A Generic and Modular Modeling Approach for Automotive Drivetrains Using a Coordinate Partitioning Method

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Abstract. Drivetrain models play an important role in state-of-the-art automotive drivetrain and control concept development. Based on a proposed set of elementary drivetrain components, this article contributes a generic straightforward approach to compute state-space models for various geared drivetrain layouts, including complex hybrid multi-mode transmissions. The modular approach follows Lagrange formalism: The free motion of rigid shafts is subsequently constrained, considering connecting elements like spur and planetary gear sets. The generalized coordinates are determined by a coordinate partitioning method, ensuring a physically reasonable coordinate system. The proposed approach features high potential for automation. This enables drivetrain modeling by non-experts in the field of mechanical engineering.

1 Introduction

Drivetrain models gained huge importance in drivetrain development over the last decades. On the one hand, they significantly contribute to fast product development cycles, enabling virtual testing and validation in MiL, SiL and HiL simulations. On the other hand, they enable development of model-based real-time observers, proposed for example in [1], and control concepts, see for example [2], as well as their parametrization, which have become state of the art due to increasing complexity of drivetrain topologies and rigorous efficiency and comfort requirements.

From the mechanical point of view a drivetrain is a constrained multibody system: The motion of rigid or flexible shafts and their coupling via geared wheels defines its dynamics. From the mathematical point of view this ends up in a system of differential and algebraic equations (DAE) of index=3, like every mechanical system with rigidly connected masses (see [3], page 143). Object-oriented modeling, e.g. Dymola/Modelica (see [4] and [5]), applies index reduction techniques (see for example [6]) in order to reduce the DAE system to a system of ordinary differential equations (ODE), if possible. Specific libraries, e.g. engine and powertrain library in [7], offer a simple application of object-oriented modeling for drivetrains. Publication [8] gives a general overview of the history of object-oriented modeling languages. In contrast to object-oriented modeling, energy-based approaches (Lagrange formalism, Hamilton equations) use the concept of generalized coordinate to compute a non-redundant ODE system. To obtain a set of such generalized coordinates, various methods have been proposed, e.g. coordinate partitioning method in [9], which is called separation of coordinates in [10], and QR decomposition of the constraint's nullspace in [11]. Drivetrain modeling using bond-graphs (for details see [12] and [13]) is possible, but not very practicable due to non-trivial causality issues (see [8]) in case of combined planetary gear sets.

This paper follows the general approach of energy-based modeling taking special advantages of drivetrain peculiarity, namely linearity of the algebraic constraints, due to spur and planetary gear sets. It targets the systematic computation of state-space models, i.e. an ODE system, for geared drivetrains. These state-space models offer a general and reasonable model structure from both mechanical engineering and control engineering perspective. Therefore, they are a suitable modeling interface between mechanical construction of a drivetrain and the modeling for various simulation, control and parametrization purposes as described above. The focus of the proposed modeling approach is on the applicability for all common geared drivetrains, including arbitrarily combined planetary gear sets, and the automation of the approach.

The first key issue is the definition of a set of elementary drivetrain components in section 2 structured into shafts, connectors and inputs. Section 3 summarizes the general application of Lagrangian formalism for drivetrain modeling in two steps: First, the equations of free motion of all shafts are computed. The second step considers the interaction between the single shafts, resulting in the so called equations of constrained motion and furthermore in a state-space model.

In this second step a systematic determination of a set of generalized coordinates, by utilization of the degrees of

Table 1. Drivetrain components and their properties

component	ports	par.	description
rigid shaft	2	J	inertia
		d	velocity dependent damping
flexible shaft	2	k	stiffness
		d	velocity dependent damping
ground	1		
spur gear set	2	z_{P}	num. of teeth on prim. side
		z_{S}	num. of teeth on sec. side
planetary	3 to 3+n_{p}	z_{S}	num. of teeth sun gear
gear set		z_{R}	num. of teeth ring gear
		z_{P_i}	num. of teeth i-th planet gear
		n_{p}	num. planets
clutch	2		
torque input	1		

freedom in the nullspace computation similar to the coordination partitioning method, is used in order to enable physical interpretation of the resulting state-space model. The detailed application of the two step approach is substantiated in section 4. Finally, in section 5, the proposed approach is applied to an exemplary drivetrain topologies.

2 Abstraction - Drivetrain Components

In order to abstract a general automotive drivetrain, first a set of elementary modeling components needs to be defined. This is a crucial point in order to lay the basis for a generic and modular modeling approach. An ill-conceived selection of components inevitably would cause serious difficulties in the modeling algorithm, especially for complex drivetrain topologies. The set is required to be nonredundant, reasonable and sufficient to define all common geared drivetrains: The set is structured into shafts, connectors and inputs. The proposed set is illustrated in Fig. 1. Additionally, Table 1 lists the set and gives all properties of the single components (c.f. section 4).

The rigid shaft component is the only optionally inert component. Therefore, inertia of gear wheels (spur and planetary gear set) have to be modeled via the connected rigid shafts. Furthermore, rigid shafts are the only possible connection between all the other components (including flexible shafts). Losses can be modeled by the velocity dependent damping of rigid shafts or by defining additional torque inputs. The planetary gear set component features a variable number of sets of planets and possible unconnected ports, as discussed in detail in section 4.1.5. In combination with spur gear sets this enables modeling of arbitrary combined planetary gear sets, which play an important role in the construction of automatic transmissions (AT) and multi-mode transmissions (MMT).

Using this set of components, it is possible to abstract a general drivetrain by defining its shafts, the connecting elements and inputs. The result of this abstraction can be either represented list-based or graphically and comprehends all information about the drivetrain topology, which shall be captured by the subsequent modeling approach.

