Total vertex irregularity strength of comb product of two cycles

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Abstract. Let $G = (V(G), E(G))$ be a graph and k be a positive integer. A total k -labeling of G is a map $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$. The vertex weight v under the labeling f is denoted by $w_f(v)$ and defined by $w_f(v) = f(v) + \sum_{uv \in E(G)} f(uv)$. A total k -labeling of G is called vertex irregular if there are no two vertices with the same weight. The total vertex irregularity strength of G , denoted by $tv_s(G)$, is the minimum k such that G has a vertex irregular total k -labeling. This labeling was introduced by Bača, Jendrol', Miller, and Ryan in 2007. Let G and H be two connected graphs. Let o be a vertex of H . The comb product between G and H , in the vertex o , denoted by $G \triangleright_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i -th copy of H at the vertex o to the i -th vertex of G . In this paper, we determine the total vertex irregularity strength of comb product of C_n and C_m where $m \in \{1, 2\}$.

1 Introduction

Mathematics is a logical, analytical, and systematic thinking framework in helping human to solve their problems. One of the topics in mathematics that is interesting to study is graph labeling. One of the applications of graph labeling are to solve network problem.

In this paper, we discuss a kind of graph labeling, which is total vertex irregular labeling.

Let $G = (V, E)$ be a graph and k be a positive integer. A total k -labeling of G is a map $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$.

In 2007, Bača et al. introduced total irregular labeling. [1]. There are two kinds of the total irregular labeling, which are an edge irregular total labeling and a vertex irregular total labeling.

Definition 1.1. [1] For an integer k , a total labeling $f: V \cup E \rightarrow \{1, 2, \dots, k\}$ is called an edge irregular total k -labelling of G if every two distinct edges $e = uv$ and $f = wx$ in E satisfy $w_f(e) \neq w_f(f)$, where $w_f(e) = f(u) + f(v) + f(e)$.

The notation $w_f(e)$ is called by the weight of edge e under the labeling f .

Definition 1.2. [1] The minimum k for which a graph G has an edge irregular total k -labeling, denoted by $tes(G)$, is called the total edge irregularity strength of G .

Bača et al [1] gave a lower bound and an upper bound on $tes(G)$ for arbitrary graph G . The bounds is given by Theorem 2.1.

Theorem 1.1 [1] Let $G = (V, E)$ be a graph with vertex set V and a non-empty edge set E . Then

$$\left\lceil \frac{|E|+2}{3} \right\rceil \leq tes(G) \leq |E|. \quad (1)$$

In the same paper, they also gave the exact value of $tes(G)$ for G are paths, cycles, stars, complete graphs, wheels, and friendships. [1]

Ivančo and S. Jendrol' [2] posed a conjecture that for arbitrary graph $G \neq K_5$,

$$tes(G) = \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}. \quad (2)$$

Ramdani, et al [3] gave an upper bound on the total edge irregularity strength of disjoint union of graphs as follows.

Theorem 1.2 [3] The total edge irregularity strength of disjoint union of graphs $G_1, G_2, \dots, G_m$, $m \geq 2$, is

$$tes(\cup_{i=1}^m G_i) \leq \sum_{i=1}^m tes(G_i) - \left\lfloor \frac{m-1}{2} \right\rfloor. \quad (3)$$

In [4], Nurdin et al. determined the total edge irregular strength of the corona product of paths with some graphs.

Definition 1.3. [1] For an integer k , a total labelling $f: V \cup E \rightarrow \{1, 2, \dots, k\}$ is called a vertex irregular total k -labelling of G if every two distinct vertices u and v in V satisfy $w_f(u) \neq w_f(v)$, where $w_f(u) = f(u) + \sum_{uz \in E} f(uz)$.

The notation $w_f(u)$ is called by the weight of vertex u under the labeling f .

Definition 1.4. [1] The minimum k for which a graph G has a vertex irregular total k -labeling, denoted by $tv_s(G)$, is called the total vertex irregularity strength of G .

In [1], Bača et al determined a lower bound and an upper bound on the total vertex irregularity strength of arbitrary graph G with p vertices, q edges, the minimum degree δ , and the maximum degree Δ , as follows.

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$$\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1. \quad (4)$$

In [5], Nurdin et al. determined another lower bound of $tvs(G)$ for G a connected graph as follows.

