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Approximation properties of a new generalized Bernstein-Kantorovich operators

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Abstract. By means of construction of suitable functions and the method of Bojanic-Cheng, the author gives the rate of convergence of a new generalized Bernstein-Kantorovich operators for some absolutely continuous functions.

1 Introduction

For a function $f(x)$ defined on the closed interval $[0, 1]$, the expression

$$B_n(f, x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x)$$

is called the Bernstein polynomial of order n, where $p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$. The polynomials $B_n(f, x)$ were introduced by S.Bernstein [1] in order to give an especially simple proof of the Weierstrass approximation theorem. Then many scholars have done a lot of relevant research work. Bernstein operators became popular for several reasons: (1) they are given explicitly and depend only on the values of a function for rational values of the variable. (2) they have various shape-preserving properties and provide the simplest means for the study of some problems. (3) they are easy to handle in computer algebra systems when the evaluation of f is difficult and time-consuming.

Lorentz [2] gave an exhaustive exposition of main facts about the Bernstein polynomials. He also discussed some of their applications in analysis.

Based on the arithmetic mean of the total variation sequence, the estimation of the convergence rate of the B_n for the bounded variation function was obtained by Cheng [3]. It is proved that the estimation is essentially the best possibility of the continuous point.

Bojanic [4] investigated the asymptotic behavior of B_n for some absolutely continuous functions whose derivatives are bounded variation functions.

King [5] defined a new type of Bernstein operators which preserve x^2. Quantitative estimates were compared with estimates of approximation by the class Bernstein polynomials B_n in [5].

In the field of approximation theory, the applications of q-calculus are new area in the last 30 years. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupas in the year 1987. In 1997 Phillips considered another q-analogue of the classical Bernstein polynomials. Next, the q-operators have become the research object of many scholars[6].

Recently, Chen et al. [7] introduced a new family of generalized Bernstein operators based on a non-negative parameter $\alpha (0 \leq \alpha \leq 1)$ as follows:

$$T_{n,\alpha}(f, x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}^{(\alpha)}(x), \quad x \in [0, 1], \quad (1)$$

where

$$p_{1,0}^{(\alpha)}(x) = 1 - x, p_{1,1}^{(\alpha)}(x) = x,$$

$$p_{n,k}^{(\alpha)}(x) = [\tbinom{n-2}{k}(1-\alpha)x + \tbinom{n-2}{k-2}(1-\alpha)(1-x) + \tbinom{n}{k}\alpha x(1-x)]x^{k-1}(1-x)^{n-k-1}$$

for $n \geq 2$ and $\binom{n}{k} = 0 (k > n)$. When $\alpha = 1$, the operators $T_{n,\alpha}$ reduces to the classical Bernstein operators $B_n(f, x)$.

In [7], the authors studied many approximaiton properties of $T_{n,\alpha}$ such as uniform convergence, rate of convergence in terms of modulus of continuity, voronovskaya-type asymptotic formula, and shape preserving properties.

To approximate Lebesgue integrable functions, Mohiuddine et al. [8] introduced the following integral modification of the operators $T_{n,\alpha}$:

$$K_{n,\alpha}(f, x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \quad (2)$$

In [8], the uniform convergence of the operators and rate of convergence in local and global sense are studied.

The rate of approximation for some absolutely continuous functions whose derivatives are bounded variation functions is an intersting topic. This is mainly originated from Bojanic-cheng [4], then many scholars have done a lot of research in this field [9-16].

Base on this, this article studies the approximation of $K_{n,\alpha}$ for some absolutely continuous functions $f \in DBV[0, 1]$, where

$$DBV[0, 1] = \{f | f(x) = f(0) + \int_0^x h(t)dt\}$$

and $x \in [0, 1], h \in BV[0, 1]$.

Let

$$R_{n,\alpha}(x, t) = \sum_{k=0}^{n}(n+1)p_{n,k}^{(\alpha)}(x)\chi_k(t),$$

where $\chi_k(t)$ is the characteristic function of the interval $[\frac{k}{n+1}, \frac{k+1}{n+1}]$ with respect to $I = [0, 1]$.

By the Lebesgue-Stieltjes integral representations, we have

$$K_{n,\alpha}(f, x) = \int_0^1 f(t)R_{n,\alpha}(x, t)dt. \quad (3)$$

2 Some lemmas

The proof of our result are based on the following lemmas.

