The braid group: redefining

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Abstract. The role of the braid group constitutes one of the invariant measurements. Through the classification of braids formed several parts of the braid group, but does not computationally distinguish them. Some characteristics have been expressed to give the features to a braid in braid group based on redefining relation between braid group and permutation group.

1 Introduction

One of technologies that has been growing ever since is the braid [1,2]. The braid is widely used in everyday life [3]. The braid is not only used by the public in general, but the academics, especially the biologists for genetic engineering [4]. As a technology, the braid has its own mathematical model [5]. The modelling is to express the privilege of a braid.

In group theory, the braid is one of the most interesting implementations [6], which provides a form of the invariant measurement about a structure [5]. Thus, a braid has the general and special characteristics, which characterizes each braid so as to give the idea of meaning to the structure formed. Therefore, this paper redefines relations between groups based on the braids in the braid groups specifically and expresses the computational characteristics associated with it.

2 The Review and Motivation

2.1 Woven and braid

Three woven patterns are σ_i^1 , σ_i^0 , and σ_i^{-1} each like Fig. 1. They build a system if there are two parallel lines L_1 and L_2 (can be hidden/not declared) in the same direction in the space R^3 , and in the line L_1 there is the points p_i , while in the line L_2 there is points q_i , where $i, j = 1, \dots, n$ so that on each of the lines the distance of adjacent points is same, and then one point p_i is connected by curve c_i just to a point q_j . L_1 and L_2 are called upper frame and bottom frame, respectively. σ_i , $i = 1, \dots, n$ are generators [7]. Computationally,

Fig. 1. Triple of woven patterns in braid

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$$\sigma_i^1 \sigma_i^{-1} = \sigma_i^0 = 1. \quad (1)$$

Notation σ_i^1 or simply written σ_i is a woven geometric shape where the curve c_i overpass c_{i+1} exactly once on it, whereas c_j for $j \neq i, i+1$ directly connects the points p_j to q_j . Instead σ_i^{-1} is a woven where the curve c_i underpass c_{i+1} exactly once on it, whereas c_j , $j \neq i-1, i$ is direct connect the points p_j to q_j , see Fig. 2 [3]. The arrangement of points in a pair of frame lines can be expressed as a permutation form as follows

Fig. 2. A braid A_1 in $\mathcal{A}_6$

$$A = \begin{vmatrix} 1 & 2 & \dots & N \\ c(1) & c(2) & \dots & c(n) \end{vmatrix} \quad (2)$$

c is the curve that maps the points at L_1 to L_2 . Number of generator σ for representing A is more than one woven (minimally is σ^0), and we denote it as $n_\sigma()$. For example, for $i = 1, 2, \dots, 6$ we have one woven arrangement $A_1 = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2$ and $n_\sigma(A_1) = |\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2| = 7$, see Fig. 2, or based on Eq. (2) we have

$$A_1 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 1 & 2 & 5 \end{vmatrix} \quad (3)$$

or $A = ((1)(3)(6)(5)(2)(4))$.

A braid is a woven together with a deformation operation where no woven being of same height that

during the deformation: (i) L_1 and L_2 remain parallel; (ii) There are no wedges between the curves; and (iii) The curve remains normal. The deformation can be expressed as a brushing, i.e. the projection of the curve c_i moving from p_i to q_j has the distance from the line L_1 will always increase.

