The kumjian-pask algebras of finite finitely aligned k-graphs without cycle as modules

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Abstract. In general, every algebra can be viewed as a module over itself. In this paper, we analyze the structure of Kumjian-Pask algebras of finite finitely aligned k-graph without cycles via module theory.

1 Introduction

For a directed graph E , it can be constructed, in general, the corresponding graph algebra. Actually, graph algebra is a C^* -algebra satisfying some specific conditions. One of the most important notion in the construction of the algebra is the Cuntz-Krieger relation. Many important results on this topic have been published by researchers. One may refer to [1] for the detailed elaboration about it.

In 2005, Gene Abrams and Aranda Pino [2] introduced the Leavitt path algebra for a directed graph, which can be viewed as an algebraic analogue of the graph algebra. Since then, the study on this area goes on to obtain a complete description about this algebra.

For many reason, the concept directed graph is extended to the so-called k-graph which is the higher rank version of it. For a k-graph Λ , one may interested in the corresponding algebra and obtains many important results. Many researchers have studied this topic for the cases row-finite without sources, locally convex, and finitely aligned k-graph [3].

Some of them concern in the algebraic analogue of it. Aranda Pino... [4] introduced this algebra and called it Kumjian-Pask Algebra, for the row-finite Λ without sources. This study is followed by L.O. Clark et.al [5], for the locally convex case. In [6], L.O. Clark and Pangalela published their results for finitely aligned k-graph.

In 2016, Larki studied the simplicity of Kumjian-Pask algebras and gave a sufficient condition for the algebra to be purely infinite. In particular, Larki obtained a characterization for a finite-dimensional Kumjian-Pask Algebra [7].

2 Preliminaries

We write $\mathbb{N}$ for the set of natural numbers, and let $k \in \mathbb{N}$. The symbol $\mathbb{N}^k$ means the set of k-tuples of non-negative integers. For $m, n \in \mathbb{N}^k$, $m \leq n$ means $m_i \leq n_i$, for all $1 \leq i \leq k$, and $m \vee n$ means their coordinate-wise maximum. Note that we can regard $\mathbb{N}^k$ as a category with one object.

A k-graph Λ is a countable category Λ together with functor (degree map) $d: \Lambda \rightarrow \mathbb{N}^k$ satisfying the factorization property: for any $\lambda \in \Lambda$ with $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. In this case, μ is denoted also by $\lambda(0, m)$.

The set of objects (vertex) in Λ is denoted by Λ^0 , and can be identified as the identity morphism at the object. Accordingly, the set of paths which is not a vertex is denoted by $\Lambda^{\neq 0}$. For $\lambda \in \Lambda$ the vertices $s(\lambda)$, $r(\lambda)$ are called the source, the range, respectively, of λ .

Here are some notations will be used throughout the paper. For $E \subset \Lambda$, $v \in \Lambda^0$, set,

$$\begin{aligned}\Lambda^n &:= \{\lambda \in \Lambda : d(\lambda) = n\} \\ vE &:= \{\lambda \in \Lambda : r(\lambda) = v\} \\ Ev &:= \{\lambda \in \Lambda : s(\lambda) = v\}\end{aligned}$$

For example, a directed graph E is actually a 1-graph. The corresponding degree map is the length function $l: E^1 \rightarrow \mathbb{N}$. It is clear that l satisfies the factorization property.

Let $k \geq 1$, and $m \in (\mathbb{N} \cup \{\infty\})^k$. Define

$$\Omega_{k,m} := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}.$$

For the path $(p, q) \in \Omega_{k,m}$, the range, source and degree are defined as follow

$$r(p, q) = p, (p, q) = q, d(p, q) = q - p.$$

One may check that $\Omega_{k,m}$ is a k-graph.

Let Λ be a k-graph, and $\lambda, \mu \in \Lambda$. The path τ is called minimal common extension of λ and μ if

$$d(\tau) = d(\lambda) \vee d(\mu), \tau(0, d(\lambda)) = \lambda, \tau(0, d(\mu)) = \mu.$$

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The set of all minimal common extension of λ and μ is denoted by $\text{MCE}(\lambda, \mu)$, and define

$$\Lambda^{\min}(\lambda, \mu) = \{(\rho, \tau) \in \Lambda \times \Lambda : \lambda\rho = \mu\tau \in \text{MCE}(\lambda, \mu)\}.$$

If $\text{MCE}(\lambda, \mu)$ is finite for all $\lambda, \mu \in \Lambda$, then Λ is said to be finitely aligned. Moreover, for $E \subset \Lambda$ and $\lambda \in \Lambda$, define

$$\text{Ext}(\lambda; E) = \bigcup_{\mu \in E} \{(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)\}.$$

A set $E \subset v\Lambda$ is exhaustive if for each $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. Now, define $\text{FE}(\Lambda) = \{E \subset v\Lambda \setminus \{v\} : E \text{ is finite and exhaustive}\}$.

Let Λ be a k -graph. If $v\Lambda^n$ is finite for each vertex v and $n \in \mathbb{N}^k$, then Λ is called row-finite. The vertex v is called source in the case $v\Lambda^n = \emptyset$ for some $n \in \mathbb{N}^k$. The k -graph Λ is said to be locally-convex if for each $v \in \Lambda^0$, $1 \leq i, j \leq k$ with $ij \neq \emptyset$, $\lambda \in v\Lambda^{e_i}$, $\mu \in v\Lambda^{e_j}$, the sets $s(\mu)\Lambda^{e_i}$, $s(\lambda)\Lambda^{e_j}$ are nonempty. One may realize that a finitely aligned k -graph is more general than the row-finite and the locally-convex ones.