3 Modeling Algorithm - Basic Concept

Direct determination of constrained equations of motion, using for example a Lagrangian function, requires a set of generalized coordinates. The declaration of such a set is rather complicated, especially in the case of complex drivetrain topologies (e.g. MMTs). Consequently, a two step approach is used: In a first step, based on Newton's laws of motion, the proposed algorithm determines the equations of free motion of the all shafts, neglecting all connectors. In a second step it applies Lagrangian formalism (see for example [14] and [15]) to obtain the constrained equations of motion, according to the connectors.

The structure of the proposed set of elementary components, selected in section 2 (see Fig. 1) was chosen in order to facilitate this two step approach. The same approach is proposed in [2] to consider the impact of locking clutches on a drivetrain's dynamic. The following section overviews this standard approach of classical mechanics, focusing on automotive drivetrain topologies.

3.1 Unconstrained equations of motion

Since in drivetrain modeling angular positions are not of interest, the equations of free motion can be described by a system of $n_{\mathrm{x}} = n_{\mathrm{rs}} + n_{\mathrm{fs}}$ ordinary differential equations first order, according to a number of n_{rs} rigid and n_{fs} flexible shafts. The restriction to linear position resp. velocity dependent torques leads to a linear system:

$$\bar{\mathbf{M}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u} \qquad (1)$$
$$\mathbf{y} := \bar{\mathbf{C}}\mathbf{x}, \qquad \text{with} \quad \mathbf{x}^T = \begin{bmatrix}\boldsymbol{\omega}^T & \Delta\boldsymbol{\varphi}^T\end{bmatrix}\mathbf{P}.$$

The vector $\boldsymbol{\omega}$ contains the angular velocities of all rigid shafts and $\Delta\boldsymbol{\varphi}$ the torsions of all flexible shafts. The sequence of elements in the state vector $\mathbf{x}$ is arbitrary but effects the representation of the final modeling results. Therefore, a permutation matrix $\mathbf{P}$ is introduced assigning specific angular velocities and torsions to elements in $\mathbf{x}$. The torque input vector $\mathbf{u}$ consists of n_{pt} propulsion torques $\boldsymbol{\tau}_{\mathrm{P}}$ and n_{c} slipping torques $\boldsymbol{\tau}_{\mathrm{C}}$ transmitted in clutches:

$$\mathbf{u}^T = \begin{bmatrix}\boldsymbol{\tau}_{\mathrm{P}}^T & \boldsymbol{\tau}_{\mathrm{C}}^T\end{bmatrix}. \qquad (2)$$

Therefore, the vector $\mathbf{u}$ consists of $n_{\mathrm{u}} = n_{\mathrm{pt}} + n_{\mathrm{c}}$ entries. Equations (1) furthermore define output quantities $\mathbf{y}$, which are supposed to represent positions of angular velocity sensors. The detailed assembling of the matrices $\bar{\mathbf{M}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}$ is covered in section 4.1.

3.2 Constrained equations of motion

The motion of shafts is constrained by connectors. These constraints form a set of n_{co} linear algebraic equations with respect to the state vector $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{J}_{\mathrm{f}}\mathbf{x} = 0. \qquad (3)$$

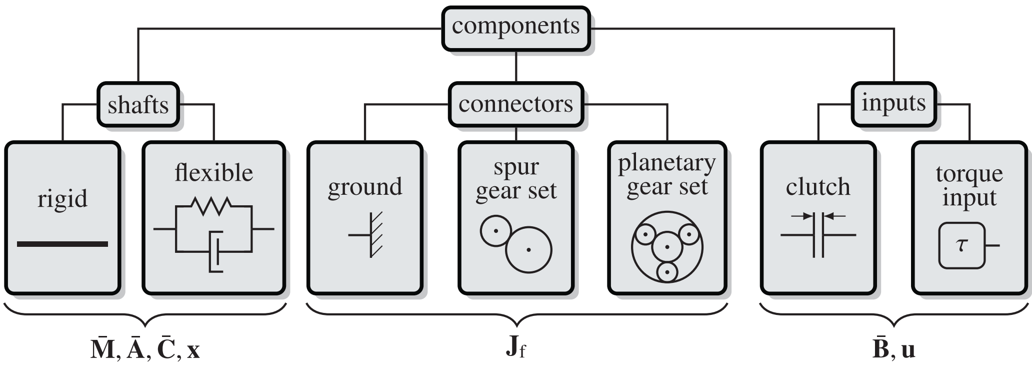

Figure 1. Set of elementary drivetrain components consisting of shafts, connectors and inputs, and their relations to the equations (1) and (3).

$\mathbf{J}_\mathrm{f}$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x})$. In order to constrain the equations of free motion in (1) according to (3), Lagrange formalism introduces the transformation to generalized coordinates $\mathbf{q}$:

$$\mathbf{x} = \mathbf{J}_\mathrm{x}\mathbf{q}. \tag{4}$$

These generalized coordinates $\mathbf{q}$ comply with the holonomic constraint for arbitrary values:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \mathbf{J}_\mathrm{f}\mathbf{J}_\mathrm{x} \equiv 0. \tag{5}$$

Equation (5) implies that $\mathbf{J}_\mathrm{x}$ forms a basis of $\mathbf{J}_\mathrm{f}$'s nullspace or kernel, denoted by $\mathcal{N}(\mathbf{J}_\mathrm{f})$ (see for example [16]):

$$\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \underbrace{\mathbf{J}_\mathrm{f}\mathbf{J}_\mathrm{x}}_{=0}\mathbf{q} \equiv 0 \quad\Rightarrow\quad R(\mathbf{J}_\mathrm{x}) = \mathcal{N}(\mathbf{J}_\mathrm{f}). \tag{6}$$