2.2 Basic of braid group

A set $\mathcal{A} \neq \emptyset$ together with binary operation $\cdot$ denoted by $\langle \mathcal{A}, \cdot \rangle$ is called a braid group if and only if: Binary operation between same ordered-braids can be expressed as a system [3], i.e. let L_1, L_2, p_i, q_j, c_i and $L_1', L_2', p_i', q_i', c_i'$ respectively as frames, points, and curves of two braids A in $\mathcal{A}$ and A' in $\mathcal{A}$, the braid $A \cdot A'$ (or AA') can be obtained through the union of L_2 and L_1' such that q_j coincides with p_i' and AA' in $\mathcal{A}$ is a new braid by removing $L_2 = L_1'$; Thus, for every same ordered-braid, the associative law applicable, i.e. $(A_1 A_2) A_3 = A_1 (A_2 A_3)$, A_1, A_2, A_3 in $\mathcal{A}$, Geometrically an identity braid, denoted by A^0 , can be expressed as a woven arrangement by which each curve c_i connects p_i directly to q_i and $i = j$, so that when the identity braid composed with any other same ordered-braid, the resulted braid will be equal to the origin braid. While, geometrically the inverse of A can be guaranteed to exist through the results of reflection of A to the line L_2 of A so that L_2 becomes L_1' of the braid A^{-1} , it constructed by the shadow of L_1 against L_2 whereby the curve c_i' will be the inverse of the curve c_i and consequently the woven on A where the curve c_i overpass become the curve c_i underpass A^{-1} , and their distance to L_2 and L_1' be equal. That is $AA^{-1} = A^0 = A^{-1}A$ and we conclude that $A^0 = 1$ where A^0 in $\mathcal{A}$ and A^{-1} in $\mathcal{A}$ [7,8].

The class of same ordered braids together with the operation is a group. This group is not determined by the length of the curve c_i of each braid, but only by the woven structure of σ in braids. We denoted $\mathcal{A}_n$ as a group of braids order n . However, the set of permutations that represent all the braids is also a group called the n -ordered permutation group and denoted by $\mathcal{P}_n$ [9]. Depending on any arrangement of σ_j can be added/inserted to any braid provided that the σ_j arrangement applies to $j < i$ while the other curve of c_i connects the same point. Thus, the adding process of the σ_j arrangement at the beginning or end of the braid is like performing a binary operation, whereas insertion is done by constructing a new line to cut and then adding new ones and combining other pieces based on appropriate binary operations.

For instance, let $A_2 = \sigma_1 \sigma_4^{-1} \sigma_2$ for $i = 1, 2, \dots, 6$ where $n_\sigma(A_2) = 3$, then $A_1 A_2 = (\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2) (\sigma_1 \sigma_4^{-1} \sigma_2) = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \sigma_1 \sigma_4^{-1} \sigma_2$ or by using Eq. (2) and Eq. (3) we obtain

$$A_1 A_2 = ((1)(3)(6)(5)(2)(4))(((1)(3)(2))((5)(4))) \\ = (((1)(2)(5))((3)(6)(4))) \quad (4)$$

Thus $A_1 A_2 \neq A_2 A_1$, but Eq. (4) is the direct proof of $(A_1 A_2) A_3 = A_1 (A_2 A_3)$. Naturally, suppose there is $A_1^{-1} = \sigma_2^{-1} \sigma_5^{-1} \sigma_3 \sigma_1 \sigma_4^{-1} \sigma_2^{-1} \sigma_3$, then $A_1 A_1^{-1} = (\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2)$

$(\sigma_2^{-1} \sigma_5^{-1} \sigma_3 \sigma_1 \sigma_4^{-1} \sigma_2^{-1} \sigma_3) = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \sigma_2^{-1} \sigma_5^{-1} \sigma_3 \sigma_1 \sigma_4^{-1} \sigma_2^{-1} \sigma_3 = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_5^{-1} \sigma_3 \sigma_1 \sigma_4^{-1} \sigma_2^{-1} \sigma_3 = \dots = 1 = A^0$, or by using Eq. (2) and Eq. (3) we get

$$A_1 A_2 = ((1)(3)(6)(5)(2)(4)) ((1)(4)(2)(5)(6)) \\ = ((1)) \quad (5)$$

Therefore, a set contains braids $\mathcal{A}_n$ based on curves c_i , $i = 1, 2, \dots, 6$ is a 6-ordered braid group, we denoted it as $\langle \mathcal{A}_6, \cdot \rangle$ or briefly $\mathcal{A}_6$ [3].

2.3 Definition and characteristics

The braid group $\mathcal{A}_n$ is group generated by $n-1$ generators σ_i , $i = 1, \dots, n-1$ with the relations [10].