Theorem 1.3 [5] Let G be a connected graph having n_i vertices of degree $(i = \delta, \delta + 1, \delta + 2, \dots, \Delta)$, where δ and Δ are the minimum and the maximum degree of G , respectively. Then

$$tvs(G) \geq \max \left\{ \left\lceil \frac{\delta+n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta+n_\delta+n_{\delta+1}}{\delta+2} \right\rceil, \dots, \left\lceil \frac{\delta+\sum_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}. \quad (5)$$

Ramdani et al., in [6], determined an upper bound on the total vertex irregularity strength of Cartesian product of P_2 and arbitrary regular graph G .

Theorem 1.4 [6]. Let G be an r -regular graph for $r \geq 1$. Then

$$tvs(P_2 \square G) \leq 2tvs(G). \quad (6)$$

In [7] and [8], Ramdani et al. determined an exact value of $tvs(G)$ for G are ladders and books. Nurdin et al. [9] gave the exact values of the total vertex irregularity strength for several types of trees and disjoint union of paths. Przybylo, in [10], gave a linear bound on $tvs(G)$.

Combining edge and vertex irregular total labeling, Marzuki et al., in [11], introduced new irregular labeling, namely totally irregular total labeling, defined as follows.

Definition 1.5 [11] For an integer k , a total labelling $f: V \cup E \rightarrow \{1, 2, \dots, k\}$ is called a totally irregular total k -labelling of G if every two distinct edges $e = uv$ and $f = wx$ in E satisfy $w_f(e) \neq w_f(f)$ and every two distinct vertices u and v in V satisfy $w_f(u) \neq w_f(v)$, where $w_f(e) = f(u) + f(e) + f(v)$ and $w_f(u) = f(u) + \sum_{uz \in E} f(uz)$.

Definition 1.6. [11] The minimum k for which a graph G has a totally irregular total k -labelling, denoted by $ts(G)$, is called the total irregularity strength of G .

In the same paper, Marzuki gave a lower bound on $ts(G)$ that

$$ts(G) \geq \max\{tes(G), tvs(G)\}. \quad (7)$$

An upper bound on $ts(G)$ was given by Ramdani et al. in [12] as follows.

Theorem 1.5 [12] Let G be a graph of order p . Let the sequence F_n of Fibonacci numbers be defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$, with seed values $F_1 = 1$ and $F_2 = 2$. Then the total irregularity strength of a graph G is

$$ts(G) \leq F_p. \quad (8)$$

In the same paper, [12], Ramdani et al. determined the exact value of $ts(G)$ for G are gears, fungus, and m copies of stars.

Other results about $ts(G)$ were given by Ramdani et al in [13]. In the paper, they gave the exact value of $ts(G)$ for G are Cartesian product of P_2 and some families of graphs. In [14], Ramdani et al. gave the total irregularity strength of freindship.

In [15], Ramdani et al. determined the total irregularity strength of regular graphs. In the paper, they gave an upper bound on total irregularity strength of m copies of arbitrary graph G . Also they gave the exact value of the total irregularity strength of m copies of path P_2 .

In this paper, we determine the exact value of the total vertex irregularity strength of comb product of two cycles. The definition of comb product graph is given below.

Definition 1.7 Let G and H be two connected graphs. Let o be a vertex of H . The comb product between G and H , in the vertex o , denoted by $G \triangleright_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i -th copy of H at the vertex o to the i -th vertex of G .

2 Main Results

In this paper, we give the exact value of the total vertex irregularity strength of $C_n \triangleright_o C_4$ and $C_n \triangleright_o C_5$. The first result is given by Theorem 2.1

Theorem 2.1 Let C_n be a cycle with n vertices and C_4 be a cycle with 4 vertices, then for $n \geq 3$,

$$tvs(C_n \triangleright_o C_4) = n + 1. \quad (9)$$

Proof. Let the vertex set of $C_n \triangleright_o C_4$ be

$$\{u_{ij} | 1 \leq i \leq n, 1 \leq j \leq 4\} \quad (10)$$

and the edge set be

$$\{u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}, u_{i4}u_{i1} | 1 \leq i \leq n\} \quad (11)$$

$$\cup \{u_{i1}u_{(i+1)1} | 1 \leq i \leq n-1\} \cup \{u_{n1}u_{11}\}. \quad (12)$$

The illustration of graph $C_n \triangleright_o C_4$ can be seen in the Fig. 1.

