Representations of cyclically ordered semigroups

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Abstract. Suppose G is a group. We equip the group G with a cyclic order in such a way so that $P(G)$, the positive cone under the cyclic order of G is a semigroup. We construct a representation of the semigroup $P(G)$ as isometries acting on a certain Hilbert space.

1 Introduction

A cyclic order on a group $(G, +)$ is a ternary relation on G satisfying some conditions. Given a group $(G, +)$, and a cyclic order R on G which is compatible with the operation on G , the pair $(G, +, R)$ is called as a cyclically ordered group. The notion of cyclically ordered group was firstly introduced by Rieger in his article series [1]. Some fundamental properties of cyclically ordered group can be found for examples in [2-5].

Given any linearly ordered group $(G, +, <)$, it can be constructed a cyclic order R induced from the linear order $<$ such that $(G, +, R)$ is a cyclically ordered group. However, it does not apply otherwise, given a cyclically ordered group $(G, +, R)$ which is not necessarily linear. Eventhough through the unwinding process [1], one can get the linearly ordered group $uw(G)$, but in general $uw(G) \neq G$, even when the cyclically ordered group $(G, +, R)$ is linear, for instance see [6]. Some cyclically ordered group can not be reversed to its linear counterpart, for example we can define a cyclic order R on the modulo group $\mathbb{Z}_n$ of integers, but there is no linear order can be induced from R . Therefore, the class of cyclically ordered group is larger than the class of linearly ordered group.

Suppose $(H, +, <)$ is a linearly ordered group. The positive cone H^+ of H is a semigroup, which is also linearly ordered under the same order $<$. If $(G, +, R)$ is a cyclically ordered group, the positive cone $P(G)$ of G is not necessarily a semigroup. In a more specific class, namely linear cyclically ordered group, the positive cone $P(G)$ could become a semigroup, and it becomes a cyclically ordered group under the cyclic order R inherited from G .

Suppose $(G, +, <)$ is a linearly ordered group. Representation of linearly ordered semigroup G^+ as operators acting on a Hilbert space is a crucial aspect in the theory of C^* -algebras crossed products by semigroups [7-11].

Any cyclically ordered group $(G, +, R)$ possesses a representation. Representation theory of cyclically ordered groups was firstly developed in [2] and [5].

However, the representation did not realise the cyclically ordered group as operators on a Hilbert space.

Given a linear cyclically ordered group $(G, +, R)$, then the positive cone $P(G)$ of G is a semigroup with identity. In this article we construct a representation of the semigroup $P(G)$ as isometries acting on a certain Hilbert space.

2 Preliminaries on Cyclically Ordered Groups

2.1 Cyclically Ordered Groups

Let $(G, +)$ be a group (it is not necessary to be abelian), and R be a ternary relation on G satisfying the following axioms:

- R1: $\forall x, y, z, R(x, y, z) \Rightarrow x \neq y \neq z \neq x$ (R is strict),
- R2: $\forall x, y, z, x \neq y \neq z \neq x \Rightarrow R(x, y, z)$ or $R(x, z, y)$ (R is total),
- R3: $\forall x, y, z, R(x, y, z) \Rightarrow R(y, z, x)$ (R is cyclic),
- R4: $\forall x, y, z, u, R(x, y, z)$ and $R(y, u, z) \Rightarrow R(x, u, z)$ (R is transitive),
- R5: $\forall x, y, z, u, v, R(x, y, z) \Rightarrow R(u + x + v, u + y + v, u + z + v)$ (R is compatible with the operation in G).

The group G which is equipped with such ternary relation R is called a cyclically ordered group, and denoted by $(G, +, R)$.

