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A Method of Calculating Principal Stress Trajectories in Powder and Porous Materials Obeying a Piece-wise Linear Yield Criterion

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Abstract. The present paper deals with the system of equations comprising the pyramid yield criterion together with the stress equilibrium equations under plane strain conditions. The stress equilibrium equations are written relative to a coordinate system in which the coordinate curves coincide with the trajectories of the principal stress directions. The general solution of the system is found giving a relation connecting the two scale factors for the coordinate curves. This relation is used for developing a method for finding the mapping between the principal lines and Cartesian coordinates with the use of a solution of a hyperbolic system of equations. In particular, the mapping between the principal lines and Cartesian coordinates is given in parametric form with the characteristic coordinates as parameters.

1 Introduction

In the case of rigid perfectly plastic solids, several efficient methods that utilize this or that property of special coordinate systems are used for solving plane strain boundary value problems. Examples are characteristic coordinates and Mikhlin's coordinates [1, 2]. An important relation between the scale factors of a principal line coordinate system has been derived in [3]. Using this property it is possible to develop an efficient method of calculating principal stress trajectories. This has been demonstrated in [4] where the Mohr-Coulomb yield criterion has been adopted. The material model used in [3] is obtained as a special case. All of the aforementioned methods for smooth yield criteria (or for a smooth portion of piece-wise smooth yield criteria). However, in the case of the pyramid yield criterion used for powder and porous materials [5] plane strain deformation occurs at an edge of the yield surface. Therefore, the aforementioned methods are not applicable. The method of Mikhlin's coordinates has been generalized on the pyramid yield criterion in [6]. In the present paper, a method for finding principal stress trajectories is proposed. It is known that the use of principal lines coordinate systems proves to be advantageous [7, 8]. In particular, a principal line theory of axially symmetric plastic deformation has been developed in [9] for the face regime of the Tresca yield criterion and its associated flow rule.

2 Geometry of principal stress trajectories

The pyramid yield criterion proposed in [5] as a generalization of Tresca's yield criterion on powder and porous materials reads

$$\frac{\left|\sigma_i-\sigma_j\right|}{2\tau_s}+\frac{\left|\sigma\right|}{p_s}=1,\quad i,j=1,2,3,\quad i\neq j. \tag{1}$$

Here σ_1, σ_2 and σ_3 are the principal stresses, σ is the hydrostatic stress, τ_s is the shear yield stress and p_s is the yield stress in hydrostatic compression. In general, both τ_s and p_s depend on the relative density. However, by assumption, the material is homogeneous. Therefore, τ_s and p_s are constant. Since the yield criterion is singular, several edge and face regimes are possible. It has been shown in [5] that plane stain deformation occurs at one of edge regimes. For definiteness, it is assumed that $\sigma<0$. A consequence of this assumption is that

$$0<-\sigma_i<p_s,\quad i=1,2,3. \tag{2}$$

With no loss of generality, it is possible to assume that the principal axis corresponding to the principal stress σ_3 is orthogonal to planes of flow and that $\sigma_1>\sigma_2$. Then, the plane strain yield criterion following from (1) is

$$\frac{\sigma_1-\sigma_2}{2\tau_s}-\frac{\sigma}{p_s}=1, \quad \frac{\sigma_3-\sigma_2}{2\tau_s}-\frac{\sigma}{p_s}=1. \tag{3}$$

It follows from this equation that $\sigma_1=\sigma_3$. This equation and (3) combine to give

$$\sigma_1\left(3p_s-4\tau_s\right)-\sigma_2\left(3p_s+2\tau_s\right)=6\tau_s p_s. \tag{4}$$

The system of equations comprising the equilibrium equations and the yield criterion (4) can be investigated without using velocity equations. If a boundary value problem is statically determinate then this system allows the stress field to be found. Let us introduce a curvilinear orthogonal coordinate system (α, β) whose coordinate curves coincide with trajectories of the principal stress directions. Denoting the principal stresses by $\sigma_\alpha \equiv \sigma_1$ and $\sigma_\beta \equiv \sigma_2$, respectively, the equilibrium equations in the (α, β) coordinate system may be written in the following form [10]

$$\frac{\partial\left(h_\beta\sigma_\alpha\right)}{\partial\alpha}-\sigma_\beta\frac{\partial h_\beta}{\partial\alpha}=0, \quad \frac{\partial\left(h_\alpha\sigma_\beta\right)}{\partial\beta}-\sigma_\alpha\frac{\partial h_\alpha}{\partial\beta}=0. \tag{5}$$