The expression $R(\mathbf{J}_\mathrm{x})$ is the range or image of $\mathbf{J}_\mathrm{x}$. The rank-nullity theorem ([16]) determines the dimension of $\mathbf{J}_\mathrm{f}$'s nullspace and hence the number of generalizes coordinates:

$$\dim \mathcal{N}(\mathbf{J}_\mathrm{f}) = n_\mathrm{x} - \dim \mathcal{R}(\mathbf{J}_\mathrm{f}) = n_\mathrm{q}. \tag{7}$$

This number n_q equals to the number of the drivetrain's actual mechanical degrees of freedom. For full rank $\mathbf{J}_\mathrm{f}$ (no redundant constraints) this:

$$n_\mathrm{q} = n_\mathrm{x} - n_\mathrm{co}. \tag{8}$$

Since (6) is just a necessary condition for $\mathbf{J}_\mathrm{x}$, it is always possible to inherit the physical meaning of the states $\mathbf{x}$ to the generalized coordinates $\mathbf{q}$. A sufficient condition to achieve this is to assume $\mathbf{J}_\mathrm{x}^T$ to be in reduced row echelon form. In this case the generalized coordinates $\mathbf{q}$ are a selection of the original states $\mathbf{x}$. This approach separates the coordinates into generalized, and redundant coordinates and hence it is a coordinate partitioning method, see e.g. [9] and [10]. Using $\mathbf{J}_\mathrm{x}$, the equations of free motion can be transformed to the constrained equations of motion:

$$\begin{aligned}\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}\mathbf{J}_\mathrm{x}\dot{\mathbf{q}} &= \mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}\mathbf{J}_\mathrm{x}\mathbf{q} + \mathbf{J}_\mathrm{x}^T\bar{\mathbf{B}}\mathbf{u},\\ \mathbf{y} &= \bar{\mathbf{C}}\mathbf{J}_\mathrm{x}\mathbf{q}.\end{aligned} \tag{9}$$

For every physically feasible parametrization the product $\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}\mathbf{J}_\mathrm{x}$ is regular and hence the corresponding state-space model is:

$$\begin{aligned}\dot{\mathbf{q}} &= \underbrace{\left[\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}\mathbf{J}_\mathrm{x}}_{\mathbf{A}}\,\mathbf{q} + \underbrace{\left[\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\bar{\mathbf{B}}}_{\mathbf{B}}\,\mathbf{u},\\ \mathbf{y} &= \underbrace{\bar{\mathbf{C}}\mathbf{J}_\mathrm{x}}_{\mathbf{C}}\,\mathbf{q}.\end{aligned} \tag{10}$$

Summarized, the proposed modeling algorithm has to cover two computational steps:

Step I: Composition of matrices $\bar{\mathbf{M}}$,$\bar{\mathbf{A}}$,$\bar{\mathbf{B}}$,$\bar{\mathbf{C}}$ and $\mathbf{J}_\mathrm{f}$

Step II: Computation of matrix $\mathbf{J}_\mathrm{x}$ (a basis of $\mathbf{J}_\mathrm{f}$'s nullspace, with $\mathbf{J}_\mathrm{x}^T$ in reduced row echelon form) and transform the matrices ($\bar{\mathbf{M}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}} \rightarrow \mathbf{A}, \mathbf{B}, \mathbf{C}$) according to (9).

4 Modeling Algorithm - Detailed Application

In order to support automation of the proposed two step approach, both steps are considered in detail in this section.

4.1 Step I - Composition of unconstrained system matrices

4.1.1 $\bar{\mathbf{M}}$

Inertia matrix $\bar{\mathbf{M}}$ contains the inertia of every rigid shaft on its diagonal, since this is the only optionally inert component. For every state corresponding to the torsion of a flexible shaft, a unity entry is placed on the diagonal:

$$\bar{\mathbf{M}} = \mathbf{P}^T \operatorname{diag}\left(\begin{bmatrix} J_1 & \ldots & J_{n_\mathrm{rs}} & \mathbf{1}_{1\times n_\mathrm{fs}}\end{bmatrix}\right)\mathbf{P}, \tag{11}$$

where J_i is the inertia of the shaft i.

4.1.2 $\bar{\mathbf{A}}$

Dynamic matrix $\bar{\mathbf{A}}$ covers both the velocity dependent damping of all rigid shafts and their interaction over n_{fs} flexible shafts. Since these effects superpose, it is useful to consider $\bar{\mathbf{A}}$ as sum of $n_{\mathrm{fs}}+1$ matrices:

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}_{\mathrm{d}} + \sum_{i=1}^{n_{\mathrm{fs}}} \bar{\mathbf{A}}_{\mathrm{fs},i}. \tag{12}$$

Matrix $\bar{\mathbf{A}}_{\mathrm{d}}$ considers damping of rigid shafts:

$$\bar{\mathbf{A}}_{\mathrm{d}} = \mathbf{P}^T \operatorname{diag}\left(-\begin{bmatrix} d_1 & \ldots & d_{n_{\mathrm{rs}}} & \mathbf{0}_{1\times n_{\mathrm{fs}}} \end{bmatrix}\right)\mathbf{P}. \tag{13}$$

Parameter d_i is the damping constant for velocity dependent damping of shaft i. The matrices $\bar{\mathbf{A}}_{\mathrm{fs},i}$ concern the flexible shaft i, with stiffness k_i and damping constant d_i. Let the corresponding torsion $\Delta\varphi_i$ be assigned to x_m, and the angular velocity of the connecting rigid shafts to x_j and x_l. The following table lists the non zero entries $a^i_{i_1,i_2}$ for $i_1, i_2 \in \{j, m, l\}$ in $\bar{\mathbf{A}}_{\mathrm{fs},i}$:

$$\begin{matrix} & j & m & l \\ j & -d_i & -k_i & d_i \\ m & 1 & 0 & -1 \\ l & d_i & k_i & -d_i \end{matrix} \tag{14}$$

