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The reduced modulus of elasticity of a layered half-space

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Abstract. A layered half-space is represented as the sum of parts of homogeneous half-spaces of coating materials and the base material, the rigidity of which corresponds to the rigidities of the coating and the base. Using a rigid model of a layered elastic body loaded with an axisymmetric distributed load, a new solution is obtained for determining its reduced elasticity modulus and the Poisson's ratio depending on the elastic characteristics of the base material, the coating material and the relative thickness of the coating. Expressions are given for the reduced elastic modulus and the Poisson's ratio of a layered body with a two-layer coating.

1 Introduction

To date, the possibility of increasing the life of joints of machine parts due to a change in the design or improvement of materials by optimizing their microstructure is almost exhausted. In this regard, one of the promising areas of increasing operational performance of joints of machine parts, including sealing joints and friction units, is the coating on their working surfaces, or the formation of modified layers based on metals, ceramics, and polymers [1]. Experience in operating friction units and seals with such coatings shows that their antifriction properties and sealing capacity are determined not only by the properties of the coating material but also by its thickness [2]. Known recommendations for choosing the thickness of the coating are based on experimental data, often contradictory. The absence of the theory of contact interaction of rough surfaces through the coating layer does not allow to develop reliable methods for predicting the friction characteristics of tribo-conjugations and seal tightness at the design stage, which requires expensive and labor-intensive experimental tests. One of the directions for solving this problem is to determine the reduced elastic characteristics of solid objects with thin coatings - the elastic modulus and Poisson's ratio.

2 State of the problem

The presence of a coating involves taking into account the change in mechanical properties as a function of the distance to the surface. Within the framework of the theory of elasticity, this means that we must consider an elastic body with varying values of the elastic modulus and Poisson's ratio [3]. Contact problems for bodies with mechanical properties that vary in depth

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have been examined by many researchers [2-7, etc.]. According to the author [3], research methods can be divided into three groups: analytical, numerical and numerically-analytical. However, it is not possible to apply the results obtained to solve practical problems of friction, wear and tightness. We should also note the paper [8], in which an approximate solution of the axisymmetric contact problem is given for an elastic layer of finite thickness, which may be of interest when using polymer coatings.

Engineering methods for solving contact problems on the basis of simplifying hypotheses, for example, the representation of a layered body as a topocomposite-constructions with special mechanical properties, depending on the mechanical properties of the base and coating materials, the thickness of the coating, should be included in a separate group. In [9, 10], it was proposed to use the Hertz theory for this purpose. On the basis of reliable results for the extreme thickness values of the coating, an expression is obtained for the dimensionless elastic geometric parameter Φ connecting the displacement of the surface of the topocomposite with the displacement of the surface of a homogeneous half-space from the base material.

The authors [11-13], with the development of the method on the basis of the rigid model of a layered body, determined the reduced elasticity modulus and the Poisson's ratio for any values of the coating thickness for an axisymmetric loading of a layered half-space.

3 Moving points inside a homogeneous half-space

Consider a homogeneous half-space under loading by an axisymmetric load of the form

$$p(r) = p_0\left(1 - r^2/a^2\right)^{\beta}, \qquad 0 \le r \le a, \tag{1}$$

where $0 \le \beta \le 0{,}5$, $p_0 = p_m(1+\beta)$, p_m is the average pressure, $P = p_m \cdot \pi a^2$.

Following the classical approach based on the application of Boussinescu's potential functions [14], for the displacement of any point along the symmetry axis into a homogeneous half-space for the case of its loading by a distributed load (1):

$$u_z = \frac{1+\nu}{2\pi E}\left[2(1-\nu)\psi - z\frac{d\psi}{dz}\right], \qquad \psi = \iint_s p(r)\frac{1}{R}rdrd\varphi,\ R = \sqrt{r^2+z^2}. \tag{2}$$