Example 1

Suppose $n \in \mathbb{N}$, the abelian group $\mathbb{Z}_n$ can be equipped with a cyclic order defined by,

$R(\bar{i}, \bar{j}, \bar{k}) \Leftrightarrow$
there exists $\bar{i}', \bar{j}', \bar{k}' \in \{\bar{0}, \dots, \bar{n-1}\}$
such that $\bar{i}' = \bar{i}, \bar{j}' = \bar{j}, \bar{k}' = \bar{k}$ and
 $i' < j' < k'$ or $j' < k' < i'$ or $k' < i' < j'$. The finite group $(\mathbb{Z}_n, R)$ is then a cyclically ordered abelian group.

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Example 2

Consider the multiplicative group $(\mathbb{T}, \cdot)$ which consists of all complex numbers of length 1. Every $z \in \mathbb{T}$ can be written as $e^{i\theta_z}$, for some $\theta_z \in [0, 2\pi)$. Define a ternary relation by $R(x, y, z)$ by,

$$\theta_x < \theta_y < \theta_z \text{ or } \theta_y < \theta_z < \theta_x \text{ or } \theta_z < \theta_x < \theta_y.$$

Then R is a cyclic order in $\mathbb{T}$, and $(\mathbb{T}, R)$ is a cyclically ordered abelian group.

In general, every linearly ordered group (under a linear order $<$) can be equipped with a cyclic order. Suppose $(G, +, <)$ is a linearly ordered group. For $x, y, z \in G$, define $R(x, y, z)$ by,

$$x < y < z \text{ or } y < z < x \text{ or } z < y < x.$$

Then R is cyclic order on G , hence $(G, +, R)$ is a cyclically ordered group, and $(G, +, R)$ is called the cyclic order group associated to $(G, +, <)$. In this case $(G, +, R)$ is called a linear cyclically ordered group.

2.2 Positive Cone of a Cyclically Ordered Group

Let $(G, +, R)$ be a cyclically ordered group. The positive cone of (G, R) is the set,

$$P(G) := \{x \in G \mid R(0, x, 2x)\} \cup \{0\}.$$

Unlike in a linearly ordered group, a positive cone of a cyclically ordered abelian group is not necessarily a semigroup. We note that, by the definition of positive cone, $P(G)$ always contains the identity 0.

Example 3:

Consider the cyclically ordered group $(\mathbb{Z}_5, +, R)$. Then we get that $P(G) = \{\bar{0}, \bar{1}, \bar{2}\}$, but $\bar{2} + \bar{2} = \bar{4} \notin P(G)$. Hence in this case, $P(G)$ is not a semigroup. In general, when $G = \mathbb{Z}_n$, the positive cone $P(G)$ is not a semigroup.

Example 4:

Consider the multiplicative group $(\mathbb{T}, \cdot)$ discussed in Example 2. Suppose $x = e^{i\frac{7}{8}\pi}$, $y = e^{i\frac{15}{16}\pi}$, then $x, y \in P(\mathbb{T})$. But $xy = e^{i\frac{29}{16}\pi}$ is not in $P(\mathbb{T})$, because $(xy)^2 = e^{i\frac{58}{16}\pi}$, but we don't have $R(1, e^{i\frac{29}{16}\pi}, e^{i\frac{58}{16}\pi}) = e^{i\frac{26}{16}\pi}$. Hence $P(\mathbb{T})$ is not a semigroup, because it is not closed under the operation on $\mathbb{T}$.

Suppose $(G, +, R)$ is a cyclically ordered group. If there is a linear order $<$ on G such that $(G, +, <)$ is linearly ordered group and the cyclic order R is induced by this linear order, $(G, +, R)$ is called linear cyclically ordered group. A necessary and sufficient condition for linear cyclically ordered group was given in [5, Lemma 3.3]. Another version of linear cyclically ordered group was given in [12] which states that a cyclically ordered group is linear if and only if the positive cone is closed under operation in G .

Theorem 1 [12]

A cyclically ordered group $(G, +, R)$ is linear if and only if $P(G)P(G) \subset P(G)$.

We note that when $(G, +, R)$ is a linear cyclically ordered group, the restriction of the ternary relation R to $P(G)$ gives a cyclic order on $P(G)$. In other words, the axioms R1-R5 are still compatible with the semigroup $P(G)$. Hence the positive cone $P(G)$ is cyclically ordered, and we call $(P(G), +, R)$ a cyclically ordered semigroup.