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for all } i, j = 1, 2, \dots, n-1, |i-j| \geq 2. \quad (6)$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, i = 1, 2, \dots, n-2. \quad (7)$$

In algebra computation, let $\mathbf{a} = \sigma_1 \sigma_2 \dots \sigma_{i-1} \sigma_i \sigma_{i+1} \dots \sigma_{n-1}$, based on Eq. (6) and Eq. (7) we have $\mathbf{a} \sigma_i = \mathbf{a}_{i+1} \mathbf{a}$, $i = 1, \dots, n-2$, or $\sigma_{i+1} = \mathbf{a} \sigma_i \mathbf{a}^{-1}$, $\mathbf{a} \mathbf{a}^{-1} = 1$. After some of iterations we obtain $\sigma_i = \mathbf{a}_{i-1} \sigma^{-1(i-1)}$.

Addition and insertion of structures such as Eq. (6) and Eq. (7) and their derivatives generally change the arrangement pattern in permutations that are an element of the $\mathcal{P}_n$ group. Therefore, there is a relation between two groups, $\mathcal{A}_n$ and $\mathcal{P}_n$, as the classifications of elements in braid group.

3 An Approach

As has been expressed in its definition and characteristics, the woven arrangement that forms the braid gives a special meaning to each braid. Therefore, the braid group can be represented by two representations of structure and frame, the presence of the relation between two forms is a linkage that gives a special meaning. Thus, an approach is used to express the meaning of the structure and frames of the braid and relation between them. It is general expressed by group theory that there is a classification between braids that have the same meaning or in vary [11].

A mapping from the braid group $\mathcal{A}_n$ to the $\mathcal{P}_n$ permutation group is to define the classification of braids into appropriate classes [9], i.e.

$$h : \mathcal{A}_n \rightarrow \mathcal{P}_n \quad (8)$$

then the specific meaning is based on its structure through the composition of the generator that builds a braid. Disclosure of σ with index composition will be different meaning in the presence of σ^{-1} as opposed to. The ratio includes a comparison that assesses the braid structure. Likewise with the permutation groups, the same point ratio of positions on both frames connected by the same curve c_i would differ in meanings of its

presence compared to the presence of curve c_i connecting the different points between the two frame lines [3]. Thus, it is a comparison about the structures or frames of braids and it gives the meaning of the existence of elements in braid group.

4 The Structure of Braid

Let σ_i is one generator of braids whereby its permutation is

$$P_i = \begin{vmatrix} \dots & i & j & \dots \\ \dots & j & i & \dots \end{vmatrix} \quad (9)$$

but for σ_i^2 the permutation is

$$P_0 = \begin{vmatrix} \dots & i & j & \dots \\ \dots & i & j & \dots \end{vmatrix} \quad (10)$$

and we conclude that for σ_i^p whereby p is even or $p = 0, 2, 4, \dots$ we have their permutations are like Eq. (10), while p is odd or $p = 1, 3, 5, \dots$ we have their permutations are like Eq. (9). The composition of generator is not only based on one index but a sequential index as $\sigma_i \sigma_{i+1} \sigma_{i+2} \dots$ [8,3]. Therefore, we have a lemma as follows.

4.1 Relation

Lemma 1. Let a braid with the composition of generators in $(\sigma_i \sigma_{i+1} \dots \sigma_{n-1})^p$, if $p = nj$, $j = 1, 2, \dots, k$ then their permutation is P_0 .