Now, we turn to the concept of Kumjian-Pask algebra. Assume that Λ is a finitely aligned k -graph, and R is a commutative ring with identity. For a path $\lambda \in \Lambda$, the ghost path $\lambda^* \in \Lambda$ is defined as that of path satisfying

$$d(\lambda^*) = -d(\lambda), r(\lambda^*) = s(\lambda), s(\lambda^*) = r(\lambda).$$

The collection of all ghost path is denoted by $G(\Lambda)$, and the composition on it is defined as $\lambda^* \mu^* = (\mu\lambda)^*$

Definition 2.1. A Kumjian-Pask Λ -family $\{S_\lambda, S_{\mu^*} : \lambda, \mu \in \Lambda\}$ in an R -algebra A consists of $S : \Lambda \cup G(\Lambda^{\neq 0}) \rightarrow A$ such that:

(KP1) $\{S_v : v \in \Lambda^0\}$ is collection of mutually orthogonal idempotents;

(KP2) for $\lambda, \mu \in \Lambda$ with $(\lambda) = r(\mu)$, $S_\lambda S_\mu = S_{\lambda\mu}$ and $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$;

(KP3) $S_\lambda S_\mu = \sum_{(\rho, \tau) \in \Lambda^{\min}(\lambda, \mu)} S_\rho S_\tau$ for all $\lambda, \mu \in \Lambda$;

(KP4) $\prod_{\lambda \in E} (S_{r(E)} - S_\lambda S_{\lambda^*}) = 0$ for all $E \in \text{FE}(\Lambda)$.

3 Method

The structure of algebra has a deep connection with that of ring and module. In this paper we use ring and module theory to analyze the structure of the Kumjian-Pask algebra.

4 Discussion

One may see that the structure of a Kumjian-Pask algebra is more complicated than the other algebras, even for the 2-graph case. But for the finite case without cycles, the algebra can be viewed as a direct sum of matrices spaces. The following is the result of Larki.

Theorem 4.1. Let Λ be a finitely aligned k -graph and R be a unital commutative ring. Then $\text{KP}_R(\Lambda)$ is a finite-dimensional R -algebra if and only if Λ is finite and contains no cycles. In particular, in the case we have

$$\text{KP}_R(\Lambda) \cong \bigoplus_{v\Lambda = \{v\}} M_{|v\Lambda|}(R) \quad (1)$$

Note that if Λ is finite and without cycles, there must exist, at least, a vertex v satisfying $v\Lambda = \{v\}$. Hence, we have the fact that for a single vertex v satisfying $v\Lambda = \{v\}$ the $\text{KP}_R(\Lambda)$ is no more than the ring $M_n(R)$ for some $n \in \mathbb{N}$, together with the multiplication in the left by elements in R . Based on this fact, we shall use the related results in ring theory to study the $\text{KP}_R(\Lambda)$.

Here are some definitions and the corresponding facts.

Definition 4.2. A ring R with identity is simple if it has no nontrivial proper ideals.

Definition 4.3. [8, Definition 1.7]. An R -module is semi-simple if it is a direct sum of simple R -modules.

Definition 4.4. [9, Definition 4.1] A ring R is primitive if it has faithful irreducible representation.

Definition 4.5. [9, Definition 4.1] A ring R is semi-primitive if for any $a \neq 0$ in R there exists an irreducible representation ρ such that $\rho(a) \neq 0$.

Proposition 4.6. Any simple ring is primitive.

Note that C is simple, hence as a ring, $M_n(C)$ is simple, and therefore, $M_n(C)$ is primitive.

Assume that Λ is a finite and finitely aligned k -graph without cycles. By the Theorem of Larki, the corresponding $\text{KP}_C(\Lambda)$ is finite dimensional. Moreover, we can view $\text{KP}_C(\Lambda)$ as a C^* -algebra. So, in general, the facts about the C^* -algebra also apply in $\text{KP}_C(\Lambda)$.

If we assume that Λ has a source v satisfying $\{v\} = v\Lambda$, then we have the fact $\text{KP}_C(\Lambda) \cong M_{|v\Lambda|}(C)$. In addition, the simplicity of $M_{|v\Lambda|}(C)$ implies the simplicity of $\text{KP}_C(\Lambda)$. Hence, there exists no non-trivial ideal in the $\text{KP}_C(\Lambda)$.

In the case Λ has sources $\{v_1, \dots, v_n\}$, we have,

$$\text{KP}_C(\Lambda) \cong \bigoplus_{i=1}^n M_{|v_i\Lambda|}(C)$$

So, as a ring, $\text{KP}_C(\Lambda)$ is semi-simple. Note that $\text{KP}_C(\Lambda)$ is finitely generated (over division ring C), hence $\text{KP}_C(\Lambda)$ is artinian [9, pp. 103]. Using

Wedderburn-Artin Theorem, $KP_C(\Lambda)$ is semi-primitive [9, pp. 203].

Proposition 4.7. Let I be an ideal in $M_{|\Lambda_{vi}|}(C) \oplus M_{|\Lambda_{vj}|}(C)$. Then $I = 0 \oplus M_{|\Lambda_{vj}|}(C)$ or $I = M_{|\Lambda_{vi}|}(C) \oplus 0$.

Corollary 4.8. The ideals in $KP_C(\Lambda)$ are of the form $I = 0 \oplus M_n(C)$ and $I = M_p(C) \oplus 0$.

5 Conclusions

We can analyze the structure of Kumjian-Pask algebra of finite finitely aligned k -graph via module and ring theory. In the case, we see that the Kumjian-Pask algebra is no other than a direct sum of matrices algebra.

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