Taking into account Eq. (1) and the fact that $\rho = r/a$ и $\bar{z} = z/a$, after integration, we have

$$\psi = \frac{\pi p_0 a}{1+\beta}\cdot\frac{1}{\sqrt{1+\bar{z}^2}}\cdot{}_2F_1\left(\frac{1}{2},\ 1+\beta;\ 2+\beta;\ \frac{1}{1+\bar{z}^2}\right), \tag{3}$$

where ${}_2F_1(a,\ b;\ c;\ x)$ is the Gaussian hypergeometric function

Substituting Eq. (3) into Eq. (2) and taking into account that $\frac{d\psi}{dz} = \frac{d\psi}{ad\bar{z}}$, we get

$$u_z = \frac{p_m a}{E^*}K(\bar{z},\beta,\nu) \tag{4}$$

$$K(\bar{z},\beta,\nu) = \frac{1}{\sqrt{1+\bar{z}^2}}\cdot{}_2F_1\left(\frac{1}{2},1+\beta;2+\beta;\frac{1}{1+\bar{z}^2}\right) - \frac{\bar{z}}{2(1-\nu)}\cdot\frac{d}{d\bar{z}}\left[\frac{1}{\sqrt{1+\bar{z}^2}}\cdot{}_2F_1\left(\frac{1}{2},1+\beta;2+\beta;\frac{1}{1+\bar{z}^2}\right)\right], \tag{5}$$

where $E^* = E/\left(1-\nu^2\right)$. When $\beta = 0.5$, an expression for $K(\bar{z},\beta,\nu)$ can be represented as

$$K(\bar{z},0.5,\nu)=\operatorname{arcctg}\bar{z}+\frac{\nu}{1-\nu}\bar{z}\left(1-\bar{z}\operatorname{arcctg}\bar{z}\right). \tag{6}$$

We simplify the notation by adopting

$$K_i(0,0.5,\nu_i)=K_i(0), \qquad K_i(\bar{\delta}_1,0.5,\nu_i)=K_i(\bar{\delta}_1),$$

As the analysis showed, the function $K(\bar{z})$ to a small extent depends on the values of the Poisson ratio. In this case $K_i(0)=\pi/2$; when $\bar{z}\to\infty$, $K(\bar{z})\to 0$.

4 Elastic characteristics of a layered half-space

Consider a layered elastic half-space (Fig. 1), which consists of a coating of thickness δ_1 with elastic characteristics ν_1 и E_1 and a base material with elastic characteristics ν_0 и E_0 and loaded with an axisymmetric load (1).

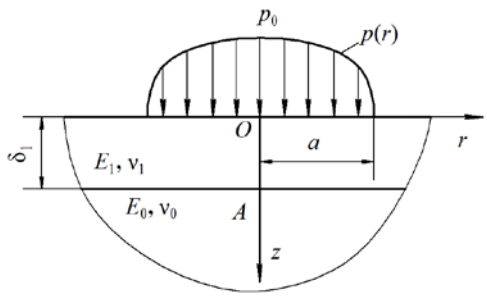

Fig. 1. Loading scheme of a layered half-space

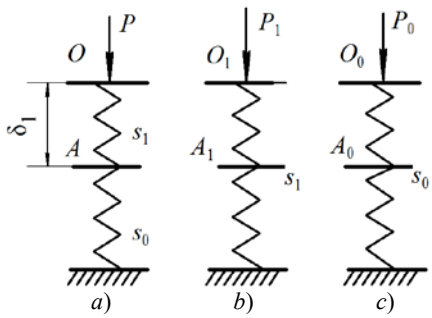

Fig. 2. Modeling a layered half-space:
a) – initial scheme of a layered half-space (Fig.1);
b), *c*) – Stiffness schemes of homogeneous half-spaces

Using the rigid model of a layered body for the reduced elasticity modulus and Poisson's ratio of the topocomposite, the authors of [12, 13] obtained:

$$E_{01}^{*}=E_1^{*}\cdot F_{1\delta}; \tag{7}$$

$$F_{1\delta}=K_1(0)\left[\frac{\left(K_1(0)-K_1(\bar{\delta}_1)\right)^2}{K_{01}(0)-K_{01}(\bar{\delta}_1)}+K_1(\bar{\delta})\frac{K_0(\bar{\delta}_1)}{K_{01}(\bar{\delta}_1)}\cdot I_e\right]^{-1}; \tag{8}$$

$$\nu_{01}=\nu_1+(\nu_0-\nu_1)\frac{1-F_1^{-1}}{1-I_e}. \tag{9}$$

where $I_e=E_1^{*}/E_0^{*}$, $I_e=I\left(1-\nu_0^2\right)/\left(1-\nu_1^2\right)$, $I=E_1/E_0$.

