $1/n$, minimization of relative entropy (7) is equivalent to the discrete optimal control problem for equation (3) with boundary condition $A \le \varphi_n \le B$ and (by virtue of (8), (10)) of the action functional (AF):

$$I = \frac{\gamma}{2} \sum_{i=n-k}^{n} v_i^2 , \ \gamma = \frac{n-k}{n} . \qquad (11)$$

3 Control and checking for continuous stochastic systems

Let it be necessary to select the r - vector of control $U = U(t)$ for object that motions are described by a weakly perturbed differential equation for n-vector of state variable $x = x(t)$:

$$\dot{x} = \alpha(x, U) + \varepsilon\, \sigma(x)\, \dot{w}, \ x(0) = x_0 \in E, \qquad (12)$$

where $\varepsilon > 0$ – small parameter, $\dot{w}$ – k-perturbation vector of the "white noise" type, α, σ – smooth matrix functions, and with respect to σ we assume, as in [8], that σ – is uniformly non-degenerate in E, where E – is operational area: $E \subset R^n$ (the PH condition). Control signals $U(t)$ in (12) are formed in the form of feedback $U = Kx$ so as to ensure a stable equilibrium state χ (attractor) нof the unperturbed system, which is obtained from (12) with $\varepsilon = 0$ and the region of attraction $O_\chi \supset E$. Further, we assume that $\chi = 0$, and the unperturbed system can be linearized at zero:

$$\dot{x} = A_0 x + B_0 U, \tag{13}$$

where matrices A_0, B_0 are partial differential matrices $\alpha(x,U)$ by arguments at zero point, and the quality of the stabilizing control is determined by a closed-loop system $\dot{x} = Ax$, with Hurwitz matrix $A = A_0 + B_0 K$. As a result of this closure in (12) we obtain the system:

$$\dot{\tilde{x}} = a(\tilde{x}) + \varepsilon\sigma(\tilde{x})\dot{w}, \quad \tilde{x}(0) = x_0 \in E. \tag{14}$$

As in the discrete case, together with equation (14) we consider the deterministic system of paths [3]:

$$\dot{\varphi} = a(\varphi) + \sigma(\varphi)v, \quad \varphi(0) = x_0 \in E. \tag{15}$$

In accordance with [8] and [3], denoting $v_t = \sigma^{-1}(\varphi_t)(\dot{\varphi}_t - a(\varphi_t))$, we write the normalized AF:

$$S_{t_0 t_f}(\varphi, v) = \frac{1}{2}\int_{t_0}^{t_f} v^T v\, dt, \tag{16}$$

that takes finite values for absolutely continuous functions on $[t_0, t_f]$. Keeping in mind the typical cases of control, we introduce the domain D, such that $E \subset D \subset O_\chi$ – is some intermediate situation between the regular process ($x \subset E$) and total loss of stability ($x \subset R^n / O_\chi$). Let us write the condition that the path belongs to the set $F = F(D)$ (implementing the event ∂_D, the probability of which is estimated) from a family of functions that are continuous on an interval: $F = \{\varphi \in C_{t_0 t_f}(R^n) : \varphi_{t_f} \in R^n \setminus D\}$. For the set $F = F(D)$ and the system (15) we have the equality [8]:

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln P\{\tilde{x}_t \in R^n \setminus D\} = -\min_{\varphi \in F} S_{t_0 t_f}(\varphi, v), \tag{17}$$

where the functional $S_{t_0 t_f} = S_{t_0 t_f}(\varphi, t)$ is determined in accordance with (16) on the solutions of a controllable system (15), for which we write the boundary condition of the escape to the critical state:

$$\varphi(t_f) \in \Delta \subset \partial_D. \tag{18}$$

For example, the boundary of an area D may be a plane in $R^n : \partial_D : \quad Cx(t_f) - y = 0$.

4 Critical state profiles for linear systems

The solution of the Lagrange-Pontryagin problem (15), (16), (18) in the linear case and for the critical boundary in the form of a plane is determined by the following equations for direct and conjugate variables, and also for optimal control $v_t = \tilde{v}_t = \sigma^T \psi_t$:

$$\dot{\varphi}_t = A\varphi_t + \sigma v_t,\ \ C\varphi(t_f) = y, \tag{19}$$

$$\dot{\psi}_t = -A^T \psi_t,\ \ \psi(t_f) = \psi_f, \tag{20}$$

Substituting the solution of the initial problem (20) into $\tilde{v}_t$ and (19), we obtain a system whose solution $\varphi(t)$ determines the profile of the CS: $x_f = \varphi(t_f)$:

$$\varphi(t) = e^{A(t-t_f)} x_f + J(t)\psi_f, \tag{21}$$

where

$$J(t) = \int_{t_f}^{t} e^{A(t-\tau)} \sigma \sigma^T e^{A^T(t_f-\tau)} d\tau$$ – controllability gramian.