Fig. 1. A graph $C_n \triangleright_o C_4$

Graph $C_n \triangleright_o C_4$ has $3n$ vertices with degree $\delta = 2$ and n vertices with degree $\Delta = 4$. By using Theorem 1.3, we have

$$tvs(C_n \triangleright_o C_4) \geq \max \left\{ \left\lceil \frac{\delta+n_\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta+n_\delta+n_{\delta+1}}{\delta+2} \right\rceil, \dots, \left\lceil \frac{\delta+\sum_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}. \quad (13)$$

$$= \max \left\{ \left\lceil \frac{2+3n}{2+1} \right\rceil, \left\lceil \frac{2+3n+n}{4+1} \right\rceil \right\} \quad (14)$$

$$= \max \left\{ \left\lceil \frac{2+3n}{3} \right\rceil, \left\lceil \frac{4n+2}{5} \right\rceil \right\} \quad (15)$$

$$= \left\lceil \frac{2+3n}{3} \right\rceil \quad (16)$$

$$= n + 1. \quad (17)$$

So that, we have

$$tvs(C_n \triangleright C_4) \geq n + 1. \quad (18)$$

Define a labeling f of $V(C_n \triangleright C_4) \cup E(C_n \triangleright C_4)$ as follows :

$$f(u_{i1}) = n + 1, \text{ for } 1 \leq i \leq n; \quad (19)$$

$$f(u_{i2}) = f(u_{i3}) = f(u_{i4}) = i, \text{ for } 1 \leq i \leq n; \quad (20)$$

$$(u_{i1}u_{i2}) = f(u_{i2}u_{i3}) = i, \text{ for } 1 \leq i \leq n; \quad (21)$$

$$f(u_{i3}u_{i4}) = f(u_{i4}u_{i1}) = i + 1, \text{ for } 1 \leq i \leq n; \quad (22)$$

$$f(u_{i1}u_{(i+1)1}) = n + 1, \text{ for } 1 \leq i \leq n - 1; \quad (23)$$

$$f(u_{n1}u_{n1}) = n + 1. \quad (24)$$

The labeling f gives weight of each vertex, for $1 \leq i \leq n$, as follows :

$$\begin{aligned} w_f(u_{i1}) &= f(u_{i1}) + f(u_{i1}u_{i2}) + f(u_{i4}u_{i1}) \\ &\quad + f(u_{i1}u_{(i+1)1}) + f(u_{(i-1)1}u_{i1}) \\ &= (n + 1) + (i) + (i + 1) \\ &\quad + (n + 1) + (n + 1) \\ &= 3n + 4 + 2i, \end{aligned} \quad (25)$$

$$\begin{aligned} w_f(u_{i2}) &= f(u_{i2}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3}) \\ &= (i) + (i) + (i) \\ &= 3i \end{aligned} \quad (26)$$

$$\begin{aligned} w_f(u_{i3}) &= f(u_{i3}) + f(u_{i2}u_{i3}) + f(u_{i3}u_{i4}) \\ &= (i) + (i) + (i + 1) \\ &= 3i + 1 \end{aligned} \quad (27)$$

$$\begin{aligned} w_f(u_{i4}) &= f(u_{i4}) + f(u_{i3}u_{i4}) + f(u_{i4}u_{i1}) \\ &= (i) + (i + 1) + (i + 1) \\ &= 3i + 2 \end{aligned} \quad (28)$$

It can be seen that there are no two vertices of the same weight.

So, f is a total vertex irregular labeling of $C_n \triangleright C_4$. The maximum label of the labeling f is $n + 1$. So, we have

$$tvs(C_n \triangleright C_4) \leq n + 1. \quad (29)$$

From (18) and (29), we can conclude that for $n \geq 3$,

$$tvs(C_n \triangleright C_4) = n + 1. \quad (30)$$

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This paper also determine the total vertex irregularity strength of $C_n \triangleright C_5$. The result can be seen in the theorem below.