Similar expressions are obtained for the indentation of a rigid spherical indenter into a layered body. As calculations have shown, the results obtained when the layered body is

loaded by an axially symmetric distributed load and when a spherical indenter is indented into it differ by not more than 1%, therefore, in the future it is recommended to use $F_{1\delta}$.

In the derivation of Eqs. (8) and (9), the equality of displacements for homogeneous bodies at point A was used [13]. Retaining the notation of [13], we consider a different approach.

The scheme (Fig. 1) can be represented in the form of Fig. 2a. Then the movements

$$w_\delta = Ps_1, \quad w_a = Ps_0, \qquad w_0 = P(s_1 + s_0),$$

where s_1, s_0 are the rigidity of the layer and the base material, $P = \pi a^2 p_0 / (1+\beta)$.

We consider two homogeneous half-spaces with elastic characteristics ν_1, E_1 и ν_0, E_0, loaded respectively by forces P_1 и P_0 (рис. 2*b* и 2*c*). Forces P_1 и P_0 and accordingly the maximum pressures p_{01} and p_{00} are chosen from the condition of equality of displacements: $\omega_\delta = \omega_1$; $\omega_a = \omega_0$.

For the scheme (Fig. 2*b*)

$$w_\delta = w_{O1} - w_{A1} = u_{zO1} - u_{zA1} = \frac{p_{01} a}{E_1^*}\left[K_1(0,\nu_1) - K_1(\bar{\delta}_1,\nu_1)\right]$$

where $\bar{\delta}_1 = \delta_1 / a$.

For the scheme (Fig. 2*c*)

$$w_{A0} = u_{zA0} = \frac{p_{00} a}{E_0^*} K_0(\bar{\delta}_1).$$

Appropriate rigidity

$$s_1 = \frac{u_{zO1} - u_{zA1}}{P_1}, \qquad s_0 = \frac{u_{zA0}}{P_0}. \tag{10}$$

For the scheme (Fig. 2*a*)

$$w_0 = \frac{P}{P_1}(u_{zO1} - u_{zA1}) + \frac{P}{P_0} u_{zA0}.$$

From the conditions for determining the forces of P_1 and P_0 it follows:

$$P = P_1 \frac{s_1}{s_1 + s_0} + P_0 \frac{s_0}{s_1 + s_0}, \qquad p_0 = p_{01} \frac{s_1}{s_1 + s_0} + p_{00} \frac{s_0}{s_1 + s_0} \tag{11}$$

The value of p_{01} is determined from the conditions for the equality of the compression of a coating of thickness δ_1 for a layered body under load P and a homogeneous material under load P_1:

$$\frac{p_0 a}{E_{01}^*}\left[K_{01}(0) - K_{01}(\bar{\delta}_1)\right] = \frac{p_{01} a}{E_1^*}\left[K_1(0) - K_1(\bar{\delta}_1)\right], \qquad p_{01} = \frac{E_1^*}{E_{01}^*} \frac{K_{01}(0) - K_{01}(\bar{\delta}_1)}{K_1(0) - K_1(\bar{\delta}_1)} p_0. \tag{12}$$

The value of p_{00} is determined from the condition of equality of displacements for $z = \delta_1$ of a layered body under load p_0 and a homogeneous material at z = δ1 under load p_{00}:

$$\frac{p_0 a}{E_{01}^*} K_{01}\left(\bar{\delta}_1\right) = \frac{p_{00} a}{E_0^*} K_0\left(\bar{\delta}_1\right), \qquad p_{00} = \frac{E_0^*}{E_{01}^*} \frac{K_{01}\left(\bar{\delta}_1\right)}{K_0\left(\bar{\delta}_1\right)} p_0 . \tag{13}$$