4.1.3 $\bar{\mathbf{B}}$

Input matrix $\bar{\mathbf{B}}$ distributes inputs $\mathbf{u}$ [see (2)] to the states $\mathbf{x}$. For each propulsion torque, $\bar{\mathbf{B}}$ contains an unity column vector, selecting the corresponding angular velocity in state vector $\mathbf{x}$, of the shafts, which is the contact point of the torque. According to Newton's third law of motion, the slipping torque of a clutch acts on both connecting rigid shafts with opposite direction. Therefore in the corresponding column of matrix $\bar{\mathbf{B}}$ a (1)- resp. (−1)-entry is added in the rows assigned to the connecting rigid shafts. Note that exchanging the signs of both entries switches the arbitrarily assigned direction of the transmitted torque.

4.1.4 $\bar{\mathbf{C}}$

Output matrix $\bar{\mathbf{C}}$ assigns states $\mathbf{x}$ equipped with sensors (angular velocity resp. torsion sensors) to output equations.

4.1.5 $\mathbf{J}_{\mathrm{f}}$

The Jacobian matrix $\mathbf{J}_{\mathrm{f}}$ assembles all constraints due to connectors (c.f. section 3.2). Each row in $\mathbf{J}_{\mathrm{f}}$ is assigned to an constraint due to a specific connector. All possible constraints due to specific connector components are investigated in this section:

Ground

The simplest possible constraint $f_g(\mathbf{x})$ occurs, if a rigid shaft (angular velocity x_i) is grounded:

$$f_{\mathrm{g}}(\mathbf{x}) = x_i = \mathbf{f}_{\mathrm{g}}^T \mathbf{x} = 0. \tag{15}$$

All constraints due to n_{g} grounded shafts are assembled in matrix $\mathbf{J}_{\mathrm{f,g}}$:

$$\mathbf{J}_{\mathrm{f,g}}^T := \begin{bmatrix} \mathbf{f}_{\mathrm{g},1} & \ldots & \mathbf{f}_{\mathrm{g},n_{\mathrm{g}}} \end{bmatrix}. \tag{16}$$

Spur gear set

A spur gear set connects two rigid shafts (angular velocities x_i and x_j) by two meshing gear wheels (number of teeth z_i and z_j). The resulting constraint $f_{\mathrm{sg}}(\mathbf{x})$ is:

$$f_{\mathrm{sg}}(\mathbf{x}) = x_i + x_j \frac{z_j}{z_i} = \mathbf{f}_{\mathrm{sg}}^T \mathbf{x} = 0. \tag{17}$$

The constraints dut to n_{sg} spur gear sets are assembled in matrix $\mathbf{J}_{\mathrm{f,sg}}$:

$$\mathbf{J}_{\mathrm{f,sg}}^T := \begin{bmatrix} \mathbf{f}_{\mathrm{sg},1} & \ldots & \mathbf{f}_{\mathrm{sg},n_{\mathrm{sg}}} \end{bmatrix}. \tag{18}$$

Planetary gear set

Planetary gear sets are a key technology in AT and MMT development. The combination of several planetary gear sets to combined planetary gear sets (e.g. a Ravigneaux set), feature different ratios with wide spreading in a compact design. The kinematic analysis of planetary gear sets is well investigated (see for example [17–19]). A structured approach to systematically assemble the corresponding constraints is mandatory, due to the diversity of possible combinations and implementations.

A planetary gear set consists of three basic elements: a sun gear (S), a planet carrier (C), and a ring gear (R) with internal gears. One or several sets of planet gears (P_i) are mounted on the planet carrier. Every set of planets consists of several planets equally distributed radially to increase torque transmission limits. In order to support readability the usage of the term planet in this article implies the existence of a full set of planets. The inner planet gear (P_1) meshes with the sun gear, the outer one (P_n) with the ring gear:

$$\begin{aligned} \mathrm{S} - \mathrm{P}_1&: \quad z_{\mathrm{S}}(\omega_{\mathrm{S}} - \omega_{\mathrm{C}}) \overset{!}{=} -z_{\mathrm{P}_1}\omega_{\mathrm{P}_1}, \\ \mathrm{R} - \mathrm{P}_{n_{\mathrm{p}}}&: \quad z_{\mathrm{R}}(\omega_{\mathrm{R}} - \omega_{\mathrm{C}}) \overset{!}{=} z_{\mathrm{P}_{n_{\mathrm{p}}}}\omega_{\mathrm{P}_{n_{\mathrm{p}}}}. \end{aligned} \tag{19}$$

Fig. 2 illustrates these meshes. If the inner and outer planet is not identical, additionally every planet gear (P_i) meshes with its neighbor (P_{i+1}):

$$\mathrm{P}_i - \mathrm{P}_{i+1}: \quad z_{\mathrm{P}_i}\omega_{\mathrm{P}_i} \overset{!}{=} -z_{\mathrm{P}_{i+1}}\omega_{\mathrm{P}_{i+1}}, \quad i = 1, \ldots, n_{\mathrm{p}} - 1. \tag{20}$$

The well known Willis equation (see for example [20]) is a special combination of (19) and (20):

$$\omega_{\mathrm{S}} + \left[(-1)^{n_{\mathrm{p}}+1}\frac{z_{\mathrm{R}}}{z_{\mathrm{S}}}\right]\omega_{\mathrm{R}} - \left[1 + (-1)^{n_{\mathrm{p}}+1}\frac{z_{\mathrm{R}}}{z_{\mathrm{S}}}\right]\omega_{\mathrm{C}} = 0. \tag{21}$$