3 Representations of Cyclically Ordered Semigroups

In [5], it was proved that based on fundamental result of [2], every cyclically ordered group possesses a representation. The cyclically ordered group is represented as a direct product of a linear cyclically ordered group, the cyclically ordered group of real numbers $(0, 1)$ with operation mod 1, and the cyclic order induced from the linear order of the group $\mathbb{R}$ of real numbers.

Given a linear cyclically ordered group $(G, +, R)$, hence the positive cone $P(G)$ is a locally ordered semigroup. In this section we construct a representation of the semigroup $P(G)$ as isometries acting on a Hilbert space.

3.1 The Hilbert Space

Suppose $(G, +, R)$ is a linear cyclically ordered group, then the positive cone $P(G)$ is a semigroup with identity. Let $\ell^2(P(G), \mathbb{C})$ be the space of all maps,

$$f: P(G) \rightarrow \mathbb{C}$$

which are norm-square summable, i.e.,

$$\sum_{x \in P(G)} |f(x)|^2 < \infty.$$

Under the norm

$$\|f\| = \left(\sum_{x \in P(G)} |f(x)|^2 \right)^{\frac{1}{2}},$$

and scalar product given by

$$\langle f | g \rangle = \sum_{x \in P(G)} f(x) \overline{g(x)},$$

$\ell^2(P(G), \mathbb{C})$ is a Hilbert space.

In the next sub section, we construct a representation of cyclically ordered group $(G, +, R)$ as isometries acting on this Hilbert space.

3.2 The Representation

Definition 1

Suppose $(S, +, R)$ is a cyclically ordered semigroup. For a Hilbert space H , let $B(H)$ be the C^* -algebra of all bounded linear operators on H . A mapping,

$$W: S \rightarrow B(H), x \mapsto W_x$$

is called an isometric representation if each W_x is an isometry, and W is a semigroup homomorphism.

Suppose $(G, +, R)$ is a linear cyclically ordered group. Since $(G, +, R)$ is linear, then $P(G)$ is a semigroup with identity. We are going to construct an isometric representation of $P(G)$.

For $x \in P(G)$, we define a mapping

$$V_x: \ell^2(P(G), \mathbb{C}) \rightarrow \ell^2(P(G), \mathbb{C})$$

which maps $f \mapsto V_x(f)$ such that

$$(V_x(f))(y) := \begin{cases} f(y-x), & \text{if } y-x \in P(G) \\ 0, & \text{other wise.} \end{cases}$$

Then V_x is a bounded linear operator on $\ell^2(P(G), \mathbb{C})$.

$$\begin{aligned} \text{Since } \|V_x(f)\|^2 &= \sum_{y \in P(G)} |V_x(f)(y)|^2 \\ &= \sum_{y \in P(G), y-x \in P(G)} |f(y-x)|^2 \\ &= \sum_{z \in P(G)} |f(z)|^2 \\ &= \|f\|^2, \end{aligned}$$

V_x is an isometry. A routine computation shows that $V: x \mapsto V_x$ is a semigroup homomorphism of $P(G)$ into $B(\ell^2(P(G), \mathbb{C}))$. Therefore, V is an isometric representation of the semigroup $P(G)$ on the Hilbert space $\ell^2(P(G), \mathbb{C})$.

Above paragraph gives a proof of our theorem:

Theorem 2

Suppose $(G, +, R)$ is a linear cyclically ordered group. Then the positive cone $P(G)$ possesses an isometric representation, and the representation is given by,

$$\begin{aligned} V: P(G) &\rightarrow B(\ell^2(P(G), \mathbb{C})) \\ x &\mapsto V_x \end{aligned}$$

such that,

$$(V_x(f))(y) := \begin{cases} f(y-x), & \text{if } y-x \in P(G) \\ 0, & \text{other wise.} \end{cases}$$

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