Lemma 2.1 ([8]) For $e_i = t^i, i = 0, 1, 2$, we have

$$K_{n,\alpha}(e_0, x) = 1, \quad K_{n,\alpha}(e_1, x) = \frac{nx}{n+1} + \frac{1}{2(n+1)},$$

$$K_{n,\alpha}(e_2, x) = \frac{n^2}{(n+1)^2}\left(x^2 + \frac{n+2(1-\alpha)}{n^2}x(1-x)\right) + \frac{nx}{(n+1)^2} + \frac{1}{3(n+1)^2}.$$

Remark 2.1 By simple applications of Lemma 2.1, we get

$$K_{n,\alpha}(t - x, x) = \frac{1-2x}{2(n+1)},$$

$$\begin{aligned} & K_{n,\alpha}((t-x)^2, x) \\ = & \frac{n+2(1-\alpha)-1}{(n+1)^2}x(1-x) + \frac{1}{3(n+1)^2} \\ \triangleq & \eta_{n\alpha}^2(x). \end{aligned}$$

Lemma 2.2 When n sufficient large, we have

$$K_{n,\alpha}(|t - x|, x) \leq \eta_{n\alpha}(x). \quad (4)$$

Proof By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & K_{n,\alpha}(|t-x|, x) \\ \leq & \sqrt{K_{n,\alpha}((t-x)^2, x)} \cdot \sqrt{K_{n,\alpha}(1, x)} \\ = & \eta_{n\alpha}(x). \end{aligned}$$

The last inequality is obtained by Lemma 2.1 and Remark 2.1.

Lemma 2.3

(i) For $0 \leq y < x < 1$, there holds

$$\widetilde{R}_{n,\alpha}(x, y) = \int_0^y R_{n,\alpha}(x, t)dt \leq \frac{\eta_{n\alpha}^2(x)}{(x-y)^2}. \quad (5)$$

(ii) For $0 < x < z \leq 1$, there holds

$$1 - \widetilde{R}_{n,\alpha}(x, z) = \int_z^1 R_{n,\alpha}(x, t)dt \leq \frac{\eta_{n\alpha}^2(x)}{(z-x)^2}. \quad (6)$$

Proof (i) By (3) and Remark 2.1, we get

$$\begin{aligned}
&\widetilde{R}_{n,\alpha}(x,y)\\
&= \int_0^y R_{n,\alpha}(x,t)dt\\
&\le \int_0^y \left(\frac{x-t}{x-y}\right)^2 R_{n,\alpha}(x,t)dt\\
&\le \frac{1}{(x-y)^2}\int_0^1 (t-x)^2 R_{n,\alpha}(x,t)dt\\
&= \frac{1}{(x-y)^2} K_{n,\alpha}((t-x)^2,x)\\
&= \frac{\eta_{n\alpha}^2(x)}{(x-y)^2}.
\end{aligned}$$

(ii) Using a similar method, we have

$$\begin{aligned}
&1-\widetilde{R}_{n,\alpha}(x,z)\\
&= \int_z^1 R_{n,\alpha}(x,t)dt\\
&\le \int_z^1 \left(\frac{x-t}{z-x}\right)^2 R_{n,\alpha}(x,t)dt\\
&\le \frac{1}{(z-x)^2}\int_0^1 (t-x)^2 R_{n,\alpha}(x,t)dt\\
&= \frac{1}{(x-y)^2} K_{n,\alpha}((t-x)^2,x)\\
&= \frac{\eta_{n\alpha}^2(x)}{(z-x)^2}.
\end{aligned}$$

3 Main results

Theorem *Let $f \in DBV[0,1]$. If $h(x+)$ and $h(x-)$ exist at a fixed point $x \in (0,1)$, then we have*

$$\left|K_{n,\alpha}(f,x) - f(x) - \frac{(1-2x)[h(x+)+h(x-)]}{4(n+1)}\right|$$

$$\le \left|\frac{h(x+)-h(x-)}{2}\right|\eta_{n\alpha}(x)$$

$$+ \frac{2\eta_{n\alpha}^2(x)}{x(1-x)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(\varphi_x) + \frac{1}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}(\varphi_x),$$

where

$$\varphi_x(t) = \begin{cases} h(t)-h(x+), & x < t \le 1;\\ 0, & t = x;\\ h(t)-h(x-), & 0 \le t < x.\end{cases}$$