Proof. Based on Eq. (9) and Eq. (10), $\sigma_i \sigma_{i+1}$ has a permutation $((i)(i+2)(i+1))$ and $(\sigma_i \sigma_{i+1})^3$ has a permutation $((i)) = ((i)(i+2)(i+1))((i)(i+2)(i+1))((i)(i+2)(i+1))$. Therefore, $n_\sigma() = 3$ and $p=3$. For $\sigma_i \sigma_{i+1} \sigma_{i+2}$ has a permutation $((i)(i+3)(i+1)(i+2))$ we have $(\sigma_i \sigma_{i+1} \sigma_{i+2})^4$ has a permutation $((i))$. Thus, $n_\sigma() = 4$ or $p = 4$. For $n_\sigma() = n$, then $p = n$ where $(\sigma_i \sigma_{i+1} \dots \sigma_{n-1})^n$ has $((i))$. Therefore, because $((i))((i)) = ((i)) = P_0$, $((i))((i))((i)) = ((i)) = P_0$, ..., we simplify that for $p = nj$, $j=1, 2, \dots, k$, we have permutations of $(\sigma_i \sigma_{i+1} \dots \sigma_{n-1})^{nj}$ are P_0 .

We call $(\sigma_i \sigma_{i+1} \dots \sigma_{n-1})^{nj}$ as the *normal braids* [3,12].

Lemma 2. For each braid in a braid group, if there is one frame line on the braid then the line cuts the braid be two braid in a braid group.

Proof. Let A in $\mathcal{A}$, the line cut the braid right on the σ_i^0 for $i = 1, 2, \dots, n$ so that there two braids: A_1, A_2 in $\mathcal{A}$ or A^0, A in $\mathcal{A}$, we call them as the *resultant braids*, and be in force $A = A_1 A_2$, $A = A^0 A$, or $A = A A^0$. Because on a new frame line there are corresponding points then each resultant braid has a permutation.

Proposition 1. For each braid can be inserted other braid.

Proof. To insert any braid into another braid is by first considering the curves c_i connecting two frame lines. If both braids are in the same order, then the insertion can be done. If the number of curves c_i of the inserted braid is smaller than the number of curves c_j of braid accept the insertion, then the number of curves c_i is adjusted to the number of curves c_j by adding curves in the

appropriate part. Also it applies vice versa. Therefore, based on Lemma 2, insertion can be done by separating one braid into two braids, and then carrying out binary operations for the three braids by order first braid, the inserted braid, and the last braid.

Theorem 1. For the normal braids, the insertion of the normal braids to any other braid does not alter its permutation.

Proof. First, based on Lemma 1 the normal braids has P_0 , based on Proposition 1 the normal braids can be inserted to any braids, and based on Lemma 2 we have a new braid with applying the binary operation on three braids. In other words, $A = A_1 A_2$ in $\mathcal{A}_n$, we have $A = A_1(A^0)A_2$ whereby the permutation of A do not change. Second, let braid A has a permutation $((i)(j))$, based on Proposition 1 we have two permutations $((\dots)(i)(k)(j)(\dots))$ and $((\dots)(k)(j)(\dots))$ for example and we have $((i)(j)) = ((\dots)(i)(k)(j)(\dots))((i))((\dots)(k)(j)(\dots))$, whereby $((i))$ is presentation of the normal braids.

Corollary 1. The insertion of un-normal braids to any other braid alters its permutation.

Proof. This is as consequence of Theorem 1 to Proposition 1.

4.2 Defining characteristics

Through Lemma 1, Lemma 2, Proposition 1 and Theorem 1 has been shown that there is a function h from $\mathcal{A}_n$ into $\mathcal{P}_n$, that is

$$h(N:1) : \mathcal{A}_n \rightarrow \mathcal{P}_n \quad (11)$$

which classifies each element of $\mathcal{A}_n$ into the same class based on elements of $\mathcal{P}_n$.

Computationally, it can be declared some scales of calculation in braid group $\mathcal{A}_n$, i.e.

[S1] A number of generators allowed in the braid group $\mathcal{A}_n$, we denoted it as $n_{n-1}()$, for example the braid group $\mathcal{A}_6$ has $n_6 = 5$, or generally the braid group $\mathcal{A}_n$ based on definition of braid group having $n_n() = n-1$.