Here h_α and h_β denote the scale factors for the $\alpha-$ and $\beta-$ curves, respectively. Equation (4) can be rewritten as

$$\sigma_\alpha\left(3p_s-4\tau_s\right)-\sigma_\beta\left(3p_s+2\tau_s\right)=6\tau_s p_s. \tag{6}$$

Eliminating σ_β in the first equation in (5) and σ_α in the second equation in (5) by means of (6) yields

$$\begin{aligned} &h_\beta\frac{\partial\sigma_\alpha}{\partial\alpha}+\frac{6\tau_s\left(\sigma_\alpha+p_s\right)}{\left(3p_s+2\tau_s\right)}\frac{\partial h_\beta}{\partial\alpha}=0,\\ &h_\alpha\frac{\partial\sigma_\beta}{\partial\beta}-\frac{6\tau_s\left(\sigma_\beta+p_s\right)}{\left(3p_s-4\tau_s\right)}\frac{\partial h_\alpha}{\partial\beta}=0. \end{aligned} \tag{7}$$

Each of these equations can be immediately integrated using (2) to give

$$\sigma_\alpha+p_s=h_\beta^{-k_s}C_1(\beta), \quad \sigma_\beta+p_s=h_\alpha^{t_s}C_2(\alpha). \tag{8}$$

Here $C_1(\beta)$ is independent of α , $C_2(\alpha)$ is independent of β , $k_s=6\tau_s/\left(3p_s+2\tau_s\right)$ and $t_s=6\tau_s/\left(3p_s-4\tau_s\right)$. Substituting (8) into (4) shows that

$$h_\alpha^{t_s}h_\beta^{k_s}=\frac{C_1(\beta)\left(3p_s-4\tau_s\right)}{C_2(\alpha)\left(3p_s+2\tau_s\right)}. \tag{9}$$

It is evident from this expression that different choices of these functions $C_1(\beta)$ and $C_2(\alpha)$ merely change the scale of the $\beta-$ and $\alpha-$ curves, respectively. Therefore, without loss of generality it is possible to choose $C_1(\beta)$ and $C_2(\alpha)$ as

$$C_1(\beta)=\left(3p_s+2\tau_s\right), \quad C_2(\alpha)=\left(3p_s-4\tau_s\right). \tag{10}$$

Equations (9) and (10) combine to give

$$h_\alpha^m h_\beta=1 \tag{11}$$

where $m=\left(3p_s+2\tau_s\right)/\left(3p_s-4\tau_s\right)$. Thus the problem of finding the field of stress has been reduced to the problem of finding an orthogonal coordinate system whose scale factors satisfy (11). Once such a system of coordinate has been found, the principal stresses are determined from (8) and (10).

3 Method of determining principal stress trajectories

Introduce a Cartesian coordinate system (x, y). Let ψ be the angle between the $\alpha-$ lines and the $x-$ axis, measured anticlockwise positive from the $x-$ axis (Fig. 1).

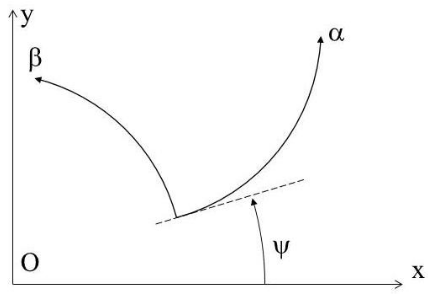

Figure 1. Principal line and Cartesian coordinate systems.

It follows from the geometry of this figure that

$$\begin{aligned} &\frac{\partial x}{\partial\alpha}=h_\alpha\cos\psi, \quad \frac{\partial x}{\partial\beta}=-h_\beta\sin\psi,\\ &\frac{\partial y}{\partial\alpha}=h_\alpha\sin\psi, \quad \frac{\partial y}{\partial\beta}=h_\beta\cos\psi. \end{aligned} \tag{12}$$