Theorem 2.2 Let C_n be a cycle with n vertices and C_5 be a cycle with 5 vertices. For $n \geq 3$,

$$tvs(C_n \triangleright C_5) = \left\lceil \frac{4n+2}{3} \right\rceil. \quad (31)$$

Proof. Let the vertex set of $C_n \triangleright C_5$ be

$$\{u_{ij} | 1 \leq i \leq n, 1 \leq j \leq 5\} \quad (32)$$

and the edge set be

$$\{u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}, u_{i4}u_{i5}, u_{i5}u_{i1} | 1 \leq i \leq n\} \quad (33)$$

$$\cup \{u_{i1}u_{(i+1)1} | 1 \leq i \leq n - 1\} \quad (34)$$

$$\cup \{u_{n1}u_{11}\}. \quad (35)$$

In the Fig. 2, we can see the illustration of graph $C_n \triangleright C_5$.

Fig. 2. A graph $C_n \triangleright C_5$

It can be seen in the Fig. 2. that graph $C_n \triangleright C_5$ has $4n$ vertices with degree $\delta = 2$ and n vertices with degree $\Delta = 4$. We use Theorem 2.3 to have

$$tvs(C_n \triangleright C_5) \geq \max \left\{ \left\lceil \frac{\delta+n\delta}{\delta+1} \right\rceil, \left\lceil \frac{\delta+n\delta+n\delta+1}{\delta+2} \right\rceil, \dots, \left\lceil \frac{\delta+\sum_{i=\delta}^{\Delta} n_i}{\Delta+1} \right\rceil \right\}. \quad (36)$$

$$= \max \left\{ \left\lceil \frac{2+4n}{2+1} \right\rceil, \left\lceil \frac{2+4n+n}{4+1} \right\rceil \right\} \quad (37)$$

$$= \max \left\{ \left\lceil \frac{2+4n}{3} \right\rceil, \left\lceil \frac{5n+2}{5} \right\rceil \right\} \quad (38)$$

$$= \left\lceil \frac{4n+2}{3} \right\rceil \quad (39)$$

So that, we have

$$tvs(C_n \triangleright C_5) \geq \left\lceil \frac{4n+2}{3} \right\rceil. \quad (40)$$

Next, define a labeling f of $V(C_n \triangleright C_5) \cup E(C_n \triangleright C_5)$ as follows :

$$f(u_{i1}) = f(u_{i5}u_{i1}) = \left\lceil \frac{4n+2}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (41)$$

$$f(u_{i2}) = \left\lceil \frac{4i-3}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (42)$$

$$f(u_{i3}) = f(u_{i1}u_{i2}) = \left\lceil \frac{4i-2}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (43)$$

$$f(u_{i4}) = f(u_{i2}u_{i3}) = \left\lceil \frac{4i-1}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (44)$$

$$f(u_{i5}) = f(u_{i3}u_{i4}) = \left\lceil \frac{4i}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (45)$$

$$f(u_{i4}u_{i5}) = \left\lceil \frac{4n+1}{3} \right\rceil, \text{ for } 1 \leq i \leq n; \quad (46)$$

$$f(u_{i1}u_{(i+1)1}) = \left\lceil \frac{4n+2}{3} \right\rceil, \text{ for } 1 \leq i \leq n-1; \quad (47)$$

$$f(u_{11}u_{n1}) = \left\lceil \frac{4n+2}{3} \right\rceil. \quad (48)$$

The weight of vertices of $C_n \triangleright_o C_5$ under labeling f vertex, as follows :

For $1 \leq i \leq n$,

$$w_f(u_{i1}) = 3 \left\lceil \frac{4n+2}{3} \right\rceil + \left\lceil \frac{4i-2}{3} \right\rceil + \left\lceil \frac{4i+2}{3} \right\rceil, \quad (49)$$

$$w_f(u_{i2}) = 4i - 1, \quad (50)$$

$$w_f(u_{i3}) = 4i, \quad (51)$$

$$w_f(u_{i4}) = 4i + 1, \text{ and} \quad (52)$$

$$w_f(u_{i5}) = 4i + 2. \quad (53)$$

It can be seen that there are no two vertices of $C_n \triangleright_o C_5$ under labeling f of the same weight.

So that, f is a total vertex irregular labeling of $C_n \triangleright_o C_5$.

5. The maximum label of labeling f is $\left\lceil \frac{4n+2}{3} \right\rceil$. So, we have

$$tvs(C_n \triangleright_o C_5) \leq \left\lceil \frac{4n+2}{3} \right\rceil. \quad (54)$$

From (40) and (54), we conclude that

$$tvs(C_n \triangleright_o C_4) = n + 1, \quad (55)$$

for $n \geq 3$.

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