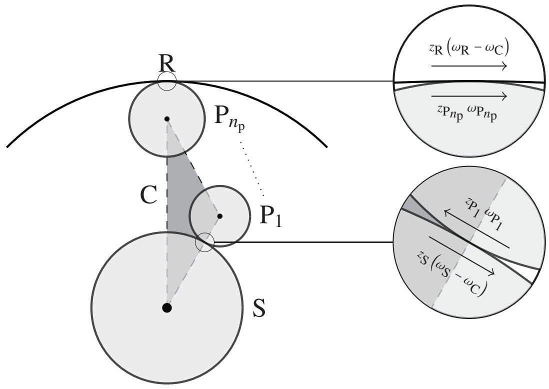

Figure 2. Planetary gear consisting of a sun gear (S), a ring gear (R) and planet gears ($P_1, \dots, P_{n_p}$) mounted on a carrier (C)

Table 2. Coefficients of constraints (24) for planetary gear sets

		k_S	k_C	k_R	$k_{P,i}$
$i = 1, \dots, n_p$	SCP	1	−1	0	$\frac{z_{P_i}}{z_S}(-1)^{i+1}$
	SPR	1	0	−1	$\frac{z_{P_i}}{z_S}(-1)^{i+1} + \frac{z_{P_i}}{z_R}(-1)^{i+n_p}$
	CPR	0	−1	1	$\frac{z_{P_i}}{z_R}(-1)^{i+n_p+1}$

Note that though this equation depends on the number of planet gears n_P, it does not depend on their number of teeth $z_{P_1}, \dots, z_{P_n}$. In the simplest case the constraint due to a planetary gear set combines three rigid shafts (angular velocities x_i, x_j and x_k) connected to the sun gear, the ring gear and the carrier:

$$x_i = \omega_S, \quad x_j = \omega_R, \quad x_k = \omega_C. \tag{22}$$

Consequently, from (21) and (22) follows the constraint:

$$f_{pg}(\mathbf{x}) = \mathbf{f}_{pg}^T \mathbf{x} = 0. \tag{23}$$

A combined planetary gear set consists of usually two single planetary gear sets with a common carrier. This enables meshing between planet gears from different planetary gear sets or coupling of their angular velocities. In order to consider such spur gear set-type constraint between the planet gears, it is necessary determine the angular velocity of these planet gears relatively to the angular velocity of the carrier. From (19) and (20) three possible constraints follow, which are mathematically equivalent, and of the type

$$k_S \omega_S + k_R \omega_R + k_C \omega_C + k_{P,i} \omega_{P_i} = 0. \tag{24}$$

Table 2 lists the coefficients $k_S, k_R, k_C, k_{P,i}$ of all three possibilities. Additionally, in case of such combined planetary gear sets possibly either the sun or the ring gear of one of the planetary gear sets is mechanically not implemented. If so, there is only one appropriate constraint (24) according to Table 2, which does not involve the respective gear. Furthermore, in this case, it is not an additional constraint to (23) but it replaces (23).

Summarized, the type and the number of constraints in case of a planetary gear set depends on the actual mechanical implementation. Since the number of mechanical degrees of freedom of a planetary gear set is always two, the total number of rigid shafts n_{pg,s_i} connected to the single planetary gear set i determines its total number of constraints n_{pg,c_i}:

$$n_{pg,c_i} = n_{pg,s_i} - 2. \tag{25}$$

Consequently, matrix $\mathbf{J}_{f,pg}$, which assembles all constraint due to planetary gear sets, consists of $n_{pg,c}$ vectors,

$$\mathbf{J}_{f,pg}^T := \begin{bmatrix} \mathbf{f}_{pg,1} & \dots & \mathbf{f}_{pg,n_{pg,c}} \end{bmatrix}, \tag{26}$$

with:

$$n_{pg,c} = \sum_{i=1}^{n_{pg}} n_{pg,s_i} - 2n_{pg}. \tag{27}$$

According to (16), (18) and (26) the complete constraint matrix structures as follows:

$$\mathbf{J}_f^T := \begin{bmatrix} \mathbf{J}_{f,g}^T & \mathbf{J}_{f,sg}^T & \mathbf{J}_{f,pg}^T \end{bmatrix}. \tag{28}$$

The total number of constraints n_{co} is:

$$n_{co} = n_g + n_{sg} + n_{pg,c}. \tag{29}$$

4.2 Step II - Computation of a Nullspace

Nullspace computation, $R(\mathbf{J}_x) = \mathcal{N}(\mathbf{J}_f)$, is a well investigated topic of linear Algebra (see for example[16]), but, as already mentioned in section 3.2, it is not unique. This degree of freedom is used to inherit physical meaning to the generalized coordinates from the states $\mathbf{x}$, requiring $\mathbf{J}_x^T$ to be in reduced row echelon form. Therefore, the generalized coordinates $\mathbf{q}$ are a unique selection of the states $\mathbf{x}$:

$$\mathbf{q}^T = \begin{bmatrix} q_1 & \dots & q_{n_q} \end{bmatrix} = \begin{bmatrix} x_{i_1} & \dots & x_{i_{n_q}} \end{bmatrix} \tag{30}$$

Due to the structure of a matrix in reduced row echelon form, the general task of determining $\mathbf{q}$ is to pick n_q linear independent states, with respect to $\mathbf{J}_f \mathbf{x} = \mathbf{0}$, from top to bottom of state vector $\mathbf{x}$. In appendix 6 these selection process described in words is stated as optimization problem. The sequence of states in $\mathbf{x}$ is purely entirely defined by the permutation matrix $\mathbf{P}$, see (1). Therefore $\mathbf{P}$ can be used to prioritize the members of the vector $\begin{bmatrix} \boldsymbol{\omega}^T & \Delta\boldsymbol{\varphi}^T \end{bmatrix}$, in order to be part of the generalized coordinates. Consequently, the first state x_1, always becomes a generalized coordinate (q_1). Note that different $\mathbf{P}$ and hence different $\mathbf{q}$ leads to the representation of the unique modeling result in different coordinates. State transformations can be used afterwards to transform the state-space model to different coordinates, corresponding to a different set of $\mathbf{q}$.