These equations and the compatibility equations in the form

$$\frac{\partial^2 x}{\partial\alpha\partial\beta}=\frac{\partial^2 x}{\partial\beta\partial\alpha}, \quad \frac{\partial^2 y}{\partial\alpha\partial\beta}=\frac{\partial^2 y}{\partial\beta\partial\alpha}.$$

combine to give

$$\begin{aligned} &\frac{\partial h_\alpha}{\partial\beta}\cos\psi-h_\alpha\sin\psi\frac{\partial\psi}{\partial\beta}=-\frac{\partial h_\beta}{\partial\alpha}\sin\psi-h_\beta\cos\psi\frac{\partial\psi}{\partial\alpha},\\ &\frac{\partial h_\alpha}{\partial\beta}\sin\psi+h_\alpha\cos\psi\frac{\partial\psi}{\partial\beta}=\frac{\partial h_\beta}{\partial\alpha}\cos\psi-h_\beta\sin\psi\frac{\partial\psi}{\partial\alpha}. \end{aligned} \tag{13}$$

It is always possible to rotate the Cartesian coordinate system such that its $x-$ axis is tangent to the $\alpha-$ curve at a given point. In this case $\psi = 0$ at this point and equation (13) becomes

$$\frac{\partial h_\alpha}{\partial \beta} + h_\beta \frac{\partial \psi}{\partial \alpha} = 0, \quad h_\alpha \frac{\partial \psi}{\partial \beta} - \frac{\partial h_\beta}{\partial \alpha} = 0. \tag{14}$$

Eliminating h_β in this equation by means of (11) results in

$$h_\alpha^m \frac{\partial h_\alpha}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} = 0, \quad h_\alpha^{m+2} \frac{\partial \psi}{\partial \beta} + m \frac{\partial h_\alpha}{\partial \alpha} = 0. \tag{15}$$

Using a standard technique it is possible to show that this system of equations is hyperbolic. The characteristic curves are determined from

$$\begin{aligned} \frac{d\beta}{d\alpha} &= -\frac{h_\alpha^{m+1}}{\sqrt{m}} \quad \text{on} \quad \xi-\text{lines} \\ \frac{d\beta}{d\alpha} &= \frac{h_\alpha^{m+1}}{\sqrt{m}} \quad \text{on} \quad \eta-\text{lines.} \end{aligned} \tag{16}$$

The characteristic relations are

$$\begin{aligned} d\psi - \sqrt{m}\frac{dh_\alpha}{h_\alpha} &= 0 \quad \text{on} \quad \xi-\text{lines,} \\ d\psi + \sqrt{m}\frac{dh_\alpha}{h_\alpha} &= 0 \quad \text{on} \quad \eta-\text{lines.} \end{aligned} \tag{17}$$

The characteristic relations can be immediately integrated to give

$$\psi - \sqrt{m}\ln h_\alpha = 2C_3(\eta), \ \psi + \sqrt{m}\ln h_\alpha = 2C_4(\xi). \tag{18}$$

Here $C_3(\eta)$ is independent of ξ and $C_4(\xi)$ is independent of η. Solving the equations in (18) for ψ and $\ln h_\alpha$ results in

$$\psi = C_3(\eta) + C_4(\xi), \ \sqrt{m}\ln h_\alpha = C_4(\xi) - C_3(\eta). \tag{19}$$

If both $\xi-$ and $\eta-$ lines are curved, then their parameterization can be chosen such that $C_3(\eta) = q\eta$ and $C_4(\xi) = q\xi$ where q is constant. In this case, equation (19) becomes

$$\psi = q(\xi + \eta), \quad \sqrt{m}\ln h_\alpha = q(\xi - \eta). \tag{20}$$

Equation (16) can be rewritten as

$$\frac{\partial \beta}{\partial \xi} + \frac{h_\alpha^{m+1}}{\sqrt{m}}\frac{\partial \alpha}{\partial \xi} = 0, \quad \frac{\partial \beta}{\partial \eta} - \frac{h_\alpha^{m+1}}{\sqrt{m}}\frac{\partial \alpha}{\partial \eta} = 0. \tag{21}$$

Eliminating h_α in this equation by means of (20) yields

$$\begin{aligned} \sqrt{m}\frac{\partial \beta}{\partial \xi} + \exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial \alpha}{\partial \xi} &= 0, \\ \sqrt{m}\frac{\partial \beta}{\partial \eta} - \exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial \alpha}{\partial \eta} &= 0. \end{aligned} \tag{22}$$