Proof Let f satisfy the conditions of Theorem, by using Bojanic-Cheng's method [4], we have

$$f(t) - f(x) = \int_x^t h(u)du, \qquad (7)$$

where

$$h(u) = \frac{h(x+)+h(x-)}{2} + \frac{h(x+)-h(x-)}{2} sign(u-x)$$

$$+\varphi_x(u) + \delta_x(u)\left[h(x) - \frac{h(x+)+h(x-)}{2}\right] \qquad (8)$$

and

$$\delta_x(u) = \begin{cases} 1, & u = x;\\ 0, & u \ne x.\end{cases}$$

$$sign(x) = \begin{cases} 1, & x > 0;\\ 0, & x = 0;\\ -1, & x < 0.\end{cases}$$

From (7), (8), and noting $\int_x^t sign(u-x)du = |t-x|$, $\int_x^t \delta_x(u)du = 0$, we find that

$$\begin{aligned}
& K_{n,\alpha}(f,x) - f(x)\\
=\ & K_{n,\alpha}(f(t)-f(x),x)\\
=\ & K_{n,\alpha}\left(\int_x^t h(u)du, x\right)\\
=\ & \frac{h(x+)+h(x-)}{2} K_{n,\alpha}(t-x,x)\\
+\ & \frac{h(x+)-h(x-)}{2} K_{n,\alpha}(|t-x|,x)\\
+\ & K_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right).
\end{aligned}$$

By Remark 2.1 and Lemma 2.2, we have

$$\left|K_{n,\alpha}(f,x) - f(x) - \frac{(1-2x)(h(x+)+h(x-))}{4(n+1)}\right|$$

$$\le \left|\frac{h(x+)-h(x-)}{2}\right|\eta_{n\alpha}(x)$$

$$+\left|K_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right)\right|. \qquad (9)$$

To complete the proof, we must estimate the term $K_{n,\alpha}(\int_x^t \varphi_x(u)du, x)$.

From (3), the term $K_{n,\alpha}(\int_x^t \varphi_x(u)du, x)$ can be stated as

$$\begin{aligned} & K_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right) \\ = & \int_0^1 (\int_x^t \varphi_x(u)du) R_{n,\alpha}(x,t)dt \\ = & \int_0^1 (\int_x^t \varphi_x(u)du) d_t \widetilde{R}_{n,\alpha}(x,t) \\ = & \int_0^x (\int_x^t \varphi_x(u)du) d_t \widetilde{R}_{n,\alpha}(x,t) \\ + & \int_x^1 (\int_x^t \varphi_x(u)du) d_t \widetilde{R}_{n,\alpha}(x,t). \end{aligned}$$

Let

$$\Delta_{1n} = \int_0^x (\int_x^t \varphi_x(u)du) d_t \widetilde{R}_{n,\alpha}(x,t),$$

$$\Delta_{2n} = \int_x^1 (\int_x^t \varphi_x(u)du) d_t \widetilde{R}_{n,\alpha}(x,t).$$

Then we have

$$K_{n,\alpha}\left(\int_x^t \varphi_x(u)du, x\right) = \Delta_{1n} + \Delta_{2n}. \qquad (10)$$

Using partial integration and noticing $\widetilde{R}_{n,\alpha}(x,0) = 0$, $\int_x^x \varphi_x(u)du = 0$, we get

$$\begin{aligned} \Delta_{1n} &= \widetilde{R}_{n,\alpha}(x,t) \int_x^t \varphi_x(u)du\Big|_0^x \\ &- \int_0^x \widetilde{R}_{n,\alpha}(x,t)\varphi_x(t)dt \\ &= -\int_0^x \widetilde{R}_{n,\alpha}(x,t)\varphi_x(t)dt \\ &= -(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^x) \widetilde{R}_{n,\alpha}(x,t)\varphi_x(t)dt. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} |\Delta_{1n}| &\le \int_0^{x-\frac{x}{\sqrt{n}}} \widetilde{R}_{n,\alpha}(x,t) \bigvee_t^x (\varphi_x)dt \\ &+ \int_{x-\frac{x}{\sqrt{n}}}^x \widetilde{R}_{n,\alpha}(x,t) \bigvee_t^x (\varphi_x)dt \end{aligned}$$

From Lemma 2.3 (i) and $0 \le \widetilde{R}_{n,\alpha}(x,t) \le 1$, we get

$$|\Delta_{1n}| \le \eta_{n\alpha}^2(x) \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\varphi_x). \qquad (11)$$

Putting $t = x - \frac{x}{u}$ for the integral of (11), we get

$$\begin{aligned} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\bigvee_t^x(\varphi_x)}{(x-t)^2}dt &= \frac{1}{x}\int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (\varphi_x)du \\ &\le \frac{2}{x}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\varphi_x). \end{aligned}$$

From (11) and the above inequality, it follows that

$$|\Delta_{1n}| \le \frac{2\eta_{n\alpha}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\varphi_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\varphi_x). \qquad (12)$$

From Lemma 2.3 (ii), using the same method, we also get

$$|\Delta_{2n}| \le \frac{2\eta_{n\alpha}^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{1-x}{k}} (\varphi_x) + \frac{1-x}{\sqrt{n}} \bigvee_x^{x+\frac{1-x}{\sqrt{n}}} (\varphi_x). \qquad (13)$$

Theorem now follows from (9),(12) and (13). This completes the proof.

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