In order to exemplify the selection of a set of generalized coordinates it is useful to consider a simple example:

$$\boldsymbol{\omega}^T = \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{J}_f = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{31}$$

$$\Rightarrow \mathbf{x}^T = \begin{bmatrix} \omega_2 & \omega_1 \end{bmatrix} \tag{32}$$

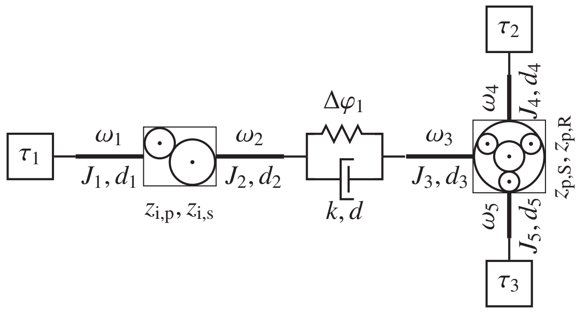

Figure 3. Exemplary drivetrain abstraction: conventional drivetrain including differential

Since $n_x = 2$ and $n_{co} = 1$ the motion of the constrained system can be described by one single generalized coordinate q. A possible choice is

$$q = \frac{\sqrt{2}}{2}\omega_1. \tag{33}$$

Although this choice forms an orthogonal basis of $\mathbf{J}_f$'s nullspace,

$$\mathbf{J}_x^T = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \tag{34}$$

there is no physical motivation for this choice. Determination of reduced row echelon form delivers:

$$\mathbf{J}_x^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad \Rightarrow \quad q = \omega_2. \tag{35}$$

Note that the possible choice $q = \omega_1$ does not reflect state priority of matrix $\mathbf{P}$ according to (31).

Once the $\mathbf{J}_x$ is computed the transformation of the state-space matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is straight forward. The dimensions of all matrices are listed in (50) and the used symbol explanations in Table 3 in Appendix 6.

5 Example

In this final section the proposed two step approach is applied to a simple, exemplary drivetrain topology. Fig. 3 shows the abstracted conventional drivetrain, including a differential, using the components defined in section 2. It consists of 5 rigid shafts, $\boldsymbol{\omega}^T = \begin{bmatrix} \omega_1 & \dots & \omega_5 \end{bmatrix}$, and 1 flexible shaft, $\Delta\boldsymbol{\varphi}^T = \begin{bmatrix} \Delta\varphi_1 \end{bmatrix}$. Permutation matrix $\mathbf{P}$ is used to give high priority to the the angular velocities $\omega_1, \omega_4, \omega_5$, representing rotational speed of the engine and the wheels:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

Consequently, the state vector $\mathbf{x}$ is:

$$\mathbf{x}^T = \begin{bmatrix} \omega_1 & \omega_4 & \omega_5 & \omega_2 & \omega_3 & \Delta\varphi_1 \end{bmatrix}. \tag{37}$$

These angular velocities shall furthermore be equipped with sensors. Composition of the matrices $\bar{\mathbf{M}}, \bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}$ follows the procedure explained in section 4.1:

$$\bar{\mathbf{M}} = \mathbf{P}^T \operatorname{diag}\left(\begin{bmatrix} J_1 & J_2 & J_3 & J_4 & J_5 & 1 \end{bmatrix}\right)\mathbf{P}, \tag{38}$$

$$\bar{\mathbf{A}}_d = \mathbf{P}^T \operatorname{diag}\left(\begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 & 1 \end{bmatrix}\right)\mathbf{P}, \tag{39}$$

$$\bar{\mathbf{A}}_{fs} = \mathbf{P}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -d & d & 0 & 0 & -k \\ 0 & d & -d & 0 & 0 & k \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \mathbf{P}, \tag{40}$$

$$\bar{\mathbf{B}}^T = \bar{\mathbf{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

The considered drivetrain features two connectors: a spur gear set and a planetary gear set. The planetary gear set is connected at sun gear, ring gear and carrier. Consequently, Willis equation, see (21), is sufficient to state the kinematic relation. Matrix $\mathbf{J}_f$, hence, consists of two rows:

$$\mathbf{J}_f = \begin{bmatrix} 1 & 0 & 0 & \frac{z_{i,s}}{z_{i,p}} & 0 & 0 \\ 0 & 1 & \frac{z_{p,S}}{z_{p,R}} & 0 & -\left(1 + \frac{z_{p,S}}{z_{p,R}}\right) & 0 \end{bmatrix}. \tag{42}$$

A differential is a special implementation of a planetary gear set with $z_{p,S} = z_{p,R}$. This further simplifies $\mathbf{J}_f$:

$$\mathbf{J}_f = \begin{bmatrix} 1 & 0 & 0 & \frac{z_{i,s}}{z_{i,p}} & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \end{bmatrix}. \tag{43}$$

The required basis for $\mathcal{N}(\mathbf{J}_f)$ is:

$$\mathbf{J}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{z_{i,p}}{z_{i,s}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{44}$$