Introduce the new quantities ε and ν by

$$\alpha = \varepsilon \exp(n_1\xi + n_2\eta), \ \sqrt{m}\beta = \nu \exp(m_1\xi + m_2\eta) \tag{23}$$

Substituting (23) into (22) gives

$$\begin{aligned} &\left(\frac{\partial \nu}{\partial \xi} + m_1\nu\right)\exp(m_1\xi + m_2\eta) + \\ &\left(\frac{\partial \varepsilon}{\partial \xi} + n_1\varepsilon\right)\exp\left[(n_1\xi + n_2\eta)\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right] = 0, \\ &\left(\frac{\partial \nu}{\partial \eta} + m_2\nu\right)\exp(m_1\xi + m_2\eta) - \\ &\left(\frac{\partial \varepsilon}{\partial \eta} + n_2\varepsilon\right)\exp\left[(n_1\xi + n_2\eta)\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right] = 0. \end{aligned} \tag{24}$$

Putting

$$m_1 = n_1 + \frac{q(m+1)}{\sqrt{m}}, \quad m_2 = n_2 - \frac{q(m+1)}{\sqrt{m}}. \tag{25}$$

Then, equation (24) becomes

$$\begin{aligned} \frac{\partial \nu}{\partial \xi} + \frac{\partial \varepsilon}{\partial \xi} + n_1\varepsilon + m_1\nu &= 0, \\ \frac{\partial \nu}{\partial \eta} - \frac{\partial \varepsilon}{\partial \eta} + m_2\nu - n_2\varepsilon &= 0. \end{aligned} \tag{26}$$

The equations in (25) are compatible if $m_1 = 1$, $n_1 = -1$, $n_2 = 1$, and $m_2 = -1$. In this case it is possible to find from (25) that

$$q = \frac{2\sqrt{m}}{m+1}. \tag{27}$$

Moreover, equation (26) transforms to

$$\frac{\partial \nu}{\partial \xi} + \frac{\partial \varepsilon}{\partial \xi} - \varepsilon + \nu = 0, \quad \frac{\partial \nu}{\partial \eta} - \frac{\partial \varepsilon}{\partial \eta} - \nu - \varepsilon = 0. \tag{28}$$

These equations can be rewritten as $\partial(\varepsilon + \nu)/\partial \xi = \varepsilon - \nu$ and $\partial(\varepsilon - \nu)/\partial \eta = -(\varepsilon + \nu)$ or

$$\frac{\partial \omega}{\partial \xi} - \theta = 0, \quad \frac{\partial \theta}{\partial \eta} + \omega = 0 \tag{29}$$

where $\omega = \varepsilon + \nu$ and $\theta = \varepsilon - \nu$. It is evident that (29) is equivalent to the equations of telegraphy:

$$\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \omega = 0, \quad \frac{\partial^2 \theta}{\partial \xi \partial \eta} + \theta = 0. \tag{30}$$

This equation is integrated by the method of Riemann. In particular, the Green's function is the Bessel function of zero order. The methods for finding solutions of the equation of telegraphy in conjunction with boundary conditions typical in plasticity theory have been well documented [1, 2, 11]. Assuming that x and y are functions of α and β and using the chain rule it is possible to find that

$$\frac{\partial x}{\partial \xi}=\frac{\partial x}{\partial \alpha}\frac{\partial \alpha}{\partial \xi}+\frac{\partial x}{\partial \beta}\frac{\partial \beta}{\partial \xi},\quad \frac{\partial x}{\partial \eta}=\frac{\partial x}{\partial \alpha}\frac{\partial \alpha}{\partial \eta}+\frac{\partial x}{\partial \beta}\frac{\partial \beta}{\partial \eta},$$
$$\frac{\partial y}{\partial \xi}=\frac{\partial y}{\partial \alpha}\frac{\partial \alpha}{\partial \xi}+\frac{\partial y}{\partial \beta}\frac{\partial \beta}{\partial \xi},\quad \frac{\partial y}{\partial \eta}=\frac{\partial y}{\partial \alpha}\frac{\partial \alpha}{\partial \eta}+\frac{\partial y}{\partial \beta}\frac{\partial \beta}{\partial \eta}. \tag{31}$$