Measures to stabilize the equilibrium state lead to a decrease of the large deviations probability. On the other hand, this makes it possible, when analyzing profiles, to move from finite intervals $[t, t_f]$ to unlimited: $t \to -\infty$. Such a transition is closely connected with the concepts of the attractor and the quasipotential of the system of paths [8].

5 Asymptotically stable linear systems, A-profiles of CS

Assuming the Hurwitz property of the matrix A, which is equivalent (under the PH condition) to the existence of a unique positive definite solution D of Lyapunov equation: $\sigma\sigma^T = -AD - DA^T$, controllability gramian $J(t)$ is expressed through D:

$$J(t) = D e^{A^T(t_f-t)} - e^{A(t-t_f)} D. \tag{22}$$

Substituting (22) into (21), we obtain the following expression for the profile:

$$\varphi(t) = e^{A(t-t_f)} x_f + \ (D e^{A^T(t_f-t)} - e^{A(t-t_f)} D)\psi_f =$$

$$= e^{A(t-t_f)}(x_f - D\psi_f) + D e^{A^T(t_f-t)} \psi_f. \tag{23}$$

The AF determines the quasipotential [8] of the system (19) (or (15)) – the function of the point x and equilibrium state χ:

$$V(\chi, x) = \inf\{\ S_{t_0 t_f}(\varphi) : \varphi \in C_{t_0\ t_f}(R^n), \varphi_{t_0} = \chi, \varphi_{t_f} = x\}.$$

The corresponding extremal $\tilde{\varphi}$, satisfying (15) and leading from the stable equilibrium state (attractor) χ, will be called the A-profile of the state x.

We require in (23): $\lim_{t\to-\infty}\varphi(t)=\chi=0$. This will be done if and only if:

$$x_f - D\psi_f = 0. \tag{24}$$

Solving (24) with respect to conjugate variables: $\tilde{\psi}_f = D^{-1}x_f$ and substituting in (23), we obtain the relation for computing the A-profile:

$$\tilde{\varphi}(t) = De^{A^T(t_f-t)}D^{-1}x_f. \tag{25}$$

Quasipotential of a stable system (19) is expressed through D:

$$V(0,x)=\frac{1}{2}x^T D^{-1}x. \tag{26}$$

In accordance with the Lagrange principle [2], in the necessary conditions of the extremum we have the problem of minimizing $V(0,x_f)$ under restriction $Cx_f - y = 0$, (C – is a full-rank matrix). Its solution gives ($x = x_f$): $x_f = DC^T(CDC^T)^{-1}y$. It remains to substitute this in (25) to obtain the A-profile:

$$\tilde{\varphi}(t) = De^{A^T(t_f-t)}C^T(CDC^T)^{-1}y. \tag{27}$$

We show that neighboring extremals, that is, profiles (23) that do not satisfy condition (24), are exponentially close to the A-profile. In this connection, the diagnostic algorithms should be based, first of all, on the A-profiles estimates. Suppose that in (23) $x_f - D\psi_f = \gamma_f \neq 0$, $\psi_f = \tilde{\psi}_f + \Delta\psi_f$, then $x_f = D\psi_f + \gamma_f$, $\psi_f = \tilde{\psi}_f + \Delta\psi_f$,

$$\varphi(t) = \tilde{\varphi}(t) + J(t)\Delta\psi_f \tag{28}$$

where $\Delta\psi_f = -D^{-1}\gamma_f$, and the function $\tilde{\varphi}(t)$ satisfies (25). In the notation $x_0 = \varphi(t_0)$, $\varphi_0 = \tilde{\varphi}(t_0)$, $J_0 = J(t_0)$, from (28) we obtain: $\Delta\psi_f = J_0^{-1}(x_0 - \tilde{\varphi}_0)$, where the non-degeneracy of the gramian follows from controllability, and with (28) it allows us to calculate the profile emanating from the point (t_0, x_0): $\varphi(t) = \tilde{\varphi}(t) + J(t)\ J_0^{-1}(x_0 - \tilde{\varphi}_0)$.

6 Conclusions

The experience of neurophysiology shows that an effective monitoring system, like any biological system for monitoring the reality and behavior of a biological organism, must have at least two-level [2]: lower, local level - the reflex control system (the first signal system), the upper one - the global control, architecturally similar to the expert system (the second signal system). Such a structure of the control system can be implemented on the basis of asymptotic methods of large deviations analysis that allow ensuring the reliability, functional stability and survivability of monitoring systems. It is shown that computations of the CS profiles should form the basis of such systems, for which in the linear case simple algorithms are proposed, implemented in real time and on-line mode. The diagnostic algorithm is reduced to comparing the current state of the controlled process with the A-profile of the CS. The final profiles are used for clarification. Some examples of the application of the proposed approach to the synthesis of control and diagnostic algorithms and, in general, control systems, can be found in [3, 9].

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