This determines the generalized coordinates $\mathbf{q}^T = \begin{bmatrix} \omega_1 & \omega_4 & \omega_5 & \Delta\varphi_1 \end{bmatrix}$ and leads to the transformed system matrices:

$$\mathbf{J}_x^T \bar{\mathbf{M}} \mathbf{J}_x = \begin{bmatrix} J_1 + J_2 \frac{z_{i,p}^2}{z_{i,s}^2} & 0 & 0 & 0 \\ 0 & \frac{J_3}{4} + J_4 & \frac{J_3}{4} & 0 \\ 0 & \frac{J_3}{4} & \frac{J_3}{4} + J_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{45}$$

$$\mathbf{J}_x^T \bar{\mathbf{A}} \mathbf{J}_x = \begin{bmatrix} \frac{-d_1 z_{i,s}^2 - (d_2+d) z_{i,p}^2}{z_{i,s}^2} & -\frac{d}{2}\frac{z_{i,p}}{z_{i,s}} & -\frac{d}{2}\frac{z_{i,p}}{z_{i,s}} & k\frac{z_{i,p}}{z_{i,s}} \\ -\frac{d}{2}\frac{z_{i,p}}{z_{i,s}} & \frac{-d-d_3-4d_4}{4} & -\frac{d}{4} - \frac{d_3}{4} & \frac{k}{2} \\ -\frac{d}{2}\frac{z_{i,p}}{z_{i,s}} & -\frac{d}{4} - \frac{d_3}{4} & \frac{-d-d_3-4d_5}{4} & \frac{k}{2} \\ -\frac{z_{i,p}}{z_{i,s}} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}, \tag{46}$$

$$\mathbf{J}_{\mathrm{x}}^{T}\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

The final state-space matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not explicitly illustrated due to space limitations:

$$\mathbf{A} = \left(\mathbf{J}_{\mathrm{x}}^{T}\mathbf{M}\mathbf{J}_{\mathrm{x}}\right)^{-1}\mathbf{J}_{\mathrm{x}}^{T}\bar{\mathbf{A}}\mathbf{J}_{\mathrm{x}}, \tag{48}$$

$$\mathbf{B} = \left(\mathbf{J}_{\mathrm{x}}^{T}\mathbf{M}\mathbf{J}_{\mathrm{x}}\right)^{-1}\mathbf{J}_{\mathrm{x}}^{T}\bar{\mathbf{B}} \quad \text{and} \quad \mathbf{C} = \bar{\mathbf{C}}\mathbf{J}_{\mathrm{x}}. \tag{49}$$

6 Conclusion and Outlook

A generic and modular modeling algorithm in state-space for geared automotive drivetrains has been proposed. Based on a compact set of elementary drivetrain components the article contributes a systematic determination of physically meaningful generalized coordinates, similar to coordinate the partitioning method, and hence a straight forward approach for the computation of a physically reasonable state-space model. These state-space models have a wide field of application in automotive industry, including for example parametrization of simulation models, drivetrain control concepts and observers. In combination with a graphical user interface (a first concept is shown in Fig. 4) this approach is predestined for full automation and consequently enables application by non-experts in the field of mechanical engineering. Future work will focus on a straight forward approach to use these state-space models for an automated drivetrain analysis. Such an analysis concerns for example transmission modes, gear ratios and gear shift maps. Symbolic representation of the state-space model enables decisions for selective hardware modifications in an early stage of the development process. This, furthermore, opens doors for future work on various drivetrain optimization purposes, e.g. optimization of the general layout or of number of teeth of the gear wheels.

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Table 3. List of symbols

symbol	description
n_{x}	number of mechanical degrees of freedom without constraints
n_{q}	number of actual mechanical degrees
n_{rs}	number of rigid shafts
n_{fs}	number flexible shafts
n_{sg}	number of spur gear sets
n_{pg}	number of planetary gear sets
$n_{\mathrm{pg,s}_i}$	number of shafts connected to a single planetary gear set
$n_{\mathrm{pg,c}}$	total number of shafts connected to all planetary gear sets
$n_{\mathrm{pg,c}_i}$	number of constraints due to a single planetary gear set
$n_{\mathrm{pg,c}}$	total number of constraints due to all planetary gear sets
n_{co}	total number of constraints
n_{g}	number of grounds
n_{c}	number of clutches
n_{pt}	number of propulsion torques
n_{u}	number of input torques
n_{y}	number of sensors
ω_{S}	angular velocity sun gear
ω_{R}	angular velocity ring gear
ω_{C}	angular velocity carrier gear
ω_{P_i}	relative angular velocity of the *i*-th set of planet gears relative to the carrier
z_{S}	teeth count sun gear
z_{R}	teeth count ring gear
z_{P_i}	teeth count of the *i*-th set of planet gears
n_{p}	number of planet groups

Appendix A

Matrix dimension:

$$\begin{aligned}
\bar{\mathbf{M}}, \bar{\mathbf{A}} &\in \mathbb{R}^{n_{\mathrm{x}}\times n_{\mathrm{x}}} = \mathbb{R}^{(n_{\mathrm{rs}}+n_{\mathrm{fs}})\times(n_{\mathrm{rs}}+n_{\mathrm{fs}})} \\
\bar{\mathbf{B}} &\in \mathbb{R}^{n_{\mathrm{x}}\times n_{\mathrm{u}}} = \mathbb{R}^{(n_{\mathrm{rs}}+n_{\mathrm{fs}})\times(n_{\mathrm{pt}}+n_{\mathrm{c}})} \\
\bar{\mathbf{C}} &\in \mathbb{R}^{n_{\mathrm{y}}\times n_{\mathrm{x}}} = \mathbb{R}^{n_{\mathrm{y}}\times(n_{\mathrm{rs}}+n_{\mathrm{fs}})} \\
\mathbf{J}_{\mathrm{f}} &\in \mathbb{R}^{n_{\mathrm{co}}\times n_{\mathrm{x}}} = \mathbb{R}^{(n_{\mathrm{sg}}+n_{\mathrm{pg,c}}+n_{\mathrm{g}})\times(n_{\mathrm{rs}}+n_{\mathrm{fs}})} \\
\mathbf{J}_{\mathrm{x}} &\in \mathbb{R}^{n_{\mathrm{q}}\times n_{\mathrm{x}}} = \mathbb{R}^{(n_{\mathrm{rs}}+n_{\mathrm{fs}}-n_{\mathrm{co}})\times(n_{\mathrm{rs}}+n_{\mathrm{fs}})} \\
\mathbf{A} &\in \mathbb{R}^{n_{\mathrm{q}}\times n_{\mathrm{q}}} = \mathbb{R}^{(n_{\mathrm{x}}-n_{\mathrm{co}})\times(n_{\mathrm{x}}-n_{\mathrm{co}})} \\
\mathbf{B} &\in \mathbb{R}^{n_{\mathrm{q}}\times n_{\mathrm{u}}} = \mathbb{R}^{(n_{\mathrm{x}}-n_{\mathrm{co}})\times(n_{\mathrm{pt}}+n_{\mathrm{c}})} \\
\mathbf{C} &\in \mathbb{R}^{n_{\mathrm{y}}\times n_{\mathrm{q}}} = \mathbb{R}^{n_{\mathrm{y}}\times(n_{\mathrm{x}}-n_{\mathrm{co}})}
\end{aligned} \tag{50}$$

Appendix B

The reduced row echelon form requirement implies that the sum of the indices $i_1, \dots, i_{n_{\mathrm{q}}}$ in

$$\mathbf{q}^{T} = \begin{bmatrix} q_1 & \dots & q_{n_{\mathrm{q}}} \end{bmatrix} = \begin{bmatrix} x_{i_1} & \dots & x_{i_{n_{\mathrm{q}}}} \end{bmatrix} \tag{51}$$

is minimal. Therefore, the unique selection of generalized coordinates is equivalent to the solution of the following optimization problem, which consequently is a mathematical statement of the reduced row echelon form require-

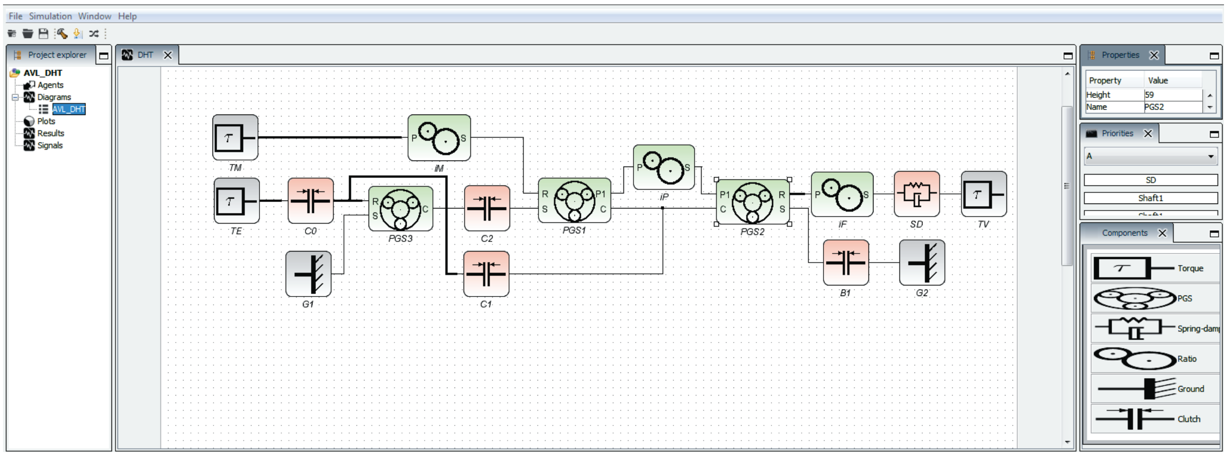

Figure 4. Concept for a graphical user interface (exemplary drivetrain topology: AVL - Dedicated Hybrid Transmission)

ment:

$$\min_{i_1,\dots,i_{n_\mathrm{q}}} \quad \sum_{i=1}^{n_\mathrm{q}} i_i \tag{52}$$

$$\text{s.t.:} \quad \mathbf{x}^T = \begin{bmatrix} x_1 & \dots & x_{n_\mathrm{x}} \end{bmatrix}$$

$$\begin{bmatrix} x_{i_1} & \dots & x_{i_{n_\mathrm{q}}} & \bar{\mathbf{q}}^T \end{bmatrix} = \mathbf{x}^T \mathbf{P}_\mathrm{q}$$

$$\mathbf{J}_\mathrm{f} \mathbf{P}_\mathrm{q} \begin{bmatrix} \mathbf{I}_{n_\mathrm{q}} \\ \mathbf{R} \end{bmatrix} = \mathbf{0},$$

$\mathbf{P}_\mathrm{q} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ is a permutation matrix, $\bar{\mathbf{q}}$ assembles the remaining coordinates, $\mathbf{I}_{n_\mathrm{q}}$ is an identity matrix of size $(n_\mathrm{q} \times n_\mathrm{q})$ and $\mathbf{R} \in \mathbb{R}^{(n_\mathrm{x}-n_\mathrm{q}) \times n_\mathrm{q}}$ is an arbitrary matrix. The according basis to $\mathbf{J}_\mathrm{f}$'s nullspace is:

$$\mathbf{J}_\mathrm{x} = \mathbf{P}_\mathrm{q} \begin{bmatrix} \mathbf{I}_{n_\mathrm{q}} \\ \mathbf{R} \end{bmatrix} \tag{53}$$

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