Equations (11) and (12) combine to give

$$\frac{\partial x}{\partial \alpha}=h_\alpha\cos\psi,\quad \frac{\partial x}{\partial \beta}=-h_\alpha^{-m}\sin\psi,$$
$$\frac{\partial y}{\partial \alpha}=h_\alpha\sin\psi,\quad \frac{\partial y}{\partial \beta}=h_\alpha^{-m}\cos\psi. \tag{32}$$

Eliminating the derivatives $\partial\beta/\partial\xi$ and $\partial\beta/\partial\eta$ in (31) by means of (22) yields

$$\frac{\partial x}{\partial \xi}=\left\{\frac{\partial x}{\partial \alpha}-\frac{1}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial x}{\partial \beta}\right\}\frac{\partial \alpha}{\partial \xi},$$
$$\frac{\partial x}{\partial \eta}=\left\{\frac{\partial x}{\partial \alpha}+\frac{1}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial x}{\partial \beta}\right\}\frac{\partial \alpha}{\partial \eta},$$
$$\frac{\partial y}{\partial \xi}=\left\{\frac{\partial y}{\partial \alpha}-\frac{1}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial y}{\partial \beta}\right\}\frac{\partial \alpha}{\partial \xi},$$
$$\frac{\partial y}{\partial \eta}=\left\{\frac{\partial y}{\partial \alpha}+\frac{1}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\frac{\partial y}{\partial \beta}\right\}\frac{\partial \alpha}{\partial \eta}. \tag{33}$$

Equations (32) and (33) combine to give

$$\frac{\partial x}{\partial \xi}=\left\{h_\alpha\cos\psi+\frac{h_\alpha^{-m}\sin\psi}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\right\}\frac{\partial \alpha}{\partial \xi},$$
$$\frac{\partial x}{\partial \eta}=\left\{h_\alpha\cos\psi-\frac{h_\alpha^{-m}\sin\psi}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\right\}\frac{\partial \alpha}{\partial \eta},$$
$$\frac{\partial y}{\partial \xi}=\left\{h_\alpha\sin\psi-\frac{h_\alpha^{-m}\cos\psi}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\right\}\frac{\partial \alpha}{\partial \xi},$$
$$\frac{\partial y}{\partial \eta}=\left\{h_\alpha\sin\psi+\frac{h_\alpha^{-m}\cos\psi}{\sqrt{m}}\exp\left[\frac{q(\xi-\eta)(m+1)}{\sqrt{m}}\right]\right\}\frac{\partial \alpha}{\partial \eta}. \tag{34}$$

Here q should be eliminated by menas of (27). Moreover, using (27) it is possible to transform (20) to

$$\psi=\frac{2\sqrt{m}}{m+1}(\xi+\eta),\quad \ln h_\alpha=\frac{2}{m+1}(\xi-\eta). \tag{35}$$

Eliminating h_α and ψ in (34) by means of (35) the coefficients of the derivatives $\partial\alpha/\partial\xi$ and $\partial\alpha/\partial\eta$ in (34) are represented as functions of ξ and η. The derivatives $\partial\alpha/\partial\xi$ and $\partial\alpha/\partial\eta$ are readely determined from (23) if the functions $\varepsilon(\xi,\eta)$ and $\nu(\xi,\eta)$ are known. The latter are found from a solution of (30) and the equations $\varepsilon=(\omega+\theta)/2$ and $\nu=(\omega-\theta)/2$. Therefore, equations (34) can be integrated numerically along any convenient path in the (ξ,η) space to determine the principal stress trajectories.

4 Conclusions

On the assumption of plane strain conditions the system of equations comprising the piramyd yield criterion (1) and the equilibrium equations has been solved in a curvilinear orthogonal coordinate system in which the coordinate curves coincide with trajectories of the principal stress directions. It has been shown that the scale factors of the coordinate curves should satisfy equation (11). The principal stresses are expressed in terms of the scale factors using equations (8) and (10). Mapping between the principal line and Cartesian coordinates is derived in parametric form with characteristic variables as parameters. In particualar, equations (34) can be integrated along any path in the (ξ,η) space giving $x(\xi,\eta)$ and $y(\xi,\eta)$. On the other hand, the functions $\alpha(\xi,\eta)$ and $\beta(\xi,\eta)$ are determined from (23) and any solution of the equations in (30). It is worthy of note here that $\varepsilon=(\omega+\theta)/2$ and $\nu=(\omega-\theta)/2$.

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