[S2] A number of generators that appears in a braid, this is expressed as the cardinality we denoted it as $n_c() = ||.||$. For example $n_c(A_1) = ||\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2|| = 5$ because only $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 that appear in a braid $A_1 = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2$, while $A_1 A_2 = \sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \sigma_1 \sigma_4^{-1} \sigma_2$ also has $n_c(A_1 A_2) = ||\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \sigma_1 \sigma_4^{-1} \sigma_2|| = 5$. Minimum of $n_c()$ is 0 (zero) because $\sigma_i^0 = 1$. Therefore, $0 \leq n_c \leq n-1$.

[S3] A size of woven arrangement in a braid, we redefined it as $n_\sigma() = |.|$. For example, $n_\sigma(A_1) = |\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2| = 7$, while $n_\sigma(A_1 A_2) = |\sigma_3^{-1} \sigma_2 \sigma_4 \sigma_1^{-1} \sigma_3^{-1} \sigma_5 \sigma_2 \sigma_1 \sigma_4^{-1} \sigma_2| = 10$. Based on $n_c()$ we have also minimum of $n_\sigma()$ is 0 (zero), but maximum of $n_\sigma()$ cannot be determined (it is uncertainty). Thus, $0 \leq n_\sigma() \leq \dots$

To generate the potential about the structure of the braids, the ratio between three measurements: S1, S2 and S3 give the different meaning. That ratio between S1 and S2,

$$r_c() = n_c() / n_{n-1}() \quad (12)$$

where $0 \leq r_c \leq 1$. It gives special meaning as a probability of a braid about structure in a braid group. In other case, we can generate the potential strength of braid in a braid group by ratio between S_1 , S_2 and S_3 ,

$$r_\sigma() = n_\sigma()/n_c() \quad (13)$$

where $0 \leq r_c \leq \dots$, and if r_σ divided by $n-1$ we have average of a braid structure or $r_{av}()$. Let A_1, A_2 in $\mathcal{A}_6$, we obtain $n_n(\mathcal{A}_6) = 5$, and then $r_c(A_1) = 5/5 = 1.0000$, $r_c(A_2) = 3/5 = 0.6000$, but $r_c(A_1A_2) = 5/5 = 1.0000$ also. Therefore, to get another meaning of structure in a braid, we use Eq. (13) and their results divided by $n-1$. That is $r_{av}(A_1) = r_\sigma(A_1)/n_n(\mathcal{A}_6) = (n_\sigma(A_1)/n_c(A_1))/n_n(\mathcal{A}_6) = (7/5)/5 = 0.2800$. Similarly, $r_{av}(A_2) = 0.2000$ and $r_{av}(A_1A_2) = 0.4000$. Thus, although $r_c(A_1) = r_c(A_1A_2)$, but $r_{av}(A_1) \neq r_{av}(A_1A_2)$.

In permutation, it can declare some scales of calculation in permutation group $\mathcal{P}_n$, i.e.

[S4] Number of points in the frame line, we denoted it as $n_n()$. For example, $n_n(\mathcal{P}_6) = 6$ for a group braid $\mathcal{A}_6$.

[S5] Number of points in same position on two frame lines, we denoted it as $n_p()$. For example, for a braid A_1 we have $n_p(A_1) = 6$.

[S6] Number of points in different position on two frame lines, we denoted it as $n_q()$. For example, for a braid A_1 we obtain $n_q(A_1) = 0$.

If probability of same position of points in two frame lines is

$$r_p() = n_p()/n_n(), \quad 0 \leq r_p() \leq 1, \quad (14)$$

and probability of different position of points in two frame lines is

$$r_q() = n_q()/n_n(), \quad 0 \leq r_q() \leq 1, \quad (15)$$

then $r_p() = 1 - r_q()$. For example, $r_p(A_1) = 6/6 = 1$ and $r_q(A_1) = 0/6 = 0$. Thus Eq. (12), (13), (14), and (15) redefine h on Eq. (11) for a braid group and a permutation group.

5 The conclusion

By involving the definition of braid groups, several characteristics are presented to reveal the meaning of each braid. A function h has given relation between braid group and permutation group, it has shown the characteristics expressed through Eq. (12), (13), (14), and (15) give different meanings for each braid structure. This also includes the braids classification.

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