Taking Eq. (4) into account, we express the Eq. (10) in the form

$$s_1 = \frac{3}{2\pi a E_1^*}\left[K_1(0) - K_1\left(\bar{\delta}_1\right)\right], \qquad s_0 = \frac{3}{2\pi a E_0^*} K_0\left(\bar{\delta}_1\right). \tag{14}$$

Substituting the Eqs. (12) - (14) into Eq. (11), we obtain

$$E_{01}^* = E_1^* \cdot \frac{K_{01}(0)}{K_1(0) - K_1\left(\bar{\delta}_1\right) + K_0\left(\bar{\delta}_1\right) I_e} . \tag{15}$$

Consequently, by analogy with Eq. (8) we have

$$F_1 = \frac{K_{01}(0)}{K_1(0) - K_1\left(\bar{\delta}_1\right) + K_0\left(\bar{\delta}_1\right) \cdot I_e} . \tag{16}$$

For the Poisson ratio, the Eq. (9) should be used.

Comparison of the dependences of the $F_{1\delta}\left(\bar{\delta}_1\right)$ and $F_1\left(\bar{\delta}_1\right)$ with the Eqs. (8) and (16) obtained when the layered body was loaded with the axisymmetric distributed load, showed their insignificant divergence (not more than 3%). Therefore, it is recommended to use Eq. (16) for engineering calculations.

For a two-layer coating with thicknesses δ_1 and δ_2, we obtain:

$$E_{02}^* = E_2^* \cdot F_2 , \qquad F_2 = \frac{K_{02}(0)}{K_2(0) - K_2\left(\bar{\delta}_2\right) + K_{01}\left(\bar{\delta}_2\right) \cdot E_2^* / E_{01}^*} , \tag{17}$$

$$\nu_{02} = \nu_2 + \left(\nu_{01} - \nu_2\right) \frac{1 - F_{02}^{-1}}{1 - E_2^* / E_{01}^*} . \tag{18}$$

For the case of a contact between a smooth rigid sphere and a layered half-space, the approach of the bodies, the contact radius and the maximum pressure are determined by the expressions:

$$w_{01} = w_1 \cdot F_1^{-\frac{2}{3}}, \; a_{01} = a_1 \cdot F_1^{-\frac{1}{3}}, \quad p_{01} = p_1 \cdot F_1^{\frac{2}{3}}, \tag{19}$$

When contacting a rough surface with an elastic layered half-space for a single asperity, the parameter $\bar{\delta}_{1i}$ can be represented in the form

$$\bar{\delta}_{1i} = \frac{\delta_1}{a_r} = \frac{\delta}{a_c} \cdot \frac{a_c}{a_r} = \gamma \cdot \eta_i^{-0.5} \quad , \tag{20}$$

where $\gamma = \delta_1 / a_c$ – relative coating thickness; $\eta_i = a_r^2 / a_c^2$ – relative contact area for a particular unevenness; a_c – radius of the area per single asperity (parameter of discrete model of roughness).

Thus, for each contacting asperities, according to Eqs. (15) and (19), we have:

$$E_{01}^*(\gamma, \eta_i) = E_1^* \cdot F_{1i}(\gamma, \eta_i); \tag{21}$$

where $F_{1i}(\gamma, \eta_i)$ is determined by the Eq. (16) with allowance for Eq. (9).

This approach is simpler and more alternative in determining the relative contact area [15], the gap density [16, 17], and the tightness [18, 19] in the interaction of a rough rough surface through a layer of elastic coating.

5 Conclusion

1. We analytically determined the dependence of the displacement of any point along the symmetry axis inside a homogeneous half-space for the case of its loading by a distributed load of the form (1), which is placed in the basis of the rigid model of a layered elastic body.
2. Using a rigid model of a layered body, a simpler solution is proposed for the reduced elasticity modulus of a layered body. When it is obtained, there is no condition on the equality of displacements for homogeneous bodies at point *A* (Figure 2).
3. The proposed approach makes it possible to determine the reduced elastic modulus and Poisson's ratio of a layered elastic body with a two-layer coating.
4. When using the presented results to describe the contact of a rigid rough surface with a layered half-space, it should be taken into account that each contacting asperity corresponds to its own effective modulus of elasticity.

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